Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 31 (1981), No. 2, 229–240

Persistent URL: `http://dml.cz/dmlcz/101740`

Czechoslovak Mathematical Journal, 31 (106) 1981, Praha

ON QUASI-RIEMANNIAN FIBER MANIFOLD

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(Received August 14, 1979)

Let $\pi : E \to M$ be a fiber bundle with a total space E, a base space M and a projection π. Let ω be a symmetric regular bilinear form on E. Denote by γ the quasi-Riemannian connection of the quasi-Riemannian fiber manifold (E, ω). Let Γ be the generalized connection on $\pi: E \to M$ the horizontal vector of which at any $u \in E$ are such vectors $X \in T_uE$ that $\omega(Y, X) = 0$ for every vertical vector $Y \in T_uE$. The purpose of this paper is to find the necessary and sufficient condition for $\gamma . \Gamma$ to be reducible to the connection $V\Gamma$ on $VE \to M$, where $V\Gamma$ is the vertical prolongation of Γ, VE is the vector bundle of vertical vectors on E and $\gamma . \Gamma$ is the composition of γ and Γ.

1. First we recall two equivalent definitions of the generalized connection Γ on a fiber bundle E.

(a) Let $J^1E \to E$ be a fiber bundle of the 1-jets of all local sections $\sigma : M \to E$. Then a generalized connection on E is a global cross-cestion $\Gamma : E \to J^1E$, see for example [4]. In the case of a vector bundle E, a connection Γ is linear if the mapping $\Gamma : E \to J^1E$ is linear on every fiber of E.

(b) A generalized connection on E is a splitting Γ of the exact sequence

$$0 \to VE \to TE \leftrightarrows^{\Gamma} TM \to 0 .$$

In local coordinate charts (x^i) on M, (x^i, y^α) on E, $(x^i, y^\alpha, \xi^i, \eta^\alpha)$ on TE, (x^i, ξ^i) on TM, $(x^i, y^\alpha, y^\alpha_i)$ on J^1E a generalized connection Γ on E is determined by

$$(x^i, y^\alpha) \mapsto (x^i, y^\alpha, y^\alpha_i = a^\alpha_i(x, y)) \quad \text{or}$$
$$(x^i, y^\alpha) \mapsto [(x^i, \xi^i) \mapsto^{\Gamma} (x^i, y^\alpha, \xi^i, \eta^\alpha = a^\alpha_i(x, y)\, \xi^i)]$$

or quite shortly by the equation

$$\mathrm{d}y^\alpha = a^\alpha_i(x, y)\, \mathrm{d}x^i .$$

Let $X \in T_mM$, $h \in E_m$. Then $\Gamma X \in T_hE$ is called a *Γ-lift of X* at h. Denote by Γ_h the subspace $\Gamma_h(T_mM) \subset T_hE$ of all the so called *Γ-horizontal vectors at h*. We have $T_hE = V_hE + \Gamma_h$ and two canonical projections $v_\Gamma : TE \to VE$, $h_\Gamma : TE \to H_\Gamma E$, where $H_\Gamma E$ is the vector bundle of all *Γ-horizontal vectors on E.*

Let us recall that the curvature of Γ is a global cross-section $\Phi : E \to VE \otimes \otimes \bigwedge^2 T^*M$, which has the coordinate form

$$\Phi = \left(\frac{\partial a_i^\alpha}{\partial x_j} + \frac{\partial a_i^\alpha}{\partial y^\beta} a_j^\beta\right) \mathrm{d}x^i \wedge \mathrm{d}x^j \otimes \partial/\partial y^\alpha . \tag{1}$$

In the case of a generalized connection Γ_1 on a subspace $\pi_1 : E_1 \to M$ of $\pi : E \to M$ we say that a generalized connection Γ on E is reducible to E_1 or to Γ_1 if $\Gamma|_{E_1}$ is a connection on E_1 or if $\Gamma|_{E_1} = \Gamma_1$, respectively.

Let $p_i : F_i \to E$, $i = 1, 2$, be vector bundles over a fiber bundle E. Let $p : F_1 \oplus \oplus F_2 \to E$ be the direct sum of F_1 and F_2 over E. Denote by $\varkappa_i : F_1 \oplus F_2 \to F_i$ the canonical projection on the i-factor. Let $Y_1 \in T_a F_1$, $Y_2 \in T_b F_2$, where $Tp_1 Y_1 = = Tp_2 Y_2$. Then there is such a unique vector $Y = Y_1 \oplus Y_2 \in T_{a+b}(F_1 \oplus F_2)$ that $T\varkappa_i(Y) = Y_i$. The construction of the direct sum $\gamma_1 + \gamma_2$ of two connections γ_1 on $F_1 \to E$ and γ_2 on $F_2 \to E$ is well known, see [3]. Now, let γ_i be a connection on $\pi . p_i : F_i \to M$, $i = 1, 2$, projectable over a connection Γ on $E \to M$, i.e., every vector $Tp_i X$ is Γ-horizontal for any γ_i-horizontal vector $X \in TF_i$. Let $\gamma_i X$ be the γ_i-lift of $X \in TM$ at $a_i \in F_i$. We will say that a connection $\gamma := \gamma_1 \oplus \gamma_2$ on $\pi p : F_1 \oplus \oplus F_2 \to M$ is the *semi-direct sum of* γ_1 *and* γ_2 if

$$\gamma X = \gamma_1 X \oplus \gamma_2 X ,$$

where γX is the γ-lift of X at $a_1 + a_2 \in F_1 \oplus F_2$. Let us recall that a connection γ_i on $F_i \to M$ projectable over Γ on E is semi-linear if the morphism $\gamma_i : F_i \to J^1 F_i$ over $\Gamma : E \to J^1 E$ is linear, see [6]. Identifying $F_1 \equiv F_1 \oplus 0 \subset F_1 \oplus F_2$, $F_2 = 0 \oplus \oplus F_2 \subset F_1 \oplus F_2$ we have

Lemma 1. *Let $\gamma_1, \gamma_2, \gamma$ be semilinear connections on $\pi p_1 : F_1 \to M$, $\pi p_2 : F_2 \to M$, $\pi p : F_1 \oplus F_2 \to M$ projectable over Γ on $\pi : E \to M$. Then $\gamma = \gamma_1 \oplus \gamma_2$ if and only if γ is reducible to γ_1 and to γ_2.*

