Jaroslav Barták

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STABILITY OF ABSTRACT DIFFERENTIAL EQUATIONS WITH THE RIGHT-HAND SIDE SMOOTH IN THE TIME VARIABLE

JAROSLAV BARTÁK, Praha

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1. INTRODUCTION

In the paper [1] the stability of solutions of the equation $u^{(n)}(t) + a_1(A) \,.\, u^{(n-1)}(t) + \ldots + a_n(A)\, u(t) = F(t, u(t))$ under the assumption $F \in \mathscr{C}(\mathscr{D}(u), \mathscr{D}(A^{1/n}))$ was investigated. The aim of this paper is to obtain similar results for this equation with the right hand side $F \in \mathscr{C}^{(1)}(\mathscr{D}(u), H)$.

We shall use all the notations involved in [1]. Let us remember the most important of them. Under the symbol A we mean a linear, selfadjoint and strictly positive operator in a Hilbert space H. The norm in H will be denoted by $\|\cdot\|$. The functions $a_i(A)$ satisfy the growth condition (1.1.1) from [1] (roughly speaking, $\|a_i(A)\, A^{-i/n}\varphi\| \leqq C_0^*\|\varphi\|$ for $i = 1, \ldots, n$, $\varphi \in H$). In [1] (Section 1.2) we introduced functions $m_i(t; t_0, s)$, $i = 0, \ldots, n-2$, and a function $m_{n-1}(t; t_0, s) = m(t; t_0, s)$. Let us recall that m solves the equation $m^{(n)}(t) + a_1(s)\, m^{(n-1)}(t) + \ldots + a_n(s)\, m(t) = 0$ and fulfils the initial conditions $m^{(i)}(t_0; t_0, s) = \delta^i_{n-1}$, $i = 0, \ldots, n-1$, $t_0 \in R^+$, $s \in \sigma(A)$. With help of the function m we introduced the notion "*type of the operator* $\mathscr{L}$", where

$$\mathscr{L}\, u(t) \equiv u^{(n)}(t) + a_1(A)\, u^{(n-1)}(t) + \ldots + a_n(A)\, u(t)\,,$$

and we distinguished three kinds of operators $\mathscr{L}$; *the exponentially stable*, *the stable* and *the instable* ones (see [1] (Section (1.4))). For the other details we recommend the reader to have a look at the notation in the paper [1].

Let $I \subseteq R^+$ be an interval of the type $[a, b)$ or $[a, b]$ and let $F : \mathcal{D}(F) \subseteq \subseteq \{(t, u(t), u'(t), \ldots, u^{(n-1)}(t)) \mid u \in \mathcal{D}(\mathcal{L})$ such that $\mathcal{D}(u) \subseteq I,\ t \in \mathcal{D}(u)\} \to H$. We shall write $F \in \mathcal{C}^{(1)}(\mathcal{D}(u_{/I}), H)$, if for every $u \in \mathcal{D}(\mathcal{L})$ such that $\mathcal{D}(u) \subseteq I$, the functions $\|F(t, u(t))\|$, $\|F'(t, u(t))\|$ depend continuously on the variable t for $t \in \mathcal{D}(u)$. The symbol F' means the total derivative of the function F with respect to the variable t. (In order to save space, we write $F(t, u(t))$ instead of $F(t, u(t), u'(t), \ldots, u^{(n-1)}(t))$.

The Cauchy problem will be given by the equation

$$\mathcal{L}\, u(t) = F(t, u(t)) \tag{1.1}$$

and by the initial conditions

$$u^{(i)}(t_0) = \varphi_i\,, \quad \varphi_i \in \mathcal{D}(A^{(n-i)/n})\,, \quad i = 0, \ldots, n-1\,, \quad t_0 \in R^+\,. \tag{1.2}$$

The stability will be investigated with respect to the norm

$$|||u(t)||| = [\sum_{i=0}^{n-1} \|A^{(n-i)/n}\, u^{(i)}(t)\|^2]^{1/2}\,.$$

2. THE CASE OF THE EXPONENTIALLY STABLE OPERATOR

Lemma 2.1. *If the operator $\mathcal{L}$ is of the type $\omega < 0$, then $|a_n^{-1}(s)| \leqq |\omega|^{-n}$ for $s \in \sigma(A)$, the operator $a_n^{-1}(A)$ exists, $\mathcal{D}(a_n^{-1}(A)) = H$ and $\|a_n^{-1}(A)\| \leqq |\omega|^{-n}$.*

Proof. By [1] (Theorem 1.5.2): $\operatorname{Re} \lambda_i(s) \leqq \omega$ for $s \in \sigma(A)$, $i = 1, \ldots, n$, and so $0 < |\omega|^n \leqq \prod_{i=1}^{n} |\operatorname{Re} \lambda_i(s)| \leqq \prod_{i=1}^{n} |\lambda_i(s)| = |a_n(s)|$, i.e.

$$|a_n^{-1}(s)| \leqq |\omega|^{-n} \quad \text{for} \quad s \in \sigma(A)\,;$$

Thus $a_n^{-1}(A)\, x = \int_{\sigma(A)} a_n^{-1}(s)\, dE(s)\, x$ for $x \in \mathcal{D}(a_n^{-1}(A)) = \{x \in H \mid \int_{\sigma(A)} |a_n^{-1}(s)|^2 \,.\, d\|E(s)\, x\|^2 < +\infty\} = \{x \in H \mid \int_{\sigma(A)} d\|E(s)\, x\|^2 < +\infty\} = H$. The last statement of the lemma follows from the relation $\|a_n^{-1}(A)\, x\|^2 = \int_{\sigma(A)} |a_n^{-1}(s)|^2\, d\|E(s)\, x\|^2 \leqq \leqq |\omega|^{-2n} \int_{\sigma(A)} d\|E(s)\, x\|^2 = (|\omega|^{-n}\, \|x\|)^2$.

According to Lemma 2.1 we can define a function $M_1(t; t_0, s)$ as follows: $(d/dt)\, M_1(t; t_0, s) = m(t; t_0, s)$, $a_n(s)\, M_1(t_0; t_0, s) = -1$, for $t_0 \in R^+$, $t \geqq t_0$, $s \in \sigma(A)$.

Lemma 2.2. *The condition*

(2.1) *there exists a constant K^* such that $\|Ax\| \leqq K^* \|a_n(A)\, x\|$ for all $x \in \mathcal{D}(A)$,*

is fulfilled if and only if $1 \leqq K^* |a_n(s)|\, s^{-1}$ for all $s \in \sigma(A)$.

Proof. Firstly, let (2.1) be fulfilled. Then $A^{-1}y \in \mathcal{D}(A)$ for every $y \in H$ and $\|AA^{-1}y\| \leqq K^* \|a_n(A)\, A^{-1}y\|$, i.e. $\|y\| \leqq K^* \|a_n(A)\, A^{-1}y\|$ for all $y \in H$. By [1] (Lemma 1.2.1): $1 \leqq K^* |a_n(s)|\, s^{-1}$ for $s \in \sigma(A)$.

Secondly, let $1 \leqq K^*|a_n(s)|\, s^{-1}$. Then by [1] (Lemma 1.2.1) $\|y\| \leqq K^*\|a_n(A)\, A^{-1}y\|$ for all $y \in H$, i.e. $\|AA^{-1}y\| \leqq K^*\|a_n(A)\, A^{-1}y\|$. Because every $x \in \mathscr{D}(A)$ can be expressed in the form $x = A^{-1}y$ for some $y \in H$, the last inequality proves the lemma.

Remark 2.1. The equation $\lambda^2 + \lambda(s^{(1-\varepsilon)/2} + s^{1/2}) + s^{(2-\varepsilon)/2} = 0$ $(\varepsilon \in (0, 1))$, which has the roots $\lambda_1(s) = -s^{1/2}$, $\lambda_2(s) = -s^{(1-\varepsilon)/2}$, shows that *the operator $\mathscr{L}$ can be of the type $\omega < 0$ and the condition* (2.1) *need not be fulfilled.* (The proof follows directly from Lemma 2.2 and [1] (Theorem 1.5.1).)

Lemma 2.3. *Let the operator $\mathscr{L}$ be of the type $\omega < 0$, let the condition* (2.1) *be fulfilled, $t_0 \in R^+$, $t \geqq t_0$, $s \in \sigma(A)$. Then*

(i) $$\mathscr{D}(M_1(t; t_0, A)) = H\,, \quad \mathscr{R}(M_1(t; t_0, A)) \subseteq \mathscr{D}(A)\,,$$

(ii) $$|M_1(t; t_0, s)| \leqq K^*\, C(\mathscr{L})\,(1 + (n-1)\, C_0^*)\, s^{-1} e^{\omega(t-t_0)}\,,$$

(iii) $$\|AM_1(t; t_0, A)\| \leqq K^*\, C(\mathscr{L})\,(1 + (n-1)\, C_0^*)\, e^{\omega(t-t_0)}\,.$$

(*Constants C_0^*, $C(\mathscr{L})$ were introduced in* [1] (*Sections* 1.1, 1.2).)

Proof. Integrating the equation $m^{(n)}(t) + a_1(s)\, m^{(n-1)}(t) + \ldots + a_n(s)\, m(t) = 0$ we obtain $m^{(n-1)}(t) + a_1(s)\, m^{(n-2)}(t) + \ldots + a_{n-1}(s)\, m(t) + a_n(s)\, M_1(t) = 0$. This with help of (1.1.1) from [1] and [1] (statement (iii) of Theorem 1.2.1) gives

$$|a_n(s)\, M_1(t; t_0, s)| \leqq C(\mathscr{L})\,(1 + (n-1)\, C_0^*)\, e^{\omega(t-t_0)}\,.$$

Using Lemma 2.2 we see that $|M_1(t; t_0, s)| \leqq K^*\, C(\mathscr{L})\,(1 + (n-1)\, C_0^*)\, s^{-1} e^{\omega(t-t_0)}$. This immediately implies the lemma.

Lemma 2.4. *Let the operator $\mathscr{L}$ be of the type $\omega < 0$, let the condition* (2.1) *be fulfilled, $F \in \mathscr{C}^{(1)}(\mathscr{D}(u_{/I}), H)$, $\mathscr{D}(u) \subseteq [t_0, +\infty) \subseteq I$, $t_0 \in \mathscr{D}(u)$, let $u : \mathscr{D}(u) \to H$ be a solution of the equation* (1.1) *which fulfils the initial conditions* (1.2). *Then for $t \in \mathscr{D}(u)$*

(i) $$u(t) = \sum_{i=0}^{n-1} m_i(t; t_0, A)\, \varphi_i + M_1(t; t_0, A)\, F(t_0, u(t_0)) - M_1(t_0; t_0, A)\, F(t, u(t)) + \int_{t_0}^{t} M_1(t + t_0 - \tau; t_0, A)\, F'(\tau, u(\tau))\, d\tau\,,$$

(ii) $$\Big|\Big|\Big| \sum_{i=0}^{n-1} m_i(t; t_0, A)\, \varphi_i \Big|\Big|\Big| \leqq C_3^* |||u(t_0)|||\; e^{\omega(t-t_0)}\,,$$

$$\Big|\Big|\Big| M_1(t; t_0, A)\, F(t_0, u(t_0)) - M_1(t_0; t_0, A)\, F(t, u(t)) + \int_{t_0}^{t} M_1(t + t_0 - \tau; t_0, A)\, F'(\tau, u(\tau))\, d\tau \Big|\Big|\Big| \leqq C_5^* \|F(t_0, u(t_0))\|\, e^{\omega(t-t_0)} + C_6^* \|F(t, u(t))\| + C_5^* \int_{t_0}^{t} e^{\omega(t-\tau)} \|F'(\tau, u(\tau))\|\, d\tau\,,$$

where $C_5^* = C(\mathscr{L})[K^*(1 + (n-1) C_0^*) + n - 1]$, $C_6^* = K^* C(\mathscr{L})(1 + (n-1) C_0^*)$ *and the constant* C_3^* *was defined in* [1] (*Theorem* 1.3.1).