2. Let T be the tangent functor from the category $\mathcal{M}$ of differentiable manifolds to the category $\mathcal{VFM}$ of vector bundles: if $M \in \mathcal{M}$ then TM is the tangent bundle of M and if $f : M \to N(M, N \in \mathcal{M})$ is differentiable then Tf is the tangent mapping of f. Let $X = a^i(x)\, \partial/\partial x^i$ be a vector field on M with a flow Φ_t. Then $T\Phi_t$ determines the field

$$TX = a^i\, \partial/\partial x^i + \frac{\partial a^i}{\partial x^k} \xi^k\, \partial/\partial \xi^i$$

on TM. For any $h \in TM$ it yields a linear morphism

$$\tau_h : J^1(TM)_{ph} \to T_h TM ,$$

where $p : TM \to M$ is the fiber projection. Let $h = (x^i, \xi^i)$, $u = (x^i, c^i, c^i_j) \in$ $\in J^1(TM)_{ph}$. There is such a vector field Y on M that $u = j^1_x Y$. Then $\tau_h(u) = T\,Y(h) =$ $= (x^i, \xi^i, c^i, c^i_j \xi^j)$. In the case of a general prolongation functor the mapping τ_h was established by Kolář [5].

Let $(x^i, c^i) \mapsto (x^i, c^i, c^i_j = a^i_j(x, c))$ be a connection on TM. Then the mapping $\tau_h\lambda : T_{\pi h}M \to T_h TM$,

$$(x^i, c^i) \to^{\lambda} (x^i, c^i, c^i_j = a^i_j(x, c) \to^{\tau_h} (x^i, \xi^i, c^i, a^i_j(x, c)\, \xi^j)\,,$$

is a connection on TM if abd only if λ is linear, i.e., iff $a^i_j(x, c)\, \xi^j = \Gamma^i_{jk}(x)\, c^k \xi^j$. This yields

Proposition 1. *If λ is a linear connection on TM, then $h \mapsto \tau_h\lambda$ is the connection transposed to λ.*

Let Γ be a connection on $\pi : E \to M$. Let X be a vector field on M and let ΓX be the Γ-lift of X on E. Denote by $J^1\Gamma X$ the set of all 1-jets of the cross-section $\Gamma X :$ $: E \to TE$. Then $h \mapsto \tau_h(J^1\Gamma X)$ is a vector field on TE. In coordinates, $\Gamma : \mathrm{d}y^\alpha =$ $= a^\alpha_i(x, y)\, \mathrm{d}x^i$, $X = a^i(x)\, \partial/\partial x^i$, $h = (x^i, y^\alpha, \xi^i, \eta^\alpha)$, hence

$$(x^i, y^\alpha, c^i, c^\alpha, c^i_j, c^i_\alpha, c^\alpha_i, c^\alpha_\beta) \mapsto^{\tau_h} (x^i, y^\alpha, \xi^i, \eta^\alpha, c^i, c^\alpha, c^i_j\xi^j + c^i_\alpha\eta^\alpha, c^\alpha_i\xi^i + c^\alpha_\beta\eta^\beta)\,.$$

So the equations of $J^1\Gamma X \subset J^1(TY \to Y)$ are

$$\bar{x}^i = x^i\,, \quad \bar{y}^\alpha = y^\alpha\,, \quad c^i = a^i\,, \quad c^\alpha = a^\alpha_i a^i\,, \quad c^i_j = \frac{\partial a^i}{\partial x^j}\,, \quad c^i_\alpha = 0\,,$$

$$c^\alpha_j = \frac{\partial a^\alpha_i}{\partial x^j}\, a^i + a^\alpha_i\, \frac{\partial a^i}{\partial x^j}\,, \quad c^\alpha_\beta = \frac{\partial a^\alpha_i}{\partial y^\beta}\, a^i\,.$$

Then

(3)
$$\tau_h(J^1\Gamma X) = a^i\, \partial/\partial x^i + a^\alpha_i a^i\, \partial/\partial y^\alpha + \frac{\partial a^i}{\partial x^j}\, \xi^j\, \partial/\partial \xi^i + \left(\frac{\partial a^\alpha_i}{\partial x^j}\, a^i \xi^j + a^\alpha_i\, \frac{\partial a^i}{\partial x^j}\, \xi^j + \frac{\partial a^\alpha_i}{\partial y^\beta}\, a^i \eta^\beta\right) \partial/\partial y^\alpha\,.$$

After restricting $\tau_h(J^1\Gamma X)$ to VE, (3) describes lifting with respect to a unique connection $V\Gamma$ on $VE \to X$, see [6]:

(A)
$$\mathrm{d}y^\alpha = a^\alpha_i(x, y)\, \mathrm{d}x^i\,, \quad \mathrm{d}\eta^\alpha = \frac{\partial a^\alpha_i}{\partial y^\beta}\, \eta^\beta\, \mathrm{d}x^i\,.$$

Let $Z = c^i\, \partial/\partial x^i + b^i\, \partial/\partial \xi^i \in T_{T\pi h}TM$. There is such a local vector field $X = a^i(x)\,.$ $.\, \partial/\partial x^i$ on M that $TX(T\pi h) = Z$, i.e. $a^i(x) = c^i$, $(\partial a^i(x)/\partial x^j)\,.\, \xi^j = b^i$. Putting $T\Gamma(Z) = \tau_h(J^1\Gamma X)$ we have a splitting $T\Gamma$ of the exact sequence

$$0 \to VTE \to TTE \leftrightarrows^{T\Gamma} TTM \to 0\,,$$

i.e., we have a connection $T\Gamma$ on $T\pi : TE \to TM$:

$$\mathrm{d}y^\alpha = a_i^\alpha(x, y)\,\mathrm{d}x^i\,, \quad \mathrm{d}\eta^\alpha = \left(\frac{\partial a_i^\alpha}{\partial x^j}\,\xi^j + \frac{\partial a_i^\alpha}{\partial y^\beta}\,\eta^\beta\right)\mathrm{d}x^i + a_i^\alpha\,\mathrm{d}\xi^i\,.$$

(Another construction of $T\Gamma$ is given in [7].)