Proof. Statement (i) is an immediate consequence of Remark 1.3.1 from [1] and of Lemma 2.3. Now we shall prove the statement (ii). According to [1] (Lemma 1.3.6) it suffices to prove the second estimate. Using Lemma 2.3 we obtain

$$(1) \quad \Big\| A[M_1(t; t_0, A) F(t_0, u(t_0)) - M_1(t_0; t_0, A) F(t, u(t)) + \\ + \int_{t_0}^{t} M_1(t + t_0 - \tau; t_0, A) F'(\tau, u(\tau)) \, d\tau] \Big\| \leqq \\ \leqq \|AM_1(t; t_0, A) F(t_0, u(t_0))\| + \|AM_1(t_0; t_0, A) F(t, u(t))\| + \\ + \int_{t_0}^{t} \|AM_1(t + t_0 - \tau; t_0, A) F'(\tau, u(\tau))\| \, d\tau \leqq \\ \leqq K^* C(\mathscr{L})(1 + (n-1) C_0^*) \|F(t_0, u(t_0))\| e^{\omega(t-t_0)} + \\ + K^* C(\mathscr{L})(1 + (n-1) C_0^*) \|F(t, u(t))\| + \\ + K^* C(\mathscr{L})(1 + (n-1) C_0^*) \int_{t_0}^{t} e^{\omega(t-\tau)} \|F'(\tau, u(\tau))\| \, d\tau \, .$$

Applying Theorem 1.2.1 from [1] we get for $i = 1, \ldots, n-1$

$$(2) \quad \Big\| A^{(n-i)/n} \frac{d^i}{dt^i} \Big[M_1(t; t_0, A) F(t_0, u(t_0)) - M_1(t_0; t_0, A) F(t, u(t)) + \\ + \int_{t_0}^{t} M_1(t + t_0 - \tau; t_0, A) F'(\tau, u(\tau)) \, d\tau \Big] \Big\| = \\ = \Big\| A^{(n-i)/n} \Big[m^{(i-1)}(t; t_0, A) F(t_0, u(t_0)) + \\ + \int_{t_0}^{t} m^{(i-1)}(t + t_0 - \tau; t_0, A) F'(\tau, u(\tau)) \, d\tau \Big] \Big\| \leqq \\ \leqq \|A^{(n-i)/n} m^{(i-1)}(t; t_0, A) F(t_0, u(t_0))\| + \\ + \int_{t_0}^{t} \|A^{(n-i)/n} m^{(i-1)}(t + t_0 - \tau; t_0, A) F'(\tau, u(\tau))\| \, d\tau \leqq \\ \leqq C(\mathscr{L}) \|F(t_0, u(t_0))\| e^{\omega(t-t_0)} + C(\mathscr{L}) \int_{t_0}^{t} e^{\omega(t-\tau)} \|F'(\tau, u(\tau))\| \, d\tau \, .$$

Now (1) and (2) yield

$$\left|\!\left|\!\left| M_1(t; t_0, A) F(t_0, u(t_0)) - M_1(t_0; t_0, A) F(t, u(t)) + \right.\right.\right.$$

$$\left. + \int_{t_0}^{t} M_1(t + t_0 - \tau; t_0, A) F'(\tau, u(\tau)) \, d\tau \right|\!\right|\!\right| \leqq C(\mathscr{L}) [K^*(1 + (n-1) C_0^*) + n - 1] \, .$$

$$. \, \|F(t_0, u(t_0))\| \, e^{\omega(t - t_0)} + K^* C(\mathscr{L}) (1 + (n-1) C_0^*) \|F(t, u(t))\| +$$

$$+ C(\mathscr{L}) [K^*(1 + (n-1) C_0^*) + n - 1] \int_{t_0}^{t} e^{\omega(t - \tau)} \|F'(\tau, u(\tau))\| \, d\tau \, .$$

The lemma is proved.

Lemma 2.5. *If* $\psi \in \mathscr{D}(A^{(n-i)/n})$, $u \in \mathscr{U} = \bigcup_{\substack{I = [a,b) \subseteq R^+ \\ I = [a,b] \subseteq R^+}} \bigcap_{i=0}^{n-1} \mathscr{C}^i(I, \mathscr{D}(A^{(n-i)/n}))$, $t \in \mathscr{D}(u)$, *then* $\|A^{(n-i-1)/n}\psi\| \leqq \delta^{-1/n} \|A^{(n-i)/n}\psi\|$, $\sum_{i=0}^{n-1} \|A^{(n-i-1)/n} u^{(i)}(t)\| \leqq n^{1/2} \delta^{-1/n} |\!|\!| u(t) |\!|\!|$.

Proof.

$$\|A^{(n-i-1)/n}\psi\|^2 = \int_{\delta}^{\infty} s^{2(n-i-1)/n} \, d\|E(s)\psi\|^2 \leqq$$

$$\leqq \delta^{-2/n} \int_{\delta}^{\infty} s^{2(n-i)/n} \, d\|E(s)\psi\|^2 = [\delta^{-1/n} \|A^{(n-i)/n}\psi\|]^2 \, .$$

The second inequality immediately follows from the first one with help of the inequality $c_1 + c_2 + \ldots + c_n \leqq n^{1/2}(c_1^2 + c_2^2 + \ldots + c_n^2)^{1/2}$, $c_i \geqq 0$ $(i = 1, \ldots, n)$.

Theorem 2.1. (Correctness Theorem.) *Let the operator* $\mathscr{L}$ *be of the type* $\omega < 0$, *let the condition* (2.1) *be fulfilled,* $F, F' \in \mathscr{C}([t_0, +\infty), H)$. *Then the maximal solution of the Cauchy problem* (1.1), (1.2)

(i) *is uniquely determined,*

(ii) *has the form* $u(t) = \sum_{i=0}^{n-1} m_i(t; t_0, A) \varphi_i + M_1(t; t_0, A) F(t_0) -$

$$- M_1(t_0; t_0, A) F(t) + \int_{t_0}^{t} M_1(t + t_0 - \tau; t_0, A) F'(\tau) \, d\tau, \; t \geqq t_0 \, ,$$

(iii) *fulfils the estimate* $|\!|\!| u(t) |\!|\!| \leqq C_3^* |\!|\!| u(t_0) |\!|\!| \, e^{\omega(t - t_0)} + C_5^* \|F(t_0)\| \, e^{\omega(t - t_0)} +$

$$+ C_6^* \|F(t)\| + C_5^* \int_{t_0}^{t} e^{\omega(t - \tau)} \|F'(\tau)\| \, d\tau \, .$$

Proof. (i) can be proved analogously as Lemma 1.3.4 from [1]. The statements (ii), (iii) are easy consequences of Lemma 2.4.

Theorem 2.2. *Let* $v : \mathscr{D}(v) \to H$ *be a maximal solution of the equation* (1.1), *let the operator* $\mathscr{L}$ *be of the type* $\omega < 0$, $F \in \mathscr{C}^{(1)}(\mathscr{D}(u_{/\mathscr{D}(v)}), H)$, *and let the conditions* (2.1), (2.2) *be fulfilled.*

(2.2) *There exist constants* $K_1, K_2, R > 0$ *such that the inequalities* $\|F(t, v(t) + u(t)) - F(t, v(t))\| \leqq K_1 \sum_{i=0}^{n-1} \|A^{(n-i-1)/n} u^{(i)}(t)\|$, $\|F'(t, v(t) + u(t)) - F'(t, v(t))\| \leqq K_2 |||u(t)|||$ *are valid whenever* u *is a solution of the equation* $\mathscr{L}\, u(t) = F(t, v(t) + u(t)) - F(t, v(t))$, $\mathscr{D}(u) \subseteq \mathscr{D}(v)$ *and* $t \in \mathscr{D}(u)$ *is such that* $|||u(t)||| \leqq R$.

Further, let

$$\omega + C_5^* K_2 + C(\mathscr{L})\, C_6^* K_1^2 n^{3/2} \delta^{-1/n} < 0$$

or

$$\omega + C_5^* K_2 + C(\mathscr{L})\, C_6^* K_1^2 n^{3/2} \delta^{-1/n} \leqq 0 \,.$$

Then the solution v *is respectively uniformly exponentially stable or uniformly stable with respect to the norm* $|||\cdot|||$. (*If moreover* $R = +\infty$ *then the solution* v *is globally uniformly exponentially stable or globally uniformly stable with respect to the norm* $|||\cdot|||$.)

Proof. Similarly as in [1] (Theorem 2.1.2) we can show that it suffices to prove the (global) uniform exponential stability or the (global) uniform stability of the zero solution $O_{/\mathscr{D}(v)}$ of the equation

$$\mathscr{L}\, u(t) = F(t, v(t) + u(t)) - F(t, v(t)) \,. \tag{1}$$

Let $t_0 \in \mathscr{D}(v)$, $\mathscr{D}(u) \subseteq [t_0, +\infty)$, $t_0 \in \mathscr{D}(u)$ and let $u : \mathscr{D}(u) \to H$ be a solution of the equation (1) satisfying the initial conditions (1.2). Let

$$\text{(2)}\quad |||u(t_0)||| = \Big[\sum_{i=0}^{n-1} \|A^{(n-i)/n} \varphi_i\|^2\Big]^{1/2} \leqq$$

$$\leqq \min\left(\frac{R}{2}, \frac{R}{1 + 2(C_3^* + C_2^* C_5^* K_1 n^{3/2} \delta^{-1/n} + C_2^* C_6^* K_1 n^{3/2} \delta^{-1/n})}\right).$$

In the case $R < +\infty$ let us suppose

(3) there exists a number $h > 0$ such that $[t_0, t_0 + h] \subseteq \mathscr{D}(u)$, $|||u(\tau)||| < R$ for $\tau \in [t_0, t_0 + h)$,

$|||u(t_0 + h)||| = R$.

Then using Lemma 2.4 and the condition (2.2) we obtain

$$\text{(4)}\quad |||u(t)||| \leqq C_3^* |||u(t_0)|||\, e^{\omega(t - t_0)} + C_5^* \|F(t_0, v(t_0) + u(t_0)) -$$

$$- F(t_0, v(t_0))\|\, e^{\omega(t - t_0)} + C_6^* \|F(t, v(t) + u(t)) - F(t, v(t))\| +$$

$$+ C_5^* \int_{t_0}^{t} e^{\omega(t-\tau)} \| F'(\tau, v(\tau) + u(\tau)) - F'(\tau, v(\tau)) \| \, d\tau \leqq$$

$$\leqq C_3^* |||u(t_0)||| \, e^{\omega(t-t_0)} + C_5^* K_1 \sum_{i=0}^{n-1} \| A^{(n-i-1)/n} u^{(i)}(t_0) \| \, e^{\omega(t-t_0)} +$$

$$+ C_6^* K_1 \sum_{i=0}^{n-1} \| A^{(n-i-1)/n} u^{(i)}(t) \| + C_5^* K_2 \int_{t_0}^{t} e^{\omega(t-\tau)} |||u(\tau)||| \, d\tau \quad \text{for} \quad t \in [t_0, t_0 + h].$$

Similarly as in [1] (Theorem 2.1.1), we can show that

$$A^{-1/n} u(t) = \sum_{j=0}^{n-1} m_j(t; t_0, A) \, A^{-1/n} \varphi_j +$$

$$+ \int_{t_0}^{t} m(t + t_0 - \tau; t_0, A) \, A^{-1/n} [F(\tau, v(\tau) + u(\tau)) - F(\tau, v(\tau))] \, d\tau$$

and thus using [1] (Theorem 1.2.1 and Remark 1.3.1), we get according to (2.2)

$$(5) \quad \| A^{(n-i-1)/n} u^{(i)}(t) \| \leqq C_2^* \sum_{j=0}^{n-1} \| A^{(n-j-1)/n} \varphi_j \| \, e^{\omega(t-t_0)} +$$

$$+ C(\mathscr{L}) \int_{t_0}^{t} e^{\omega(t-\tau)} \| F(\tau, v(\tau) + u(\tau)) - F(\tau, v(\tau)) \| \, d\tau \leqq$$

$$\leqq C_2^* \sum_{j=0}^{n-1} \| A^{(n-j-1)/n} \varphi_j \| \, e^{\omega(t-t_0)} +$$

$$+ C(\mathscr{L}) \, K_1 \sum_{j=0}^{n-1} \int_{t_0}^{t} e^{\omega(t-\tau)} \| A^{(n-j-1)/n} u^{(j)}(\tau) \| \, d\tau \text{ for } t \in [t_0, t_0 + h],$$

$$i = 0, \ldots, n - 1.$$

The relations (4), (5) and Lemma 2.5 imply

$$|||u(t)||| \leqq (C_3^* + C_2^* C_5^* K_1 n^{3/2} \delta^{-1/n} + C_2^* C_6^* K_1 n^{3/2} \delta^{-1/n}) \, |||u(t_0)||| \, e^{\omega(t-t_0)} +$$

$$+ (C_5^* K_2 + C(\mathscr{L}) \, C_6^* K_1^2 n^{3/2} \delta^{-1/n}) \int_{t_0}^{t} e^{\omega(t-\tau)} |||u(\tau)||| \, d\tau \quad \text{for} \quad t \in [t_0, t_0 + h]$$

and so (see [1] (Theorem 2.1.3))

$$(6) \quad |||u(t)||| \leqq (C_3^* + C_2^* C_5^* K_1 n^{3/2} \delta^{-1/n} +$$

$$+ C_2^* C_6^* K_1 n^{3/2} \delta^{-1/n}) \, |||u(t_0)||| \, e^{(\omega + C_5^* K_2 + C(\mathscr{L}) C_6^* K_1^2 n^{3/2} \delta^{-1/n})(t - t_0)}$$

for $t \in [t_0, t_0 + h]$. Because of $\omega + C_5^* K_2 + C(\mathscr{L}) \, C_6^* K_1^2 n^{3/2} \delta^{-1/n} \leqq 0$ and (2) we get $|||u(t)||| \leqq (C_3^* + C_2^* C_5^* K_1 n^{3/2} \delta^{-1/n} + C_2^* C_6^* K_1 n^{3/2} \delta^{-1/n}) \, |||u(t_0)||| < R$ for $t \in [t_0, t_0 + h]$. But this contradicts (3). So we have proved: the inequality (6) holds for all $t \in \mathscr{D}(u)$. This proves the uniform exponential stability in the case $R < +\infty$. The proof of the other assertions of the theorem is similar and thus we omit it.

Theorem 2.3. *Let* $v : \mathscr{D}(v) \to H$ *be a maximal solution of the equation* (1.1). *Let the operator* $\mathscr{L}$ *be of the type* $\omega < 0$, $F \in \mathscr{C}^{(1)}(\mathscr{D}(u_{/\mathscr{D}(v)}), H)$, *and let the conditions* (2.1), (2.3), (2.4) *be fulfilled*:

(2.3) $F(t, v(t) + u(t)) = F(t, v(t)) + F_L(t, u(t)) + F_N(t, u(t))$ *for* $u \in \mathscr{U}$ *such that* $\mathscr{D}(u) \subseteq \mathscr{D}(v)$ *and* $t \in \mathscr{D}(u)$, *where* $F_L, F_N \in \mathscr{C}^{(1)}(\mathscr{D}(u_{/\mathscr{D}(v)}), H)$.

(2.4) *There exist numbers* $C_1, C_2, C_3, C_4, R_1 > 0, \nu_1 > 0, \nu_2 > 0$ *such that if* u *is a solution of the equation* $\mathscr{L}\, u(t) = F(t, v(t) + u(t)) - F(t, v(t))$ *fulfilling* $\mathscr{D}(u) \subseteq \mathscr{D}(v)$ *and* $t \in \mathscr{D}(u)$ *fulfils the relation* $|||u(t)||| \leqq R_1$ *then*

$$\|F_L(t, u(t))\| \leqq C_1 \sum_{i=0}^{n-1} \|A^{(n-i-1)/n}\, u^{(i)}(t)\|\,,$$

$$\|F_N(t, u(t))\| \leqq C_2 |||u(t)|||^{\nu_1} \sum_{i=0}^{n-1} \|A^{(n-i-1)/n}\, u^{(i)}(t)\|\,,$$

$$\|F'_L(t, u(t))\| \leqq C_3 |||u(t)|||\,,$$

$$\|F'_N(t, u(t))\| \leqq C_4 |||u(t)|||^{1+\nu_2}\,.$$

Further, let $\omega + C_5^* C_3 + C(\mathscr{L})\, C_6^* C_1^2 n^{3/2} \delta^{-1/n} < 0$.

Then the solution v *is uniformly exponentially stable with respect to the norm* $|||\cdot|||$.

Proof. Let us choose a number $R \in (0, R_1]$ so small that

(1) $$\omega + C_5^*(C_3 + C_4 R^{\nu_2}) + C(\mathscr{L})\, C_6^*(C_1 + C_2 R^{\nu_1})^2\, n^{3/2} \delta^{-1/n} < 0\,.$$

Then if u is a solution of the equation $\mathscr{L}\, u(t) = F(t, v(t) + u(t)) - F(t, v(t))$ such that $\mathscr{D}(u) \subseteq \mathscr{D}(v)$ and $t \in \mathscr{D}(u)$ fulfils $|||u(t)||| \leqq R$, we can write according to (2.3), (2.4)

(2) $$\|F(t, v(t) + u(t)) - F(t, v(t))\| \leqq \|F_L(t, u(t))\| + \|F_N(t, u(t))\| \leqq$$
$$\leqq [C_1 + C_2 |||u(t)|||^{\nu_1}] \sum_{i=0}^{n-1} \|A^{(n-i-1)/n}\, u^{(i)}(t)\| \leqq$$
$$\leqq (C_1 + C_2 R^{\nu_1}) \sum_{i=0}^{n-1} \|A^{(n-i-1)/n}\, u^{(i)}(t)\|,$$

(3) $$\|F'(t, v(t) + u(t)) - F'(t, v(t))\| \leqq (C_3 + C_4 R^{\nu_2})\, |||u(t)|||\,.$$

The theorem now immediately follows from (1), (2), (3) with help of Theorem 2.2.

3. THE CASE OF THE STABLE OPERATOR

Let $M(t; t_0, s)$ be defined by the relations $(d/dt) M(t; t_0, s) = m(t; t_0, s)$, $M(t_0; t_0, s) = 0$ for $t_0 \in R^+$, $t \geqq t_0$, $s \in \sigma(A)$.

Lemma 3.1. *Let the operator $\mathscr{L}$ be of the type ω, let the condition* (2.1) *be fulfilled, $t_0 \in R^+$, $t \geqq t_0$, $s \in \sigma(A)$. Then*

(i) $$\mathscr{D}(M(t; t_0, A)) = H\,, \quad \mathscr{R}(M(t; t_0, A)) \subseteq \mathscr{D}(A)\,,$$

(ii) $$|M(t; t_0, s)| \leqq K^*[1 + ((n-1) C_0^* + .1) C(\mathscr{L}) e^{\omega(t-t_0)}] s^{-1}\,,$$

(iii) $$\|AM(t; t_0, A)\| \leqq K^*[1 + ((n-1) C_0^* + 1) C(\mathscr{L}) e^{\omega(t-t_0)}]\,.$$

Proof. Similarly as in the proof of Lemma 2.3 we obtain

$$m^{(n-1)}(t) + a_1(s) m^{(n-2)}(t) + \ldots + a_{n-1}(s) m(t) + a_n(s) M(t) = 1\,.$$

Using [1] (the relation (1.1.1) and Theorem 1.2.1) we get $|a_n(s) M(t; t_0, s)| \leqq 1 + ((n-1) C_0^* + 1) C(\mathscr{L}) e^{\omega(t-t_0)}$. Now applying Lemma 2.2 we can conclude

$$|M(t; t_0, s)| \leqq K^*[1 + ((n-1) C_0^* + 1) C(\mathscr{L}) e^{\omega(t-t_0)}] s^{-1}\,.$$

This inequality proves the lemma.

Lemma 3.2. *Let the operator $\mathscr{L}$ be of the type ω, let the condition* (2.1) *be fulfilled, $F \in \mathscr{C}^{(1)}(\mathscr{D}(u_{/I}), H)$, $\mathscr{D}(u) \subseteq [t_0, +\infty) \subseteq I$, $t_0 \in \mathscr{D}(u)$, let $u : \mathscr{D}(u) \to H$ be a solution of the equation* (1.1) *which fulfils the initial conditions* (1.2). *Then for $t \in \mathscr{D}(u)$*

(i) $$u(t) = \sum_{i=0}^{n-1} m_i(t; t_0, A) \varphi_i + M(t; t_0, A) F(t_0, u(t_0)) + \int_{t_0}^{t} M(t + t_0 - \tau; t_0, A) F'(\tau, u(\tau)) \, d\tau,$$

(ii) $$\left|\left|\left| \sum_{i=0}^{n-1} m_i(t; t_0, A) \varphi_i \right|\right|\right| \leqq C_3^* |||u(t_0)||| e^{\omega(t-t_0)},$$

$$\left|\left|\left| M(t; t_0, A) F(t_0, u(t_0)) + \int_{t_0}^{t} M(t + t_0 - \tau; t_0, A) F'(\tau, u(\tau)) \, d\tau \right|\right|\right| \leqq$$
$$\leqq (K^* + C_7^* e^{\omega(t-t_0)}) \|F(t_0, u(t_0))\| + \int_{t_0}^{t} (K^* + C_7^* e^{\omega(t-\tau)}) \|F'(\tau, u(\tau))\| \, d\tau,$$

where $C_7^* = C(\mathscr{L}) [((n-1) C_0^* + 1) K^* + n - 1]$ and the constant C_3^* was defined in [1] (*Theorem* 1.3.1).