Let $\lambda : \mathrm{d}\xi^i = a_j^i(x, \xi)\,\mathrm{d}x^j$ be a generalized connection on TM. Then the composition $T\Gamma\,.\,\lambda$

(4)
$$\mathrm{d}\xi^i = a_j^i(x, \xi)\,\mathrm{d}x^j\,,$$
$$\mathrm{d}y^\alpha = a_i^\alpha(x, y)\,\mathrm{d}x^i\,,$$
$$\mathrm{d}\eta^\alpha = \left(\frac{\partial a_i^\alpha}{\partial x^j}\,\xi^j + \frac{\partial a_i^\alpha}{\partial y^\beta}\,\eta^\beta + a_j^\alpha a_i^j\right)\mathrm{d}x^i$$

is a connection on $TE \to E \to M$, restricting $T\pi$ to $H_\Gamma E$ we obtain the morphism $\varphi : H_\Gamma E \to TM$ which on $H_\Gamma E \to E$ determines the induced connection $\varphi^*\lambda$. As $\eta^\alpha = a_i^\alpha(x, y)\,\xi^i$ are the equations of the subspace $H_\Gamma E \subset TE$, then a vector $\mathrm{d}x^i\,\partial/\partial x^i + \mathrm{d}y^\alpha\,\partial/\partial y^\alpha + \mathrm{d}\xi^i\,\partial/\partial\xi^i + \mathrm{d}\eta^\alpha\,\partial/\partial\eta^\alpha$ is tangent to $H_\Gamma E$ if and only if

(5)
$$\mathrm{d}\eta^\alpha = \frac{\partial a_i^\alpha}{\partial x^k}\,\xi^i\,\mathrm{d}x^k + \frac{\partial a_i^\alpha}{\partial y^\beta}\,\xi^i\,\mathrm{d}y^\beta + a_i^\alpha\,\mathrm{d}\xi^i\,.$$

That yields the following equations of $\varphi^*\lambda$:

(6)
$$\mathrm{d}\xi^i = a_j^i(x, \xi)\,\mathrm{d}x^j\,,$$
$$\mathrm{d}\eta^\alpha = \left(\frac{\partial a_j^\alpha}{\partial x^i}\,\xi^j + a_j^\alpha a_i^j\right)\mathrm{d}x^i + \frac{\partial a_j^\alpha}{\partial y^\beta}\,\xi^j\,\mathrm{d}y^\beta\,.$$

Then

(7)
$$\mathrm{d}\xi^i = a_j^i\,\mathrm{d}x^j\,,$$
$$\mathrm{d}y^\alpha = a_i^\alpha\,\mathrm{d}x^i\,,$$
$$\mathrm{d}\eta^\alpha = \left(\frac{\partial a_j^\alpha}{\partial x^i}\,\xi^j + \frac{\partial a_j^\alpha}{\partial y^\beta}\,\xi^j a_i^\beta + a_j^\alpha a_i^j\right)\mathrm{d}x^i$$

are the equations of the connection $\varphi^*\lambda\,.\,\Gamma$ on $H_\Gamma E \to E \to M$. The connections $V\Gamma$ and $\varphi^*\lambda\,.\,\Gamma$ are projectable over Γ. Since $TE = VE \oplus H_\Gamma E$, $(x^i, y^\alpha, \xi^i, \eta^\alpha) = (x^i, y^\alpha, 0, \eta^\alpha - a_i^\alpha\xi^i) + (x^i, y^\alpha, \xi^i, a_i^\alpha\xi^i)$, then

(8)
$$\mathrm{d}y^\alpha = a_i^\alpha\,\mathrm{d}x^i\,,$$
$$\mathrm{d}\xi^i = a_j^i(x, \xi)\,\mathrm{d}x^j\,,$$
$$\mathrm{d}\eta^\alpha = \left[\frac{\partial a_i^\alpha}{\partial y^\beta}(\eta^\beta - a_j^\beta\xi^j) + \frac{\partial a_j^\alpha}{\partial x^i}\,\xi^j + a_j^\alpha a_i^j + \frac{\partial a_j^\alpha}{\partial y^\beta}\,\xi^j a_i^\beta\right]\mathrm{d}x^i$$

is the semi-direct sum $V\Gamma \oplus \varphi^*\lambda \,.\, \Gamma$ of $V\Gamma$ and $\varphi^*\lambda \,.\, \Gamma$. Comparing (4) with (8) we obtain

Proposition 2. *$V\Gamma \oplus \varphi^*\lambda \,.\, \Gamma = T\Gamma \,.\, \lambda$ if and only if the connection Γ is integrable.*

3. Let γ:

(9)
$$\mathrm{d}\eta^\alpha = (A^\alpha_{ij}\xi^j + A^\alpha_{i\beta}\eta^\beta)\,\mathrm{d}x^i + (A^\alpha_{\beta k}\xi^k + A^\alpha_{\beta\gamma}\eta^\gamma)\,\mathrm{d}y^\beta\,,$$
$$\mathrm{d}\xi^i = (A^i_{jk}\xi^k + A^i_{j\beta}\eta^\beta)\,\mathrm{d}x^j + (A^i_{\beta k}\xi^k + A^i_{\beta\gamma}\eta^\gamma)\,\mathrm{d}y^\beta\,,$$

be a connection on E, i.e., a linear connection on $TE \to E$. Denoting the absolute derivative with respect to γ by ∇ we have

$$\nabla_{\partial/\partial x^i}(\partial/\partial x^j) = -A^k_{ij}\,\partial/\partial x^k - A^\alpha_{ij}\,\partial/\partial y^\alpha\,,$$
$$\nabla_{\partial/\partial x^i}(\partial/\partial y^\alpha) = -A^k_{i\alpha}\,\partial/\partial x^k - A^\beta_{i\alpha}\,\partial/\partial y^\beta\,,$$
$$\nabla_{\partial/\partial y^\alpha}(\partial/\partial x^i) = -A^k_{\alpha i}\,\partial/\partial x^k - A^\beta_{\alpha i}\,\partial/\partial y^\beta\,,$$
$$\nabla_{\partial/\partial y^\alpha}(\partial/\partial y^\beta) = -A^i_{\alpha\beta}\,\partial/\partial x^i - A^\gamma_{\alpha\beta}\,\partial/\partial y^\gamma\,.$$

Let us recall that γ is symmetric if and only if $A^k_{ij} = A^k_{ji}$, $A^k_{i\alpha} = A^k_{\alpha i}$, $A^\alpha_{ij} = A^\alpha_{ji}$, $A^\alpha_{i\beta} = A^\alpha_{\beta i} A^i_{\alpha\beta} = A^i_{\alpha\beta}$, $A^\alpha_{\beta\gamma} = A^\alpha_{\gamma\beta}$. From (9) it follows that γ is reducible to VE if and only if

(10)
$$A^i_{j\beta} = 0\,, \quad A^i_{\beta\gamma} = 0\,,$$

i.e., if and only if $\nabla_X Y$ is vertical for any vector X on E and any vertical vector field Y on E.