Proof. We shall prove the second estimate of the statement (ii) only (for the proof of the other statements see the proof of Lemma 2.4). Using Lemma 3.1 we get

$$(1)\quad \left\| A\left[M(t; t_0, A)\, F(t_0, u(t_0)) + \int_{t_0}^{t} M(t + t_0 - \tau; t_0, A)\, F'(\tau, u(\tau))\, d\tau \right] \right\| \leqq$$
$$\leqq \| AM(t; t_0, A)\, F(t_0, u(t_0)) \| +$$
$$+ \int_{t_0}^{t} \| AM(t + t_0 - \tau; t_0, A)\, F'(\tau, u(\tau)) \|\, d\tau \leqq$$
$$\leqq K^*[1 + ((n-1)\, C_0^* + 1)\, C(\mathscr{L})\, e^{\omega(t - t_0)}]\, \| F(t_0, u(t_0)) \| +$$
$$+ K^* \int_{t_0}^{t} [1 + ((n-1)\, C_0^* + 1)\, C(\mathscr{L})\, e^{\omega(t-\tau)}]\, \| F'(\tau, u(\tau)) \|\, d\tau\,.$$

Now, using [1] (Theorem 1.2.1) we obtain

$$(2)\quad \left\| A^{(n-i)/n} \frac{d^i}{dt^i} \left[M(t; t_0, A)\, F(t_0, u(t_0)) + \int_{t_0}^{t} M(t + t_0 - \tau; t_0, A)\, F'(\tau, u(\tau))\, d\tau \right] \right\| =$$
$$= \left\| A^{(n-i)/n} \left[m^{(i-1)}(t; t_0, A)\, F(t_0, u(t_0)) + \right.\right.$$
$$\left.\left. + \int_{t_0}^{t} m^{(i-1)}(t + t_0 - \tau; t_0, A)\, F'(\tau, u(\tau))\, d\tau \right] \right\| \leqq$$
$$\leqq C(\mathscr{L})\, e^{\omega(t - t_0)} \| F(t_0, u(t_0)) \| +$$
$$+ C(\mathscr{L}) \int_{t_0}^{t} e^{\omega(t-\tau)} \| F'(\tau, u(\tau)) \|\, d\tau \quad \text{for} \quad i = 1, 2, \ldots, n-1\,.$$

Now (1), (2) imply the inequality

$$\left|\left|\left| M(t; t_0, A)\, F(t_0, u(t_0)) + \int_{t_0}^{t} M(t + t_0 - \tau; t_0, A)\, F'(\tau, u(\tau))\, d\tau \right|\right|\right| \leqq$$
$$\leqq [K^* + C(\mathscr{L})\, [((n-1)\, C_0^* + 1)\, K^* + n - 1]\, e^{\omega(t - t_0)}]\, \| F(t_0, u(t_0)) \| +$$
$$+ \int_{t_0}^{t} [K^* + C(\mathscr{L})\, [((n-1)\, C_0^* + 1)\, K^* + n - 1]\, e^{\omega(t-\tau)}]\, \| F'(\tau, u(\tau)) \|\, d\tau\,.$$

The lemma is proved.

Theorem 3.1. (Correctness Theorem.) *Let the operator $\mathscr{L}$ be of the type ω, let the condition* (2.1) *be fulfilled, $F, F' \in \mathscr{C}([t_0, +\infty), H)$. Then the maximal solution of the Cauchy problem* (1.1), (1.2)

(i) *is uniquely determined*,

(ii) *has the form* $u(t) = \sum_{i=0}^{n-1} m_i(t; t_0, A)\, \varphi_i + M(t; t_0, A)\, F(t_0) +$

$$+ \int_{t_0}^{t} M(t + t_0 - \tau; t_0, A)\, F'(\tau)\, d\tau \quad \textit{for} \quad t \geqq t_0\,,$$

(iii) *fulfils the estimate* $|||u(t)||| \leqq C_3^* |||u(t_0)||| \, e^{\omega(t-t_0)} +$

$$+ (K^* + C_7^* e^{\omega(t-t_0)}) \|F(t_0)\| + \int_{t_0}^{t} (K^* + C_7^* e^{\omega(t-\tau)}) \|F'(\tau)\| \, d\tau \quad \textit{for} \quad t \geqq t_0 .$$

Proof. For (i) see [1] (Lemma 1.3.4). The statements (ii), (iii) are easy consequences of Lemma 3.2.

Theorem 3.2. *Let* $v : \mathscr{D}(v) \to H$ *be a maximal solution of the equation* (1.1). *Let the operator* $\mathscr{L}$ *be of the type* 0, $F \in \mathscr{C}^{(1)}(\mathscr{D}(u_{/\mathscr{D}(v)}), H)$ *and let the conditions* (2.1), (2.3), (3.1), (3.2) *be fulfilled:*

(3.1) *There exist constants* K_1, K_2, K_3, K_4, $R > 0$, $\nu_1 > 0$, $\nu_2 > 0$ *such that if* u, u_i, $(i = 1, 2)$ *are solutions of the equation* $\mathscr{L} u(t) = F(t, v(t) + u(t)) - F(t, v(t))$ *fulfilling* $\mathscr{D}(u) \subseteq \mathscr{D}(v)$, $\mathscr{D}(u_i) \subseteq \mathscr{D}(v)$ *then*

(i) $\|F_L(t, u(t))\| \leqq K_1 |||u(t)|||$ *for* $t \in \mathscr{D}(u)$ *such that* $|||u(t)||| \leqq R$, $\|F_L'(t, u_1(t)) - F_L'(t, u_2(t))\| \leqq K_2 |||u_1(t) - u_2(t)|||$ *for* $t \in \mathscr{D}(u_1) \cap \mathscr{D}(u_2)$ *such that* $|||u_i(t)||| \leqq R$, $(i = 1, 2)$,

(ii) $\|F_N(t, u(t))\| \leqq K_3 |||u(t)|||^{1+\nu_1}$, $\|F_N'(t, u(t))\| \leqq K_4 |||u(t)|||^{1+\nu_2}$ *for* $t \in \mathscr{D}(u)$ *such that* $|||u(t)||| \leqq R$.

(3.2) *There exists a number* $\varkappa > 0$ *such that if* $\varphi_i \in \mathscr{D}(A^{(n-i)/n})$, $(i = 0, \ldots, n-1)$, $[\sum_{i=0}^{n-1} \|A^{(n-i)/n} \varphi_i\|^2]^{1/2} \leqq \varkappa$ *and* $t_0 \in \mathscr{D}(v)$ *then there exists a maximal solution of the equation* $\mathscr{L} u(t) = F_L(t, u(t))$ *which fulfils the initial conditions* (1.2).

Further, let $F_L(t, O_{/I}) = 0$, $F_L'(t, O_{/I}) = 0$ *for every* $I \subseteq \mathscr{D}(v)$. *Finally, let the zero solution* $O_{/\mathscr{D}(v)}$ *of the equation*

$$\mathscr{L} u(t) = F_L(t, u(t)) \tag{3.3}$$

be uniformly exponentially stable with respect to the norm $|||\cdot|||$. *Then the solution* v *is uniformly exponentially stable with respect to the norm* $|||\cdot|||$.

Proof. Clearly, it suffices to prove the uniform exponential stability of the zero solution $O_{/\mathscr{D}(v)}$ of the equation

$$\mathscr{L} u(t) = F_L(t, u(t)) + F_N(t, u(t)) . \tag{1}$$

If u_L is a solution of the equation (3.3), $\mathscr{D}(u_L) \subseteq [t_0, +\infty)$, $t_0 \in \mathscr{D}(v) \cap \mathscr{D}(u_L)$, then by Lemma 3.2 and by the assumptions of the theorem,

$$\begin{aligned} (2) \quad u_L(t) &= \sum_{i=0}^{n-1} m_i(t; t_0, A) \, u_L^{(i)}(t_0) + M(t; t_0, A) \, F_L(t_0, u_L(t_0)) + \\ &+ \int_{t_0}^{t} M(t + t_0 - \tau; t_0, A) \, F_L'(\tau, u_L(\tau)) \, d\tau \quad \text{for} \quad t \in \mathscr{D}(u_L). \end{aligned}$$

(3) There exist positive constants C, α, ϱ (independent of the choice of $t_0 \in \mathscr{D}(v)$) such that

$$|||u_L(t_0)||| \leqq \varrho \Rightarrow |||u_L(t)||| \leqq C e^{-\alpha(t-t_0)} |||u_L(t_0)||| \quad \textit{for} \ t \in \mathscr{D}(u_L).$$

We shall use the notation $B = K^* + C_7^*$ in this proof. Let $\alpha_1 \in (0, \alpha)$, $C_1 > C$. For this α_1, C_1, let us choose constants $h > 0$, $R_1 \in (0, \min(\varkappa, \varrho, R)]$, $\varrho_1 \in (0, R_1)$ in such a way that $C_1 \varrho_1 < R_1$ and

$$(4) \quad C + [BK_3\varrho_1^{\nu_1} + hBK_4R_1^{\nu_2}(C_3^* + BK_1 + BK_3\varrho_1^{\nu_1})\, e^{B(K_2+K_4R_1^{\nu_2})h}]\, e^{(BK_2+\alpha_1)h} \leqq C_1,$$

$$(5) \quad Ce^{-(\alpha-\alpha_1)h} + [BK_3\varrho_1^{\nu_1} + hBK_4R_1^{\nu_2}(C_3^* + BK_1 + BK_3\varrho_1^{\nu_1})\, e^{B(K_2+K_4R_1^{\nu_2})h}]\, e^{(BK_2+\alpha_1)h} \leqq 1.$$

Now let $t_0 \in \mathscr{D}(v)$ and let u_N be a solution of the equation (1) for which $\mathscr{D}(u_N) \subseteq [t_0, +\infty)$, $t_0 \in \mathscr{D}(u_N)$, $|||u_N(t_0)||| \leqq \varrho_1$. According to Lemma 3.2 this solution fulfils

$$(6) \quad u_N(t) = \sum_{i=0}^{n-1} m_i(t; t_0, A)\, u_N^{(i)}(t_0) + M(t; t_0, A)\,[F_L(t_0, u_N(t_0)) + F_N(t_0, u_N(t_0))] + \int_{t_0}^{t} M(t + t_0 - \tau; t_0, A)\,[F_L'(\tau, u_N(\tau)) + F_N'(\tau, u_N(\tau))]\, d\tau \ \text{for} \ t \in \mathscr{D}(u_N).$$

Let u_L be a maximal solution of the equation (3.3) with $\mathscr{D}(u_L) = [t_0, +\infty)$, fulfilling the same initial conditions as the solution u_N (such a solution exists in virtue of (3.2)). Let us suppose

(7) there exists a number $\tilde{h} \leqq h$ such that $[t_0, t_0 + \tilde{h}] \subseteq \mathscr{D}(u_N)$,
$|||u_N(\tau)||| < R_1$ for $\tau \in [t_0, t_0 + \tilde{h})$,
$|||u_N(t_0 + \tilde{h})||| = R_1$.