Let Γ, $\mathrm{d}y^\alpha = a^\alpha_i(x, y)\,\mathrm{d}x^i$, be a generalized connection on $E \to M$. Then the composition $\gamma \,.\, \Gamma$ of γ and Γ is a semilinear connection on $TE \to E \to M$, projectable over Γ. Putting $\mathrm{d}y^\alpha = a^\alpha_i\,\mathrm{d}x^i$ in (7), we obtain the equations of $\gamma \,.\, \Gamma$. Then the necessary and sufficient conditions

(11)
$$A^i_{j\gamma} + A^i_{\beta\gamma}a^\beta_j = 0$$

for $\gamma \,.\, \Gamma$ to be reducible to VE yield

Lemma 2. *The connection $\gamma \,.\, \Gamma$ is reducible to VE if and only if $\nabla_X Y$ is vertical for any vertical vector field Y on E and any Γ-horizontal vector X on E.*

Restricting the equations of γ and $\gamma \,.\, \Gamma$ to $H_\Gamma E$ and using (5) we obtain the following coordinate necessary and sufficient conditions:

(12)
$$A^\alpha_{ij} + A^\alpha_{i\beta}a^\beta_j = \frac{\partial a^\alpha_j}{\partial x^i} + a^\alpha_k(A^k_{ij} + A^k_{i\beta}a^\beta_j)\,,$$

$$A^\alpha_{\beta j} + A^\alpha_{\beta\gamma}a^\gamma_j = \frac{\partial a^\alpha_j}{\partial y^\beta} + a^\alpha_k(A^k_{\beta j} + A^k_{\beta\gamma}a^\gamma_j)$$

for γ to be reducible to $H_\Gamma E$, and

$$(13) \qquad A^\alpha_{ij} + A^\alpha_{i\beta} a^\beta_j - \frac{\partial a^\alpha_j}{\partial x^i} - a^\alpha_k (A^k_{ij} + A^k_{i\beta} a^\beta_j) + \\ + \left[A^\alpha_{\beta j} + A^\alpha_{\beta\gamma} a^\gamma_j - \frac{\partial a^\alpha_j}{\partial y^\beta} - a^\alpha_k (A^k_{\beta j} + A^k_{\gamma\beta} a^\gamma_j) \right] a^\beta_i = 0$$

for $\gamma . \Gamma$ to be reducible to $H_\Gamma E$.

Lemma 3. *$\gamma . \Gamma$ is reducible to $H_\Gamma E$ if and only if $\nabla_X Y$ is Γ-horizontal for any Γ-horizontal vector field Y and any Γ-horizontal vector X on E.*

Proof. For $X = \partial/\partial x^i + a^\alpha_i \, \partial/\partial y^\alpha$, $Y = \partial/\partial x^j + a^\alpha_j/\partial y^\alpha$, $\nabla_X Y$ is Γ-horizontal if and only if the relations (13) hold. This gives our assertion because $\nabla_X fY = X(f) \, Y + f \, \nabla_X Y$.

Lemma 4. *Let γ be symmetric. Then $\gamma . \Gamma$ is reducible to $H_\Gamma E$ iff Γ is integrable and $\nabla_X Y + \nabla_Y X$ is Γ-horizontal for any Γ-horizontal vector fields X, Y on E.*

Proof. Denote by (13′) the relations which follow from (13) by interchanging $i \leftrightarrow j$. Using the symmetry of γ and calculating (13)−(13′) we obtain

$$(*) \qquad \frac{\partial a^\alpha_i}{\partial x^j} - \frac{\partial a^\alpha_j}{\partial x^i} + \frac{\partial a^\alpha_i}{\partial y^\beta} a^\beta_j - \frac{\partial a^\alpha_j}{\partial y^\beta} a^\beta_i = 0 \, .$$

For $X = \partial/\partial x^i + a^\alpha_i \, \partial/\partial y^\alpha$, $Y = \partial/\partial x^j + a^\alpha_j \, \partial/\partial y^\alpha$, $\nabla_X Y + \nabla_Y X$ is Γ-horizontal if and only if the equations (13) + (13′) hold. This completes our proof.

Let $\lambda : \mathrm{d}\xi^i = a^i_j \, \mathrm{d}x^j$, $a^i_j = \Gamma^i_{jk}(x) \, \xi^k$, be a linear connection on TM. As above we construct the connection $\varphi^*\lambda$ on $H_\Gamma E$. Using (A), (9) and (6), (9) and (7), (9) we obtain: $\gamma . \Gamma$ is reducible to $V\Gamma$ iff

$$(14) \qquad A^\alpha_{i\beta} + A^\alpha_{\gamma\beta} a^\gamma_i = \frac{\partial a^\alpha_i}{\partial y^\beta} \, , \quad A^i_{k\beta} + A^i_{\gamma\beta} a^\gamma_k = 0 \, ,$$

γ is reducible to $\varphi^*\lambda$ iff

$$(15) \qquad A^i_{\beta k} + A^i_{\beta\gamma} a^\gamma_k = 0 \, , \quad A^i_{jk} + A^i_{j\beta} a^\beta_k = \Gamma^i_{jk} \, , \\ A^\alpha_{ij} + A^\alpha_{i\beta} a^\beta_j = \frac{\partial a^\alpha_j}{\partial x^i} + a^\alpha_k \Gamma^k_{ij} \, , \quad A^\alpha_{\beta i} + A^\alpha_{\beta\gamma} a^\gamma_i = \frac{\partial a^\alpha_i}{\partial y^\beta} \, ,$$

$\gamma . \Gamma$ is reducible to $\varphi^*\lambda . \Gamma$ iff

$$(16) \qquad A^\alpha_{ij} + A^\alpha_{i\beta} a^\beta_j + (A^\alpha_{\beta j} + A^\alpha_{\beta\gamma} a^\gamma_j) \, a^\beta_i = \frac{\partial a^\alpha_j}{\partial x^i} + \frac{\partial a^\alpha_j}{\partial y^\beta} a^\beta_i + a^\alpha_k \Gamma^k_{ij} \, , \\ A^i_{jk} + A^i_{j\beta} a^\beta_k + (A^i_{\beta k} + A^i_{\beta\gamma} a^\gamma_k) \, a^\beta_j = \Gamma^i_{jk} \, .$$

Let γ be transposed to γ^t. Then the conditions (14), (15), (16) yield

Proposition 3. *Let Γ be a generalized connection on E. Let γ or λ be a linear connection on TE or on TM, respectively. Then γ is reducible to $\varphi^*\lambda$ if and only if $\gamma \,.\, \Gamma$ is reducible to $\varphi^*\lambda \,.\, \Gamma$ and $\gamma^t \,.\, \Gamma$ is reducible to $V\Gamma$.*

Corollary. *A symmetric connection γ is reducible to $\varphi^*\lambda$ if and only if $\gamma \,.\, \Gamma$ is reducible to $\varphi^*\lambda \,.\, \Gamma$ and to $V\Gamma$.*

By Lemma 1, $\gamma \,.\, \Gamma = V\Gamma \oplus \varphi^*\lambda \,.\, \Gamma$ iff $\gamma \,.\, \Gamma$ is reducible to $V\Gamma$ and to $\varphi^*\lambda \,.\, \Gamma$. Then we have

Proposition 4. *If γ is symmetric then $\gamma \,.\, \Gamma = V\Gamma \oplus \varphi^*\lambda \,.\, \Gamma$ iff γ is reducible to $\varphi^*\lambda$.*

4. The first order absolute differentiation with respect to a generalized connection Γ on E is of the same form as in the classical case, see [8], [1], [3]. For example, in the case of a vertical vector field $Y = b^\alpha(x, y)\, \partial/\partial y^\alpha$ and $X = a^i\, \partial/\partial x^i \in T_m M$, the author [1] established at $h \in E$, $\pi h = m$:

$$\nabla^\Gamma_X Y = I v_{V\Gamma}(TY(\Gamma X)) = \left(\frac{\partial b^\alpha}{\partial y^\beta}\, a^\beta_i + \frac{\partial b^\alpha}{\partial x^i} - \frac{\partial a^\alpha_i}{\partial y^\beta}\, b^\beta\right) a^i\, \partial/\partial y^\alpha\,,$$

where ΓX is the Γ-lift of X at h and I is the canonical identification $I : V_u V_h \to V_h E$.

Considering $\omega = a_{\alpha\beta}(x, y)\, \mathrm{d}y^\alpha \otimes \mathrm{d}y^\beta : E \to VE^* \otimes VE^*$ we put

$$\nabla^\Gamma_X \omega(Y, Z) = \Gamma X(\omega(Y, Z)) - \omega(\nabla^\Gamma_X Y, Z) - \omega(Y, \nabla^\Gamma_X Z) = \tag{17}$$

$$= \left(\frac{\partial a_{\alpha\beta}}{\partial x^i} + \frac{\partial a_{\alpha\beta}}{\partial y^\gamma}\, a^\gamma_i + a_{\gamma\beta}\, \frac{\partial a^\gamma_i}{\partial y^\alpha} + a_{\alpha\gamma}\, \frac{\partial a^\gamma_i}{\partial y^\beta}\right) b^\alpha c^\beta a^i\,,$$

where $Y = b^\alpha\, \partial/\partial y^\alpha$, $Z = c^\alpha\, \partial/\partial y^\alpha$ are vertical vector fields on E, $X = a^i\, \partial/\partial x^i \in T_m M$ and ΓX is the Γ-lift of X at $h \in E$, $\pi h = m$. It means that $\nabla^\Gamma \omega$ is a section $E \to \to (VE^* \otimes VE^*) \otimes T^*M$. We say that ω is Γ-parallel if $\nabla^\Gamma \omega = 0$.

Let (E, ω) be a quasi-Riemannian space, where ω is a symmetric regular bilinear form on E. Let γ be the quasi-Riemannian connection on E determined by (E, ω), i.e. $\gamma^t = \gamma$ and $\nabla\omega = 0$, where ∇ denotes the absolute differentiation with respect to γ. If $\omega = a_{ij}(x, y)\, \mathrm{d}x^i \otimes \mathrm{d}x^j + a_{i\alpha}(\mathrm{d}x^i \otimes \mathrm{d}y^\alpha + \mathrm{d}y^\alpha \otimes \mathrm{d}x^i) + a_{\alpha\beta}\, \mathrm{d}y^\alpha \otimes \mathrm{d}y^\beta$ and (9) are the equations of γ then the well known classical relations between the coefficients of ω and γ, see for example [9], in the case of the quasi-Riemannian connection on (E, ω) have the following form:

$$\frac{\partial a_{jk}}{\partial x^i} + \frac{\partial a_{ik}}{\partial x^j} - \frac{\partial a_{ji}}{\partial x^k} + 2a_{sk}A^s_{ij} + 2a_{k\alpha}A^\alpha_{ij} = 0\,, \tag{18}$$

$$\frac{\partial a_{j\alpha}}{\partial x^i} + \frac{\partial a_{i\alpha}}{\partial x^j} - \frac{\partial a_{ji}}{\partial y^\alpha} + 2a_{s\alpha}A^s_{ij} + 2a_{\beta\alpha}A^\beta_{ij} = 0\,, \tag{19}$$

$$(20)\qquad \frac{\partial a_{j\alpha}}{\partial x^i} - \frac{\partial a_{i\alpha}}{\partial x^j} + \frac{\partial a_{ji}}{\partial y^\alpha} + 2a_{js}A^s_{\alpha i} + 2a_{j\beta}A^\beta_{i\alpha} = 0\,,$$