Using Lemma 3.2 and the relations (6), (7), (3.1) and $F_L'(t, O_{/\mathscr{D}(u_N)}) = 0$ we obtain

$$|||u_N(t)||| \leqq C_3^* |||u_N(t_0)||| + B(\|F_L(t_0, u_N(t_0))\| + \|F_N(t_0, u_N(t_0))\|) + B\int_{t_0}^{t} (\|F_L'(\tau, u_N(\tau))\| + \|F_N'(\tau, u_N(\tau))\|)\, d\tau \leqq C_3^* |||u_N(t_0)||| + B(K_1|||u_N(t_0)||| + K_3|||u_N(t_0)|||^{1+\nu_1}) + B\int_{t_0}^{t} (K_2|||u_N(\tau)||| + K_4|||u_N(\tau)|||^{1+\nu_2})\, d\tau \leqq (C_3^* + BK_1 + BK_3\varrho_1^{\nu_1})\, |||u_N(t_0)||| + B(K_2 + K_4R_1^{\nu_2}) \int_{t_0}^{t} |||u_N(\tau)|||\, d\tau \quad \text{for} \quad t \in [t_0, t_0 + \tilde{h}].$$

This implies (see [1] (Theorem 2.1.3))

$$(8) \quad |||u_N(t)||| \leqq (C_3^* + BK_1 + BK_3\varrho_1^{\nu_1})\, |||u_N(t_0)|||\, e^{B(K_2+K_4R_1^{\nu_2})(t-t_0)} \ \text{for} \ t \in [t_0, t_0 + \tilde{h}].$$

Subtracting (2) from (6), using Lemma 3.2, the relations (7), (8), (3.1) and $|||u_L(t)||| \leqq$ $\leqq Ce^{-\alpha(t-t_0)}|||u_L(t_0)||| \leqq C_1\varrho_1 < R_1 \leqq R$ for $t \geqq t_0$, we get

$$|||u_N(t) - u_L(t)||| \leqq B\|F_N(t_0, u_N(t_0))\| + B\int_{t_0}^{t}(\|F'_L(\tau, u_N(\tau)) - F'_L(\tau, u_L(\tau))\| +$$

$$+ \|F'_N(\tau, u_N(\tau))\|)\,\mathrm{d}\tau \leqq BK_3|||u_N(t_0)|||^{1+\nu_1} +$$

$$+ B\int_{t_0}^{t}(K_2|||u_N(\tau) - u_L(\tau)||| + K_4|||u_N(\tau)|||^{1+\nu_2})\,\mathrm{d}\tau \leqq$$

$$\leqq [BK_3\varrho_1^{\nu_1} + \tilde{h}BK_4R_1^{\nu_2}(C_3^* + BK_1 + BK_3\varrho_1^{\nu_1})\,\mathrm{e}^{B(K_2+K_4R_1^{\nu_2})\tilde{h}}]\,|||u_N(t_0)||| +$$

$$+ BK_2\int_{t_0}^{t}|||u_N(\tau) - u_L(\tau)|||\,\mathrm{d}\tau \quad \text{for} \quad t \in [t_0, t_0 + \tilde{h}]\,.$$

This yields

$$(9)\quad |||u_N(t) - u_L(t)||| \leqq [BK_3\varrho_1^{\nu_1} + \tilde{h}BK_4R_1^{\nu_2}(C_3^* + BK_1 + BK_3\varrho_1^{\nu_1})\,\mathrm{e}^{B(K_2+K_4R_1^{\nu_2})\tilde{h}}]\,.$$

$$.\,|||u_N(t_0)|||\,\mathrm{e}^{BK_2(t-t_0)} \leqq [BK_3\varrho_1^{\nu_1} + \tilde{h}BK_4R_1^{\nu_2}(C_3^* + BK_1 + BK_3\varrho_1^{\nu_1})\,\mathrm{e}^{B(K_2+K_4R_1^{\nu_2})\tilde{h}}]\,.$$

$$.\,\mathrm{e}^{(BK_2+\alpha_1)\tilde{h}}\mathrm{e}^{-\alpha_1(t-t_0)}|||u_N(t_0)||| \quad \text{for } t \in [t_0, t_0 + \tilde{h}].$$

It follows from (3), (9) that

$$(10)\quad |||u_N(t)||| \leqq |||u_L(t)||| + |||u_N(t) - u_L(t)||| \leqq$$

$$\leqq \{C + [BK_3\varrho_1^{\nu_1} + \tilde{h}BK_4R_1^{\nu_2}(C_3^* + BK_1 + BK_3\varrho_1^{\nu_1})\,\mathrm{e}^{B(K_2+K_4R_1^{\nu_2})\tilde{h}}]\,\mathrm{e}^{(BK_2+\alpha_1)\tilde{h}}\}\,.$$

$$.\,\mathrm{e}^{-\alpha_1(t-t_0)}|||u_N(t_0)||| \quad \text{for } t \in [t_0, t_0 + \tilde{h}].$$

Because $\tilde{h} \leqq h$, we get with help of (4) $|||u_N(t)||| \leqq C_1\varrho_1 < R_1$ for $t \in [t_0, t_0 + \tilde{h}]$, which contradicts (7). So $|||u_N(t)||| < R_1 \leqq R$ for $t \in [t_0, t_0 + h] \cap \mathscr{D}(u_N)$. Moreover, if we write h instead of $\tilde{h}$ the relations (9), (10) are again valid on $[t_0, t_0 + h] \cap \mathscr{D}(u_N)$. Thus (4), (10) yield

$$(11)\qquad |||u_N(t)||| \leqq C_1\mathrm{e}^{-\alpha_1(t-t_0)}|||u_N(t_0)||| \quad \text{for} \quad t \in [t_0, t_0 + h] \cap \mathscr{D}(u_N)\,.$$

If $t_0 + h \in \mathscr{D}(u_N)$, then by (3), (9), (5),

$$(12)\quad |||u_N(t_0 + h)||| \leqq \{Ce^{-(\alpha-\alpha_1)h} + [BK_3\varrho_1^{\nu_1} + hBK_4R_1^{\nu_2}(C_3^* + BK_1 +$$

$$+ BK_3\varrho_1^{\nu_1})\,\mathrm{e}^{B(K_2+K_4R_1^{\nu_2})h}]\,\mathrm{e}^{(BK_2+\alpha_1)h}\}\,\mathrm{e}^{-\alpha_1 h}|||u_N(t_0)||| \leqq \mathrm{e}^{-\alpha_1 h}|||u_N(t_0)|||\,.$$

Now, the uniform exponential stability of the zero solution $O_{/\mathscr{D}(v)}$ follows from (11), (12). Therefore, if $|||u_N(t_0)||| \leqq \varrho_1$, $t \in \mathscr{D}(u_N)$, we can find a natural number k and a number $s \in [0, h)$ such that $t = t_0 + kh + s$. Then using (11) and k-times (12) we get $|||u_N(t_0 + kh + s)||| \leqq C_1\mathrm{e}^{-\alpha_1 s}|||u_N(t_0 + kh)||| \leqq C_1\mathrm{e}^{-\alpha_1 s}\mathrm{e}^{-\alpha_1 kh}|||u_N(t_0)||| =$ $= C_1\mathrm{e}^{-\alpha_1(t-t_0)}|||u_N(t_0)|||$. The theorem is proved.

Theorem 4.1. *Let $v : \mathscr{D}(v) = [t_0, +\infty) \to H$ be a maximal solution of the equation (1.1). If $+\infty$ is a limit point of $\sigma(A)$ then we suppose that there exist limits $\lim_{\substack{s \to +\infty \\ s \in \sigma(A)}} a_i(s)\, s^{-i/n} = a_i^*$, $(i = 1, \ldots, n)$ and that the equation $\Lambda^n + a_1^* \Lambda^{n-1} + \ldots$ $\ldots + a_n^* = 0$ has simple roots only. Let $F \in \mathscr{C}^{(1)}(\mathscr{D}(u_{/\mathscr{D}(v)}), H)$. Further, let us assume:*

(4.1) *There exist numbers $K_1 > 0$, $K_2 > 0$, $R > 0$, $\nu_1 > 0$, $\nu_2 > 0$ such that if u is a solution of the equation $\mathscr{L}\, u(t) = F(t, v(t) + u(t)) - F(t, v(t))$ with $\mathscr{D}(u) = \mathscr{D}(v)$ then*
$\|F(t_0, v(t_0) + u(t_0)) - F(t_0, v(t_0))\| \leqq K_1 |||u(t_0)|||^{1+\nu_1}$,
$\|F'(t, v(t) + u(t)) - F'(t, v(t))\| \leqq K_2 |||u(t)|||^{1+\nu_2}$ for $t \in \mathscr{D}(u)$ *such that* $|||u(t)||| \leqq R$.

(4.2) *If $\lambda_i(s)$ are solutions of the equation $\lambda^n(s) + a_1(s)\, \lambda^{n-1}(s) + \ldots + a_n(s) = 0$ then $\lambda_i(s) \neq \lambda_j(s)$ for $i \neq j$, $(i, j = 1, \ldots, n)$, $s \in \sigma(A)$.*

(4.3) *There exists an eigenvalue s_0 of the operator A and an index $i_0 \in \{1, \ldots, n\}$ such that $\operatorname{Re} \lambda_{i_0}(s_0) > 0$ and $\max_{i=1,\ldots,n} \sup_{s \in \sigma(A)} \operatorname{Re} \lambda_i(s) \leqq \operatorname{Re} \lambda_{i_0}(s_0)$.*

(4.4) *There exists a number $\varkappa > 0$ such that if $\varphi_i \in \mathscr{D}(A^{(n-i)/n})$, $(i = 0, \ldots, n-1)$, $[\sum_{i=0}^{n-1} \|A^{(n-i)/n} \varphi_i\|^2]^{1/2} \leqq \varkappa$ then there exists a maximal solution of the equation $\mathscr{L}\, u(t) = F(t, v(t) + u(t)) - F(t, v(t))$ fulfilling the initial conditions (1.2).*

Then the solution v is instable with respect to the norm $|||\cdot|||$.

Proof. Clearly, (see [1] (Theorem 2.1.2)), it suffices to prove the instability of the zero solution $O_{/\mathscr{D}(v)}$ of the equation

$$\mathscr{L}\, u(t) = F(t, v(t) + u(t)) - F(t, v(t))\,. \tag{1}$$

By [1] (Theorem 1.5.1), the following assertion holds:

$$\text{The operator } \mathscr{L} \text{ is of the type } \quad \omega = \operatorname{Re} \lambda_{i_0}(s_0) > 0\,. \tag{2}$$

We shall use the notation

$$A = K_1(K^* + C_7^*) + 1\,, \quad B = K_2 \left(\frac{K^*}{\omega(1 + \nu_2)} + \frac{C_7^*}{\omega \nu_2} \right) + 1$$

in this proof. (Remember that C_7^* was defined in Lemma 3.2.)