$$(21)\qquad \frac{\partial a_{i\beta}}{\partial y^\alpha} + \frac{\partial a_{i\alpha}}{\partial y^\beta} - \frac{\partial a_{\alpha\beta}}{\partial x^i} + 2a_{is}A^s_{\alpha\beta} + 2a_{i\gamma}A^\gamma_{\alpha\beta} = 0\,,$$

$$(22)\qquad \frac{\partial a_{i\beta}}{\partial y^\alpha} - \frac{\partial a_{i\alpha}}{\partial y^\beta} + \frac{\partial a_{\alpha\beta}}{\partial x^i} + 2a_{s\beta}A^s_{\alpha i} + 2a_{\delta\beta}A^\delta_{\alpha i} = 0\,,$$

$$(23)\qquad \frac{\partial a_{\gamma\beta}}{\partial y^\alpha} + \frac{\partial a_{\alpha\beta}}{\partial y^\gamma} - \frac{\partial a_{\gamma\alpha}}{\partial y^\beta} + 2a_{k\beta}A^k_{\alpha\gamma} + 2a_{\delta\beta}A^\delta_{\alpha\gamma} = 0\,.$$

Being regular, ω determines on E a unique generalized connection Γ, the horizontal tangent vectors of which at $h \in E$ are such vectors $X \in T_h Y$ that $\omega(Y, X) = 0$ for any vertical vector $Y \in T_h Y$. In [2] some properties of Γ were found in the more general case of ω, when only the restriction $\varpi = \omega|_{VE}$ is regular. It is easy to see that Γ is given by

$$\mathrm{d}y^\alpha = a^\alpha_i(x, y)\,\mathrm{d}x^i\,, \quad a^\alpha_i = -A^{\alpha\beta}a_{i\beta}\,,$$

where $a_{\alpha\beta}A^{\beta\gamma} = \delta^\gamma_\alpha$. We say that Γ is conjugate to ω. Throughout the remainder of the paper, γ and Γ always denote the quasi-Riemannian connection of (E, ω) and the connection conjugate to ω, respectively. The relation (17) implies

$$(24)\qquad \frac{\partial a_{\alpha\beta}}{\partial x^i} - \frac{\partial a_{i\beta}}{\partial y^\alpha} - \frac{\partial a_{i\alpha}}{\partial y^\beta} + a_{i\delta}A^{\gamma\delta}\left(\frac{\partial a_{\alpha\gamma}}{\partial y^\beta} + \frac{\partial a_{\beta\gamma}}{\partial y^\alpha} - \frac{\partial a_{\alpha\beta}}{\partial y^\gamma}\right) = 0$$

for $\bar{w}$ to be Γ-parallel.

Proposition 5. *Let ∇ denote the absolute differentiation with respect to γ. Then the restriction ϖ of ω to VE is Γ-parallel iff $\nabla_Y Z$ is vertical for any vertical vector fields Y, Z on E.*

Proof. Setting $A^\delta_{\alpha\gamma}$ evaluated from (23) in (21) we obtain

$$(25)\qquad \frac{\partial a_{i\beta}}{\partial y^\alpha} + \frac{\partial a_{i\alpha}}{\partial y^\beta} - \frac{\partial a_{\alpha\beta}}{\partial x^i} + 2(a_{is} - a_{i\delta}A^{\delta\gamma}a_{s\gamma})\,A^s_{\alpha\beta} -$$
$$- a_{i\delta}A^{\delta\gamma}\left(\frac{\partial a_{\beta\gamma}}{\partial y^\alpha} + \frac{\partial a_{\alpha\gamma}}{\partial y^\beta} - \frac{\partial a_{\beta\alpha}}{\partial y^\gamma}\right) = 0\,.$$

Then (24) holds iff

$$(26)\qquad 2(a_{is} - a_{i\delta}A^{\delta\gamma}a_{s\gamma})\,A^s_{\alpha\beta} = 0\,.$$

As γ is uniquely determined by the equations (18), ..., (23), we deduce from (25) that $\det(a_{is} - a_{i\delta}A^{\delta\gamma}a_{s\gamma})| \neq 0$. Then (26) is fulfilled iff $A^s_{\alpha\beta} = 0$ and thus $\nabla_{\partial/\partial y^\alpha}(\partial/\partial y^\beta) = -A^i_{\alpha\beta}\,\partial/\partial x^i - A^\gamma_{\alpha\beta}\,\partial/\partial y^\gamma$ completes our proof.

Proposition 6. *Let Γ be conjugate to ω. Then the quasi-Riemannian connection γ of (E, ω) is reducible to VE if and only if $\gamma . \Gamma$ is reducible to VE and ϖ is Γ-parallel.*

Proof. By the proof of Proposition 5, ϖ is Γ-parallel iff $A^s_{\alpha\beta} = 0$. then (10) and (11) give the desired result.

Let $Y = b^\alpha \partial/\partial y^\alpha$ be a vertical vector field on E. Let $L_Y\omega$ be the Lie differentiation of ω with respect to Y. Let $hL_Y\omega$ or ε_Y denote the bilinear form on E determined by

$$hL_Y\omega(X, Z) = L_Y\omega(hX, hY) \quad \text{or} \quad \varepsilon_Y(X, Y) = \omega\,(\nabla^\Gamma_X Y, Z) + \omega(X, \nabla^\Gamma_Z Y)\,.$$

Calculate explicitly

$$hL_Y\omega - \varepsilon_Y = \left(\frac{\partial a_{ij}}{\partial y^\alpha} + \frac{\partial a_{i\beta}}{\partial y^\alpha}\, a^\beta_j + \frac{\partial a_{j\beta}}{\partial y^\alpha}\, a^\beta_i + \right. \\ \left. + \frac{\partial a_{\gamma\beta}}{\partial y^\alpha}\, a^\gamma_i a^\beta_j\right) b^\alpha(\mathrm{d}x^i \otimes \mathrm{d}x^j + \mathrm{d}x^j \otimes \mathrm{d}x^i)\,. \tag{27}$$

Recall that a 1-form ψ on E is semi-basic if $\psi(Y) = 0$ for any vertical vector Y on E. Let $B(E)$ be the vector bundle of all semi-basic 1-forms on E. (27) yields

Lemma 5. *The map $\varrho_\omega : VE \to O^2\, B(E)$, $Y \mapsto hL_Y\omega - \varepsilon_Y$, is a linear morphism.*

Proposition 7. *The connection $\gamma . \Gamma$ is reducible to VE iff Γ is integrable and $\varrho_\omega = 0$.*

Proof. Denote by B the equations which we obtain from (20) putting here $A^\gamma_{\alpha i}$ evaluated from (22). Then (25) and B give

$$2(a_{js} - a_{j\delta}A^{\delta\gamma}a_{s\gamma})\,(A^s_{\alpha i} + A^s_{\alpha\beta}a^\beta_i) + \frac{\partial a_{j\alpha}}{\partial x^i} - \frac{\partial a_{i\alpha}}{\partial x^j} + \frac{\partial a_{ji}}{\partial y^\alpha} - \\ - a_{j\beta}A^{\beta\gamma}\left(\frac{\partial a_{i\gamma}}{\partial y^\alpha} - \frac{\partial a_{i\alpha}}{\partial y^\gamma} + \frac{\partial a_{\alpha\gamma}}{\partial x^i}\right) + \left(\frac{\partial a_{j\beta}}{\partial y^\alpha} + \frac{\partial a_{j\alpha}}{\partial y^\beta} - \frac{\partial a_{\alpha\beta}}{\partial x^j}\right) a^\beta_i - \\ - a_{j\delta}A^{\delta\gamma}\left(\frac{\partial a_{\beta\gamma}}{\partial y^\alpha} + \frac{\partial a_{\beta\gamma}}{\partial y^\beta} - \frac{\partial a_{\beta\alpha}}{\partial y^\gamma}\right) a^\beta_i = 0\,. \tag{28}$$

Comparing (11) with (28) we find that the connection $\gamma . \Gamma$ is reducible to VE iff

$$\frac{\partial a_{j\alpha}}{\partial x^i} - \frac{\partial a_{i\alpha}}{\partial x^j} + \frac{\partial a_{ji}}{\partial y^\alpha} + a^\gamma_j\left(\frac{\partial a_{i\gamma}}{\partial y^\alpha} - \frac{\partial a_{i\alpha}}{\partial y^\gamma} + \frac{\partial a_{\alpha\gamma}}{\partial x^i}\right) + \\ + \left(\frac{\partial a_{j\beta}}{\partial y^\alpha} + \frac{\partial a_{j\alpha}}{\partial y^\beta} - \frac{\partial a_{\alpha\beta}}{\partial x^j}\right) a^\beta_i - a^\gamma_j\left(\frac{\partial a_{\beta\gamma}}{\partial y^\alpha} + \frac{\partial a_{\alpha\gamma}}{\partial y^\beta} - \frac{\partial a_{\beta\alpha}}{\partial y^\gamma}\right) a^\beta_i = 0\,. \tag{29}$$

Let (29′) be the equation obtained from (29) by interchanging $i \leftrightarrow j$. Because of (27) and (1) the equations (29) + (29′) and (29)−(29′) are fulfilled iff $\varrho_\omega = 0$ and Γ is integrable.

Proposition 8. *The connection $\gamma \,.\, \Gamma$ is reducible to $V\Gamma$ if and only if $\varrho_\omega = 0$, $\Phi_\Gamma = 0$ and $\nabla^\Gamma \varpi = 0$.*

Proof. The equations (23) and (22) imply

$$(30)\qquad \frac{\partial a_{i\beta}}{\partial y^\alpha} - \frac{\partial a_{i\alpha}}{\partial y^\beta} + \frac{\partial a_{\alpha\beta}}{\partial x^i} + \left(\frac{\partial a_{\gamma\beta}}{\partial y^\alpha} + \frac{\partial a_{\alpha\beta}}{\partial y^\gamma} - \frac{\partial a_{\gamma\alpha}}{\partial y^\beta}\right) . \, a_i^\gamma + $$
$$+ 2a_{s\beta}(A^s_{\alpha i} + A^s_{\alpha\gamma}a^\gamma_i) + 2a_{\delta\beta}(A^\delta_{\alpha i} + A^\delta_{\alpha\gamma}a^\gamma_i) = 0\,.$$

Let $\gamma \,.\, \Gamma$ be reducible to $V\Gamma$. By Proposition 7, $\varrho_\omega = 0$, $\Phi_\Gamma = 0$. In virtue of $a^\alpha_i = -A^{\alpha\beta}a_{i\beta}$ and (14) the equations (30) give (24). Conversely, let $\Phi_\Gamma = 0$, $\varrho_\omega = 0$, $\nabla^\Gamma \varpi = 0$. Then by means of (11) and (24), the relations (30) imply $A^\delta_{\alpha i} + A^\delta_{\alpha\gamma}a^\gamma_i = \partial a^\delta_i / \partial y^\alpha$. This and (11) together give (14). Q.E.D.

Corollary of Proposition 6, 7, 9. *The connection $\gamma \,.\, \Gamma$ is reducible to $V\Gamma$ if and only if γ is reducible to VE.*

Proposition 9. *The connection $\gamma \,.\, \Gamma$ is reducible to $H_\Gamma E$ iff it is red. cible to VE.*

Proof. Interchanging $\alpha \leftrightarrow \beta$ in (22) and replacing i by j in (30) we get the equations (22′) and (30′). Then the equations (22′), (19), (30′) yield

$$\left(\frac{\partial a_{i\alpha}}{\partial y^\beta} - \frac{\partial a_{i\beta}}{\partial y^\alpha} + \frac{\partial a_{\beta\alpha}}{\partial x^i}\right) a^\beta_j + \left(\frac{\partial a_{j\alpha}}{\partial y^\beta} - \frac{\partial a_{j\beta}}{\partial y^\alpha} + \frac{\partial a_{\alpha\beta}}{\partial x^j}\right) a^\beta_i + \frac{\partial a_{j\alpha}}{\partial x^i} + \frac{\partial a_{i\alpha}}{\partial x^j} -$$
$$- \frac{\partial a_{ji}}{\partial y^\alpha} + \left(\frac{\partial a_{\gamma\alpha}}{\partial y^\beta} + \frac{\partial a_{\alpha\beta}}{\partial y^\gamma} - \frac{\partial a_{\gamma\beta}}{\partial y^\alpha}\right) a^\gamma_j a^\beta_i + 2a_{\alpha\delta}[-a^\delta_s(A^s_{ij} + A^s_{\beta i}a^\beta_j) +$$
$$+ A^\delta_{ij} + A^\delta_{\beta i}a^\beta_j - a^\delta_s(A^s_{\beta j} + A^s_{\beta\gamma}a^\gamma_j)\, a^\beta_i + (A^\delta_{\beta j} + A^\delta_{\beta\gamma}a^\gamma_j)\, a^\beta_i] = 0\,.$$

Then, because of $a_{s\alpha} = -a_{\alpha\delta}a^\delta_s$, (13) holds iff (29) is satisfied. Q.E.D.