Let us choose a number C satisfying

$$C \in (1, \min(2, 1 + BR^{\nu_2}))\,. \tag{3}$$

Obviously, there exists a number

$$\eta_0 \in \left(0, \min\left(1, \varkappa, \left[\frac{C-1}{AC^{1+\nu_1}}\right]^{1/\nu_1}\right)\right)$$

such that if $\eta \in (0, \eta_0]$, the equation

$$C - 1 = A\eta^{\nu_1} + B\eta^{\nu_2}C^{1+\nu_2}e^{\omega\nu_2 h} \tag{4}$$

has a unique solution $h = h(\eta) > 0$.

Let $\varphi_0 \in H$ be such that

$$(5)\quad A\varphi_0 = s_0\varphi_0,$$
$$\|(\varphi_0, \lambda_{i_0}(A)\,\varphi_0, \ldots, \lambda_{i_0}^{n-1}(A)\,\varphi_0)\|_{\mathscr{D}(A)\times\mathscr{D}(A^{(n-1)/n})\times\ldots\times\mathscr{D}(A^{1/n})} = 1.$$

If u_0 is a maximal solution of the equation (1) fulfilling the initial conditions $u_0^{(j)}(t_0) = \eta\lambda_{i_0}^j(A)\,\varphi_0$, $(j = 0, \ldots, n-1)$, $\eta \in (0, \eta_0]$, then obviously

$$\begin{aligned}(6)\quad u_0(t) &= \eta e^{\lambda_{i_0}(A)(t-t_0)}\varphi_0 + \\ &+ M(t; t_0, A)\left[F(t_0, v(t_0) + u_0(t_0)) - F(t_0, v(t_0))\right] + \\ &+ \int_{t_0}^t M(t + t_0 - \tau; t_0, A)\left[F'(\tau, v(\tau) + u_0(\tau)) - F'(\tau, v(\tau))\right] d\tau\end{aligned}$$

and according to (5),

$$(7)\quad \left|\left|\left|\eta e^{\lambda_{i_0}(A)(t-t_0)}\varphi_0\right|\right|\right| = \eta\left[\sum_{i=0}^{n-1}\left\|A^{(n-i)/n}\lambda_{i_0}^i(A)\,e^{\lambda_{i_0}(A)(t-t_0)}\varphi_0\right\|^2\right]^{1/2} = \eta e^{\omega(t-t_0)}.$$

In particular, $|||u_0(t_0)||| = \eta \leqq \varkappa$ and thus by (4.4) such a solution u_0 really exists. Now, let $\eta \in (0, \eta_0]$, $h = h(\eta)$. Let $\tilde{h} \in (0, h]$ be such that $|||u_0(t)||| \leqq \eta C e^{\omega(t-t_0)}$ for $t \in [t_0, t + \tilde{h}]$. Then by (3), (4),

$$|||u_0(t)||| \leqq \eta C e^{\omega(t-t_0)} \leqq \eta C e^{\omega h} =$$
$$= \left[\frac{C - 1 - A\eta^{\nu_1}}{BC}\right]^{1/\nu_2} \leqq \left[\frac{C-1}{B}\right]^{1/\nu_2} \leqq \left[\frac{BR^{\nu_2}}{B}\right]^{1/\nu_2} = R$$

for $t \in [t_0, t_0 + \tilde{h}]$ and thus using Lemma 3.2 and condition (4.1) we obtain

$$\begin{aligned}(8)\quad &\left|\left|\left| M(t_0 + \tilde{h}; t_0, A)\left[F(t_0, v(t_0) + u_0(t_0)) - F(t_0, v(t_0))\right] + \right.\right.\right.\\ &\left.\left.\left. + \int_{t_0}^{t_0+\tilde{h}} M(2t_0 + \tilde{h} - \tau; t_0, A)\left[F'(\tau, v(\tau) + u_0(\tau)) - F'(\tau, v(\tau))\right] d\tau \right|\right|\right| \leqq \\ &\leqq K_1(K^* + C_7^* e^{\omega\tilde{h}})\,|||u_0(t_0)|||^{1+\nu_1} + \\ &+ K_2\int_{t_0}^{t_0+\tilde{h}} (K^* + C_7^* e^{\omega(t_0+\tilde{h}-\tau)})\,|||u_0(\tau)|||^{1+\nu_2}\,d\tau \leqq K_1(K^* + C_7^* e^{\omega\tilde{h}})\,\eta^{1+\nu_1} +\end{aligned}$$

$$+ K_2(\eta C)^{1+\nu_2}\left[\frac{K^*}{\omega(1+\nu_2)} + \frac{C_7^*}{\omega\nu_2}\right] e^{\omega(1+\nu_2)\tilde{h}} \leqq \eta e^{\omega\tilde{h}}(A\eta^{\nu_1} + B\eta^{\nu_2}C^{1+\nu_2}e^{\omega\nu_2\tilde{h}}) .$$

Now, we shall suppose that

(9) there exists a number $\tilde{h} \in (0, h)$ such that

$$|||u_0(t)||| < \eta C e^{\omega(t-t_0)} \text{ for } t \in [t_0, t_0 + \tilde{h}),$$

$$|||u_0(t_0 + \tilde{h})||| = \eta C e^{\omega\tilde{h}}.$$

Then by (4), (6), (7), (8),

$$|||u_0(t_0 + \tilde{h})||| \leqq \eta e^{\omega\tilde{h}}[1 + A\eta^{\nu_1} + B\eta^{\nu_2}C^{1+\nu_2}e^{\omega\nu_2\tilde{h}}] <$$

$$< \eta e^{\omega\tilde{h}}[1 + A\eta^{\nu_1} + B\eta^{\nu_2}C^{1+\nu_2}e^{\omega\nu_2 h}] = \eta C e^{\omega\tilde{h}} ,$$

which contradicts (9). So

$$|||u_0(t)||| \leqq \eta C e^{\omega(t-t_0)} \leqq R \quad \text{for} \quad t \in [t_0, t + h]$$

and thus the estimate (8) holds also for $\tilde{h} = h$. Finally, with help of (4), (6), (7), (8) we get

$$(10) \quad |||u_0(t_0 + h)||| \geqq \eta e^{\omega h}[1 - (A\eta^{\nu_1} + B\eta^{\nu_2}C^{1+\nu_2}e^{\omega\nu_2 h})] =$$

$$= \eta e^{\omega h}(2 - C) = \left[\frac{C - 1 - A\eta^{\nu_1}}{BC^{1+\nu_2}}\right]^{1/\nu_2}(2 - C) \geqq$$

$$\geqq \left[\frac{C - 1 - A\eta_0^{\nu_1}}{BC^{1+\nu_2}}\right]^{1/\nu_2}(2 - C) = \text{const} > 0 .$$

Because $|||u_0(t_0)||| = \eta$, the inequality (10) proves the theorem.

5. TWO EXAMPLES

Example A. Let H be a real Hilbert space of real vector functions $h = h(x) = (h_1(x), \ldots, h_k(x))$ $(k \geqq 1)$ that are defined on a subset Ω of a Euclidean space E_N. Let $q_i \geqq 0$ $(i = 0, \ldots, n - 1)$ be natural numbers and f_{ij} $(i = 0, \ldots, n - 1,\ j = 1, \ldots, q_i)$ functions fulfilling the condition

(5.1) $f_{ij} : \sigma(A) \to R$ are continuous functions and there exists a constant F^* such that $|f_{ij}(s)| \leqq F^* s^{(n-i-1)/n}$ for $s \in \sigma(A)$, $i = 0, \ldots, n - 1$, $j = 1, \ldots, q_i$.

Further, let $t_0 \in R^+$ and let $a_{ij} = a_{ij}(t, x)$ be such mappings (defined on $[t_0, +\infty) \times \bar{\Omega}$) that $a_{ij}(t, x) f_{ij}(A)\, u^{(i)}(t, x) \in \mathscr{C}^{(1)}(\mathscr{D}(u_{/[t_0,+\infty)}), H)$, $(i = 0, \ldots, n - 1,\ j = 1, \ldots, q_i)$.

Theorem 5.1. *Let the condition* (2.1) *be fulfilled. Let* $O_{/[t_0,+\infty)}$ *solve the equation*

(5.2) $$\mathscr{L}u(t,x) = \sum_{i=0}^{n-1} \sum_{j=1}^{q_i} a_{ij}(t,x) f_{ij}(A)\, u^{(i)}(t,x)\,.$$

Let

(5.3) *there exist constants* $\bar{a}_{ij}$, M_{ij}, N_{ij} *such that*

(i) $\|a_{ij}(t,x) f_{ij}(A)\, u^{(i)}(t,x) - \bar{a}_{ij} f_{ij}(A)\, u^{(i)}(t,x)\| \leqq$
$\leqq M_{ij} \|A^{(n-i-1)/n}\, u^{(i)}(t)\|$,

(ii) $\left\|\dfrac{\mathrm{d}}{\mathrm{d}t}\left[a_{ij}(t,x) f_{ij}(A)\, u^{(i)}(t,x) - \bar{a}_{ij} f_{ij}(A)\, u^{(i)}(t,x)\right]\right\| \leqq N_{ij} |||u(t)|||$

for solutions u *of the equation* (5.2) *for which* $\mathscr{D}(u) \subseteq [t_0, +\infty)$, $t \in \mathscr{D}(u)$, $x \in \Omega$, $i = 0, \ldots, n-1$, $j = 1, \ldots, q_i$.

Finally, let the coefficients of the operator

$$\bar{\mathscr{L}}\, u(t) \equiv \mathscr{L}\, u(t) - \sum_{i=0}^{n-1} \sum_{j=1}^{q_i} \bar{a}_{ij} f_{ij}(A)\, u^{(i)}(t)$$

fulfil the condition (1.1.1) *from* [1] *with a constant* $\bar{C}_0$ *instead of* C_0^*, *let* $\bar{\mathscr{L}}$ *be of the type* $\omega < 0$ *and*

$$\omega + C(\bar{\mathscr{L}})\left[K^*(1 + (n-1)\bar{C}_0) + n - 1\right] \sum_{i=0}^{n-1} \sum_{j=1}^{q_i} N_{ij} +$$
$$+ K^* C^2(\bar{\mathscr{L}})(1 + (n-1)\bar{C}_0) \max_{i=0,\ldots,n-1}{}^2 \left(\sum_{j=1}^{q_i} M_{ij}\right) n^{3/2} \delta^{-1/n} < 0 \quad (or \leqq 0)\,.$$

Then the solution $O_{/[t_0,+\infty)}$ *is globally uniformly exponentially stable with respect to the norm* $|||\cdot|||$ (*or globally uniformly stable with respect to the norm* $|||\cdot|||$, *respectively*).

Proof. We can rewrite the equation (5.2) in the form

$$\bar{\mathscr{L}}u(t,x) = \sum_{i=0}^{n-1} \sum_{j=1}^{q_i} \left[a_{ij}(t,x) f_{ij}(A)\, u^{(i)}(t,x) - \bar{a}_{ij} f_{ij}(A)\, u^{(i)}(t,x)\right].$$

The statement of the theorem directly follows from Theorem 2.2.