Proposition 10. *The connection γ is reducible to $H_\Gamma E$ if and only if it is reducible to VE.*

Proof. Using $a_{j\alpha} = -a_{\alpha\beta}a^\beta_j$, from the equations (22′) a^β_i + (19) and (30) we deduce that (12) is fulfilled if and only if the equations (29) and (24) are satisfied. Q.E.D.

5. Let $\Gamma : \mathrm{d}y^\alpha = a^\alpha_i(x, y)\,\mathrm{d}x^i$ be generalized connection on E. A bilinear form ω on E will be called a *(Γ, ϖ, g)-form* if there are such a section $\varpi : E \to V^*E \otimes V^*E$

and a bilinear form g on M that

$$\omega(X, Y) = \varpi(v_\Gamma X, v_\Gamma Y) + g(T\pi X, T\pi Y).$$

In coordinates, if $\omega = a_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j + a_{i\alpha}\,\mathrm{d}x^i \otimes \mathrm{d}y^\alpha + a_{\alpha i}\,\mathrm{d}y^\alpha \otimes \mathrm{d}x^i + a_{\alpha\beta}\,\mathrm{d}y^\alpha \otimes \otimes\,\mathrm{d}y^\beta$, $\varpi = A_{\alpha\beta}\,\mathrm{d}y^\alpha \otimes \mathrm{d}y^\beta$, $g = g_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j$ then ω is a (Γ, ϖ, g) – form iff

$$a_{\alpha\beta} = A_{\alpha\beta}\,, \quad a_{i\alpha} = -A_{\beta\alpha}a_i^\beta\,, \quad a_{\alpha i} = -a_{\alpha\beta}a_i^\beta\,, \quad a_{ij} = A_{\gamma\beta}a_i^\alpha a_j^\beta + g_{ij}\,.$$

Hence it follows that if ω is a (Γ, ϖ, g) – form then

(a) Γ is conjugate to ω,
(b) ω is symmetric iff ϖ and g are symmetric,
(c) ω is regular iff ϖ and g are both regular.

We assume that M is paracompact in what follows.

Proposition 11. *Let (E, ω) be a quasi-Riemannian structure. Let Γ be conjugate to ω. Let ϖ be the restriction of ω to VE. Then there is such a bilinear form g on M that ω is a (Γ, ϖ, g) – form if and only if $\varrho_\omega = 0$.*

Proof. As Γ is conjugate to ω, then $a_{i\beta} = -a_{\gamma\beta}a_i^\gamma$. Therefore from (27)

$$\varrho_\omega = \left(\frac{\partial a_{ij}}{\partial y^\alpha} + \frac{\partial a_{j\gamma}}{\partial y^\alpha}a_i^\gamma + a_{j\gamma}\frac{\partial a_i^\gamma}{\partial y^\alpha}\right)(\mathrm{d}x^i \otimes \mathrm{d}x^j + \mathrm{d}x^j \otimes \mathrm{d}x^i) \otimes \mathrm{d}y^\alpha\,.$$

Then $\varrho_\omega = 0$ iff $a_{ij} = -a_{j\gamma}a_i^\gamma + g_{ij}(x) = a_{\alpha\beta}a_i^\alpha a_j^\beta + g_{ij}(x)$. Q.E.D.

A quasi-Riemannian structure (E, ω) will be said to be reducible if there is such a bilinear symmetric regular form g on M that ω is a (Γ, ϖ, g) – form and $\gamma\,.\,\Gamma = V\Gamma \oplus \varphi^*\lambda\Gamma$, where γ or λ is the quasi-Riemannian connection of (E, ω) or of (M, g), respectively, Γ is conjugate to ω and ϖ is the restriction of ω to VE.

Theorem. *A quasi-Riemannian structure (E, ω) is reducible if and only if the quasi-Riemannian connection γ of (E, ω) is reducible to VE.*

Proof. Let γ be reducible to VE. Then by Proposition 6, $\Phi_\Gamma = 0$, $\varrho_\omega = 0$, $\nabla^\Gamma\varpi = 0$. Consequently, on account of Proposition 11, there is a bilinear form g on M such that ω is a $(\Gamma, \bar{\omega}, g)$ – form, where ϖ is the restriction of ω to VE. Putting $2a_{\beta\alpha}A_{ij}^\beta$ evaluated from (19) in (18) and using $\Phi_\Gamma = 0$, $\nabla^\Gamma\varpi = 0$, $a_{ij} = -a_{\beta i}a_j^\beta + g_{ij}$, $a_{k\alpha} = -a_{\beta\alpha}a_k^\beta$ we get

$$\frac{\partial g_{ij}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ik}}{\partial x^k} + 2g_{sk}A_{ij}^s = 0\,.$$

Then the well known equations for the Christoffel symbols Γ_{ij}^s of the quasi-Riemannian connection λ of (M, g) induce

$$A_{ij}^s = \Gamma_{ij}^s\,.$$

By Proposition 10, γ is reducible to $H_\Gamma E$. Hence because of (10) and (12) the equations (16) are fulfilled and thus $\gamma . \Gamma$ is reducible to $\varphi^*\lambda . \Gamma$. According to Proposition 8, $\gamma . \Gamma$ is redusible to $V\Gamma$. Then by Lemma 1, $\gamma . \Gamma = V\Gamma \oplus \varphi^*\lambda . \Gamma$. Conversely, if (E, ω) is reducible then $\gamma . \Gamma$ is reducible to $V\Gamma$ and hence by Proposition 8, γ is reducible to VE. Q.E.D.

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