Example B. Let $H = L_2(\Omega)$, where $\Omega = (0, \pi c_1) \times (0, \pi c_2) \times \ldots \times (0, \pi c_N)$, $c_i > 0$ $(i = 1, \ldots, N)$. Let $p \geqq 1$ be a natural number such that $2p/n$ is an integer. The operator A will be given by

(5.4) $A\, v(x) = (-1)^p \left[\sum_{i=1}^{N} D_i^2\right]^p v(x)$ for $v \in \mathscr{D}(A) =$

$$= \left\{ u(x) \in L_2(\Omega) \mid u(x) = \sum_k u_k \sin \frac{k_1 x_1}{c_1} \dots \sin \frac{k_N x_N}{c_N}, \right.$$

$$\left. k = (k_1, \dots, k_N), \sum_k \left[\left(\frac{k_1}{c_1}\right)^2 + \dots + \left(\frac{k_N}{c_N}\right)^2 \right]^{2p} u_k^2 < +\infty \right\},$$

$$D_i = \frac{\partial}{\partial x_i} \text{ (in the sense of distributions).}$$

The symbol $\sum_k$ stays for $\sum_{\substack{1 \leqq k_i < +\infty \\ i = 1, \dots, N}}$.

We shall suppose that

(5.5) $F(t, u(t)) = f(t, x, f_{01}(A)\, u(t), \dots, f_{0q_0}(A)\, u(t), f_{11}(A)\, u'(t), \dots, f_{1q_1}(A)\, u'(t), \dots$ $\dots, f_{n-1\,1}(A)\, u^{(n-1)}(t), \dots, f_{n-1\,q_{n-1}}(A)\, u^{(n-1)}(t))$, where the functions f_{ij} fulfil the condition (5.1).

We shall need the following lemmas (see [1] Lemmas 3.2.5, 3.2.6).

Lemma 5.1. *There exists a constant* K_1^* *such that if* $u \in \mathscr{D}(A^{1/n})$ *then* $\|u\|_{W_2^{2p/n}(\Omega)} \leqq K_1^* \|A^{1/n} u\|$.

Lemma 5.2. (The Sobolev Embedding Theorem.) *Put* $s^* = (N + 1)/2$ *if* N *is odd,* $s^* = (N + 2)/2$ *if* N *is even. Then there exists a constant* K_2^* *such that for each* $u \in W_2^{s^*}(\Omega)$ *there exists a continuous representant of this element satisfying* $\|u\|_{\mathscr{C}(\bar{\Omega})} = \max_{x \in \bar{\Omega}} |u(x)| \leqq K_2^* \|u\|_{W_2^{s^*}(\Omega)}$.

In the rest of this section, we shall consider only such solutions that $f_{ij}(A)\, u^{(i)}(t)$ $(i = 0, \dots, n-1,\ j = 1, \dots, q_i)$ are continuous for $t \in \mathscr{D}(u)$, $x \in \bar{\Omega}$. We shall call them *continuous representants* (of solutions).

Suppose that

(5.6) there exist continuous **G**-derivatives of the function f with respect to the variables $f_{ij}(A)\, u^{(i)}$ $(i = 0, \dots, n-1,\ j = 1, \dots, q_i)$, up to the second order.

Then

(5.7) $F(t, v(t) + u(t)) = F(t, v(t)) + F_L(t, u(t)) + F_N(t, u(t))$,

where

$$F_L(t, u(t)) = \sum_{i=0}^{n-1} \sum_{j=1}^{q_i} a_{ij}(t, v(t))\, f_{ij}(A)\, u^{(i)}(t),$$

$$F_N(t, u(t)) = \sum_{i=0}^{n-1} \sum_{j=1}^{q_i} \sum_{k=0}^{n-1} \sum_{l=1}^{q_k} r_{ijkl}(t, u(t))\, f_{ij}(A)\, u^{(i)}(t)\, f_{kl}(A)\, u^{(k)}(t),$$

$$a_{ij}(t, \boldsymbol{v}(t)) = \frac{\partial f}{\partial f_{ij}}(t, x, f_{01}(A)\, v(t), \dots, f_{n-1\, q_{n-1}}(A)\, v^{(n-1)}(t)),$$

$$r_{ijkl}(t, \boldsymbol{u}(t)) = \int_0^1 \int_0^1 \tilde{r}_{ijkl}(t, \boldsymbol{v}(t) + \vartheta\sigma\, \boldsymbol{u}(t))\, \sigma\, d\vartheta\, d\sigma\,,$$

$$\tilde{r}_{ijkl}(t, \boldsymbol{u}(t)) = \frac{\partial^2 f}{\partial f_{ij}\, \partial f_{kl}}(t, x, f_{01}(A)\, u(t), \dots, f_{n-1\, q_{n-1}}(A)\, u^{(n-1)}(t))$$

$(i, k = 0, \dots, n-1,\ j = 1, \dots, q_i,\ l = 1, \dots, q_k)$ for $u, v \in \mathscr{U}$ such that $\mathscr{D}(u) \subseteq \mathscr{D}(v)$ and for $t \in \mathscr{D}(u)$. (For the sake of simplicity we have put

$$\boldsymbol{v}(t) = [f_{01}(A)\, v(t), \dots, f_{n-1\, q_{n-1}}(A)\, v^{(n-1)}(t)],$$
$$\boldsymbol{u}(t) = [f_{01}(A)\, u(t), \dots, f_{n-1\, q_{n-1}}(A)\, u^{(n-1)}(t)].)$$

Theorem 5.2. *Let $v : \mathscr{D}(v) = [t_0, +\infty) \to H$ be a maximal solution of the equation* (1.1) *such that* $\sup_{t \in \mathscr{D}(v)} |||v(t)||| < +\infty$. *Put* $\varrho^* = [\sum_{i=0}^{n-1} \sum_{j=1}^{q_i} \sup^2_{t \in \mathscr{D}(v), x \in \bar{\Omega}} |f_{ij}(A)\, v^{(i)}(t, x)|]^{1/2}$. *Let $F \in \mathscr{C}^{(1)}(\mathscr{D}(u_{/\mathscr{D}(v)}), H)$ satisfy the conditions* (5.5), (5.6) *and let the functions $a_{ij}, r_{ijkl}, \tilde{r}_{ijkl}, F_L, F_N$ be defined by* (5.7). *Let the operator A be defined by the relation* (5.4). *Further, let there exist a number $\varrho > \varrho^*$ such that the functions*

$$a_{ij}(t, \boldsymbol{v}(t))\,, \quad \frac{\partial}{\partial t} a_{ij}(t, \boldsymbol{v}(t))\,, \quad \frac{\partial}{\partial f_{\alpha\beta}} a_{ij}(t, \boldsymbol{v}(t))\,, \quad \tilde{r}_{ijkl}(t, \boldsymbol{u}(t))\,,$$

$$\frac{\partial}{\partial t} \tilde{r}_{ijkl}(t, \boldsymbol{u}(t))\,, \quad \frac{\partial}{\partial f_{\alpha\beta}} \tilde{r}_{ijkl}(t, \boldsymbol{u}(t))$$

exist and are continuous and bounded on $\mathscr{D}(v) \times \bar{\Omega} \times K_\varrho$, $i, k, \alpha \in \{0, \dots, n-1\}$, $j = 1, \dots, q_i$, $l = 1, \dots, q_k$, $\beta = 1, \dots, q_\alpha$, where $K_\varrho = \{y \in E_{\boldsymbol{q}} |\ \|y\|_{E_{\boldsymbol{q}}} \leqq \varrho\}$, $\boldsymbol{q} = \sum_{i=0}^{n-1} q_i$. Finally, let $s^* \leqq 2p/n$.

Then $F_L(t, O_{/I}) = 0$, $F'_L(t, O_{/I}) = 0$ for every $I \subseteq \mathscr{D}(v)$, the conditions (2.2), (2.3), (2.4), (3.1), (5.3) *are fulfilled with some constants K_i, C_i $(i = 1, 2, 3, 4)$, $R = R_1 > 0$, $v_1 = v_2 = 1$, $\bar{a}_{ij}$, M_{ij}, N_{ij} $(i = 0, \dots, n-1,\ j = 1, \dots, q_i)$ (in the equation* (5.2), *$a_{ij}(t, x) = a_{ij}(t, \boldsymbol{v}(t, x))$, of course).*

If moreover (2.1), (3.2) *hold, the operator $\mathscr{L}$ is of the type* 0, *the coefficients of the operator $\bar{\mathscr{L}}\, u(t) \equiv \mathscr{L}\, u(t) - \sum_{i=0}^{n-1} \sum_{j=1}^{q_i} \bar{a}_{ij} f_{ij}(A)\, u^{(i)}(t)$ fulfil the condition* (1.1.1) *from* [1] *with a constant $\tilde{C}_0$ instead of C_0^*, the operator $\bar{\mathscr{L}}$ is of the type $\omega < 0$ and*

$$\omega + C(\bar{\mathscr{L}})\, [K^*(1 + (n-1)\, \tilde{C}_0) + n - 1] \sum_{i=0}^{n-1} \sum_{j=1}^{q_i} N_{ij} +$$
$$+ K^*\, C^2(\bar{\mathscr{L}})\, (1 + (n-1)\, \tilde{C}_0) \max^2_{i=0,\dots,n-1} (\sum_{j=1}^{q_i} M_{ij})\, n^{3/2} \delta^{-1/n} < 0\,,$$

then the solution v is uniformly exponentially stable with respect to the norm $|||\cdot|||$.

Proof. By Lemmas 5.1, 5.2

$$(1)\quad \|f_{ij}(A)\,u^{(i)}(t)\|_{\mathscr{C}(\bar{\Omega})} \leqq K_2^*\|f_{ij}(A)\,u^{(i)}(t)\|_{W_2^{s^*}(\Omega)} \leqq$$
$$\leqq K_2^*\|f_{ij}(A)\,u^{(i)}(t)\|_{W_2^{2p/n}(\Omega)} \leqq K_1^*K_2^*\|\,A^{1/n} f_{ij}(A)\,u^{(i)}(t)\| \leqq$$
$$\leqq K_1^*K_2^*F^*\|A^{(n-i)/n}\,u^{(i)}(t)\| \leqq K_1^*K_2^*F^*|||u(t)|||,\ \text{for } i = 0, \ldots, n-1,$$
$$j = 1, \ldots, q_i,\ u \in \mathscr{U},\ t \in \mathscr{D}(u).$$

So for every $u \in \mathscr{U}$ there exists its continuous representant and $\varrho^* < +\infty$. Let us choose numbers $R = R_1 > 0$ so small that

(2) $[f_{01}(A)\,(v(t, x) + \vartheta\sigma u(t, x)), \ldots, f_{n-1\, q_{n-1}}(A)\,(v(t, x) + \vartheta\sigma u(t, x))] \in K_\varrho$ for $\vartheta, \sigma \in [0, 1]$, $u \in \mathscr{U}$ such that $\mathscr{D}(u) \subseteq \mathscr{D}(v)$, $x \in \bar{\Omega}$ and $t \in \mathscr{D}(u)$ such that $|||u(t)||| \leqq R = R_1$.

(This is possible according to (1).)

Let $\bar{a}_{ij}$ be constants $(i = 0, \ldots, n-1,\ j = 1, \ldots, q_i)$. Then in virtue of the boundedness of the functions a_{ij} we get

$$(3)\quad \|a_{ij}(t, v(t))\,f_{ij}(A)\,u^{(i)}(t, x) - \bar{a}_{ij} f_{ij}(A)\,u^{(i)}(t, x)\| \leqq$$
$$\leqq c_1\|f_{ij}(A)\,u^{(i)}(t, x)\| \leqq c_1F^*\|A^{(n-i-1)/n}u^{(i)}(t, x)\|.$$

In this proof c_i will mean suitable constants. Putting $\bar{a}_{ij} = 0$ in (3) we obtain

$$(4)\quad \|F_L(t, u(t))\| = \|\sum_{i=0}^{n-1}\sum_{j=1}^{q_i} a_{ij}(t, v(t))\,f_{ij}(A)\,u^{(i)}(t)\| \leqq$$
$$\leqq \sum_{i=0}^{n-1}\sum_{j=1}^{q_i} c_2\|A^{(n-i-1)/n}\,u^{(i)}(t)\| \leqq c_3\sum_{i=0}^{n-1}\|A^{(n-i-1)/n}\,u^{(i)}(t)\|.$$

Let us restrict ourselves to such $u \in \mathscr{U}$ that $\mathscr{D}(u) \subseteq \mathscr{D}(v)$ and such $t \in \mathscr{D}(u)$ that $|||u(t)||| \leqq R$. Then the boundedness of the functions $\tilde{r}_{ijkl}$ on $\mathscr{D}(v) \times \bar{\Omega} \times K_\varrho$ together with (1), (2), (5.1) yields

$$\|r_{ijkl}(t, u(t))\,f_{ij}(A)\,u^{(i)}(t)\,f_{kl}(A)\,u^{(k)}(t)\| \leqq$$
$$\leqq c_4\|f_{ij}(A)\,u^{(i)}(t)\,f_{kl}(A)\,u^{(k)}(t)\| \leqq c_5|||u(t)|||\,.\,\|f_{kl}(A)\,u^{(i)}(t)\| \leqq$$
$$\leqq c_6|||u(t)|||\,.\,\|A^{(n-i-1)/n}\,u^{(i)}(t)\|\,,\quad i, k \in \{0, \ldots, n-1\}\,,$$
$$j = 1, \ldots, q_i\,,\quad l = 1, \ldots, q_k\,.$$

The last inequality implies

$$(5)\quad \|F_N(t, u(t))\| \leqq \sum_{i=0}^{n-1}\sum_{j=1}^{q_i}\sum_{k=0}^{n-1}\sum_{l=1}^{q_k}\|r_{ijkl}(t, u(t))\,f_{ij}(A)\,u^{(i)}(t)\,f_{kl}(A)\,u^{(k)}(t)\| \leqq$$
$$\leqq c_7|||u(t)|||\sum_{i=0}^{n-1}\|A^{(n-i-1)/n}\,u^{(i)}(t)\|.$$

Using (4), (5) we obtain

$$(6)\quad \|F(t, v(t) + u(t)) - F(t, v(t))\| = \|F_L(t, u(t)) + F_N(t, u(t))\| \leqq$$
$$\leqq (c_3 + c_7 |||u(t)|||) \sum_{i=0}^{n-1} \|A^{(n-i-1)/n} u^{(i)}(t)\| \leqq$$
$$\leqq (c_3 + c_7 R) \sum_{i=0}^{n-1} \|A^{(n-i-1)/n} u^{(i)}(t)\|.$$

As a consequence of (4), (5) and of Lemma 2.5 we get

$$(7)\qquad \|F_L(t, u(t))\| \leqq c_3 n^{1/2} \delta^{-1/n} |||u(t)|||,$$

$$(8)\qquad \|F_N(t, u(t))\| \leqq c_7 n^{1/2} \delta^{-1/n} |||u(t)|||^2.$$

In the sequel we shall suppose that the function u solves the equation $\mathscr{L}\, u(t) = F(t, v(t) + u(t)) - F(t, v(t))$ and either the condition (6) or the conditions (4) and (5) or (7) and (8) are fulfilled, or that the function u solves the equation $\mathscr{L}\, u(t) = \sum_{i=0}^{n-1} \sum_{j=1}^{q_i} a_{ij}(t, x) f_{ij}(A)\, u^{(i)}(t)$ and the condition (3) is fulfilled. (In the last case $a_{ij}(t, x) = a_{ij}(t, v(t, x))$, of course, and we omit the assumption $|||u(t)||| \leqq R$.) Then, because

$$u^{(n)}(t) = F(t, v(t) + u(t)) - F(t, v(t)) - \sum_{i=1}^{n} a_i(A)\, u^{(n-i)}(t)$$

or

$$u^{(n)}(t) = \sum_{i=0}^{n-1} \sum_{j=1}^{q_i} a_{ij}(t, x) f_{ij}(A)\, u^{(i)}(t) - \sum_{i=1}^{n} a_i(A)\, u^{(n-i)}(t),$$

we can derive with help of Lemma 2.5 and of the properties of the functions $a_i(A)$:

(9) There exists a constant c_8 such that $\|u^{(n)}(t)\| \leqq c_8 |||u(t)|||$ for $t \in \mathscr{D}(u)$.

An easy consequence of (9) and (5.1) is the following estimate:

(10) $\|f_{\alpha\beta}(A)\, u^{(\alpha+1)}(t)\| \leqq c_9 |||u(t)|||$, where

$$c_9 = \begin{cases} F^*\,, & \text{if } 1 \leqq \alpha + 1 \leqq n - 1 \\ F^* c_8, & \text{if } \alpha = n - 1. \end{cases}$$

Using the boundedness of the functions $a_{ij} - \bar{a}_{ij}$, $\partial/\partial t\, a_{ij}$, $\partial/\partial f_{\alpha\beta}\, a_{ij}$, the relations (1) and (10), we obtain

$$(11)\quad \left\| \frac{\mathrm{d}}{\mathrm{d}t} \left[a_{ij}(t, v(t)) f_{ij}(A)\, u^{(i)}(t) - \bar{a}_{ij} f_{ij}(A)\, u^{(i)}(t) \right] \right\| =$$

$$= \left\| \left[\frac{\partial}{\partial t} a_{ij}(t, v(t)) + \sum_{\alpha=0}^{n-1} \sum_{\beta=1}^{q_\alpha} \frac{\partial}{\partial f_{\alpha\beta}} a_{ij}(t, v(t)) \,. \right. \right.$$

$$\left. \left. . f_{\alpha\beta}(A) v^{(\alpha+1)}(t) \right] f_{ij}(A) u^{(i)}(t) + (a_{ij}(t, v(t)) - \bar{a}_{ij}) f_{ij}(A) u^{(i+1)}(t) \right\| \leqq$$

$$\leqq (c_{10} + c_{11} |||v(t)|||) |||u(t)||| + c_{12} |||u(t)||| \leqq c_{13} |||u(t)|||.$$

Because $F'_L(t, u(t)) = \sum_{i=0}^{n-1} \sum_{j=1}^{q_i} \frac{\mathrm{d}}{\mathrm{d}t} [a_{ij}(t, v(t)) f_{ij}(A) u^{(i)}(t)]$, we obtain from (11) by putting $\bar{a}_{ij} = 0$

$$\|F'_L(t, u(t))\| \leqq c_{14} |||u(t)||| \,. \tag{12}$$

In virtue of the boundedness of the functions $\tilde{r}_{ijkl}$, $\frac{\partial}{\partial t} \tilde{r}_{ijkl}$, $\frac{\partial}{\partial f_{\alpha\beta}} \tilde{r}_{ijkl}$ we get using (1), (10),

$$\left\| \frac{\mathrm{d}}{\mathrm{d}t} [r_{ijkl}(t, u(t)) f_{ij}(A) u^{(i)}(t) f_{kl}(A) u^{(k)}(t)] \right\| =$$

$$= \left\| \int_0^1 \int_0^1 \left[\frac{\partial}{\partial t} \tilde{r}_{ijkl}(t, v(t) + \vartheta\sigma u(t)) + \right. \right.$$

$$+ \sum_{\alpha=0}^{n-1} \sum_{\beta=1}^{q_\alpha} \frac{\partial}{\partial f_{\alpha\beta}} \tilde{r}_{ijkl}(t, v(t) + \vartheta\sigma u(t)) f_{\alpha\beta}(A) (v^{(\alpha+1)}(t) +$$

$$\left. + \vartheta\sigma u^{(\alpha+1)}(t)) \right] \sigma \,\mathrm{d}\vartheta \,\mathrm{d}\sigma \, f_{ij}(A) u^{(i)}(t) f_{kl}(A) u^{(k)}(t) +$$

$$\left. + r_{ijkl}(t, u(t)) [f_{ij}(A) u^{(i+1)}(t) f_{kl}(A) u^{(k)}(t) + f_{ij}(A) u^{(i)}(t) f_{kl}(A) u^{(k+1)}(t)] \right\| \leqq$$

$$\leqq (c_{15} + c_{16} \sup_{\vartheta,\sigma \in (0,1)} |||v(t) + \vartheta\sigma u(t)|||) |||u(t)|||^2 + c_{17} |||u(t)|||^2 \leqq$$

$$\leqq [c_{15} + c_{16}(\sup_{t \in \mathscr{D}(v)} |||v(t)||| + R) + c_{17}] |||u(t)|||^2 \,.$$

Because

$$F'_N(t, u(t)) = \sum_{i=0}^{n-1} \sum_{j=1}^{q_i} \sum_{k=0}^{n-1} \sum_{l=1}^{q_k} \frac{\mathrm{d}}{\mathrm{d}t} [r_{ijkl}(t, u(t)) f_{ij}(A) u^{(i)}(t) f_{kl}(A) u^{(k)}(t)] \,,$$

the last inequality yields

$$\|F'_N(t, u(t))\| \leqq c_{18} |||u(t)|||^2 \,. \tag{13}$$

The relations (12), (13) give

$$\|F'(t, v(t) + u(t)) - F'(t, v(t))\| = \|F'_L(t, u(t)) + F'_N(t, u(t))\| \leqq \tag{14}$$

$$\leqq (c_{14} + c_{18} R) |||u(t)||| \,.$$

Now obviously the relation (2.3) is fulfilled. Moreover, $F_L(t, O_{/I}) = 0$, $F'_L(t, O_{/I}) = 0$ for every $I \subseteq \mathscr{D}(v)$. The condition (2.2) is fulfilled according to (6), (14). The relations (4), (5), (12), (13) imply (2.4). The relations (7), (8), (12), (13) with help of the relation $F'_L(t, u_1(t)) - F'_L(t, u_2(t)) = F'_L(t, u_1(t) - u_2(t))$ give (3.1). The relation (5.3) is an easy consequence of (3) and (11). The last statement of the theorem directly follows from Theorems 3.2 and 5.1. The theorem is proved.

References

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Author's address: 115 67 Praha 1, Žitná 25, ČSSR (Matematický ústav ČSAV).