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TERTIARY DECOMPOSITION IN GROTHENDIECK CATEGORIES

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0. INTRODUCTION

The aim of this paper is to strengthen some results proved in [1] in the framework of commutative Grothendieck categories by generalizing them to arbitrary Grothendieck categories. The main difference between our set-up and that in [1] is that the categories we consider are a priori locally noetherian, while this restriction is irrelevant in the commutative situation. General properties of Grothendieck categories may be found in [3, 4, 7, 13] but, since this note may be viewed as a sequel to [15], we will briefly recollect some of the main results and definitions stated in loc. cit. Mimicing the commutative situation we will then first be concerned with some generalities on associated primes in Grothendieck categories, which leads us in natural way to tertiary decompositions. Our main result states that in a noetherian Grothendieck category each noetherian object yields for its subobjects tertiary decompositions, which are reduced and essentially unique. This generalizes theorem 2.19 in [1].

1. GENERALITIES

(1.1.) Unless mentioned otherwise $\boldsymbol{C}$ will denote a Grothendieck category and G is a fixed generator for $\boldsymbol{C}$. The representable functor $\mathrm{Hom}_{\boldsymbol{C}}(G, -)$ will be denoted by q_G and R will be the ring $q_G(G)$. Thus q_G is a covariant functor from $\boldsymbol{C}$ to R-mod, the category of left R-modules. The Gabriel-Popescu theorem (cf. [4]) states that q_G has an exact left adjoint T_G possessing some additional properties. In particular, if σ is the idempotent kernel functor associated to Ker T_G, the localizing subcategory of R-mod which consists of all left R-modules M such that $T_G M = o$, and if (R, σ)-mod denotes the corresponding quotient category, then it is well-known that T_G induces an equivalence between $\boldsymbol{C}$ and (R, σ)-mod.

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(1.2.) An object C of $\boldsymbol{C}$ is *noetherian* if the lattice of its subobjects is noetherian, i.e. if every subobject of C is finitely generated. The category $\boldsymbol{C}$ is *locally noetherian* if it has a family of noetherian generators. In the present context it will cause no confusion to call $\boldsymbol{C}$ a *noetherian* category if it has a noetherian generator. If $\boldsymbol{C}$ is noetherian, then we will assume that the generator G, fixed above, is noetherian. In particular, the category R-mod is noetherian (or even locally noetherian) if and only if the ring R is left noetherian. If $\mathscr{R}$ is a noetherian sheaf of rings, then $\mathscr{R}$-mod, the category of sheaves of left $\mathscr{R}$-modules is locally noetherian; it is noetherian if the topological space, on which $\mathscr{R}$ is defined, is finite. In a locally noetherian category every direct sum of injective objects is injective and every injective object is a direct sum of indecomposable injective objects.

(1.3.) The Krull-Remak-Schmidt-Azumaya theorem (cf. [11]) asserts that in an arbitrary Grothendieck category the following holds. If A_i, C_j are a finite number of objects of $\boldsymbol{C}$ such that each A_i has a local endomorphism ring and each C_j is indecomposable, then $A_1 \oplus \ldots \oplus A_m$ and $C_1 \oplus \ldots \oplus C_n$ are isomorphic if and only if $m = n$ and there exists a permutation π of $\{1, \ldots, n\}$ such that A_i and $C_{\pi(i)}$ are isomorphic for each i. Since indecomposable injective objects have local endomorphism rings, it follows easily that an object C of $\boldsymbol{C}$ has finite rank if and only if its injective hull may be decomposed as $E(C) = E_1 \oplus \ldots \oplus E_n$, into a finite number of indecomposable injectives.

(1.4.) An object C is irreducible if it cannot be written as the intersection of two strictly larger objects; a subobject D of an arbitrary object C in $\boldsymbol{C}$ is *coirreducible* if C/D is irreducible. One easily verifies that an injective object E in $\boldsymbol{C}$ is decomposable iff each subobject of E is coirreducible which is exactly the case when E is the injective envelope of a coirreducible object. Furthermore, if C is noetherian, then each subobject D of C has an irreducible decomposition $D = C_1 \cap \ldots \cap C_n$. If this decomposition is irredundant, then $E(C/D) = E(C/C_1) \oplus \ldots \oplus E(C/C_n)$, each $E(C/C_i)$ being indecomposable. It then follows that if $D = C_1 \cap \ldots \cap C_n = C'_1 \cap \ldots \cap C'_m$, two irredundant decompositions, then $m = n$ and for some permutation π of $\{1, \ldots, n\}$ we have that $E(C/C_i)$ and $E(C/C'_{\pi(i)})$ are isomorphic.

(1.5.) For generalities concerning localization in Grothendieck categories the reader is referred to [3, 4, 5, 7, 11, 13]. Let us only recall the following. By $\mathscr{L}(C; \varkappa)$ resp. $\mathscr{K}(C; \varkappa)$ we denote for each object C in $\boldsymbol{C}$ and each idempotent kernel functor in $\boldsymbol{C}$ the class of all subobjects C' of C such that C/C' is $\varkappa$-torsion resp. $\varkappa$-torsion free. One usually calls $\mathscr{L}(G; \varkappa)$ the *idempotent filter* on G associated to $\varkappa$; $\mathscr{K}(C, \varkappa)$ is a complete modular lattice. If $\boldsymbol{C} = R$-mod then we write $\mathscr{L}(\varkappa)$ resp $\mathscr{K}(\varkappa)$ for $\mathscr{L}(R; \varkappa)$ resp $\mathscr{K}(R; \varkappa)$.

With these conventions, an idempotent kernel functor $\varkappa$ in $\boldsymbol{C}$ is said to be G-*noetherian* if its associated filter $\mathscr{L}(G; \varkappa)$ has the following property (1.5.1). If,

$I_1 < I_2 < \dots$ is an ascending chain of subobjects of G whose union lies in $\mathscr{L}(G; \varkappa)$, then there exists an index n such that $I_n \in \mathscr{L}(G; \varkappa)$. One easily checks that a kernel functor $\varkappa$ is G-noetherian if and only if the direct sum of $\varkappa$-closed objects (i.e. faithfully $\varkappa$-injective in the terminology of [6]), is $\varkappa$-closed. This is also equivalent to asserting that $Q_\varkappa$, the *localization* at $\varkappa$, commutes with direct sums. Obviously, the condition of being G-noetherian is independent of G, allowing us to speak of a *noetherian* kernel functor. If $\boldsymbol{C}$ is noetherian, then every idempotent kernel functor is noetherian.

(1.6.) Let us identify $\boldsymbol{C}$ with (R, σ)-mod by the Gabriel-Popescu theorem applied to the fixed generator G. We may view (R, σ)-mod as the full subcategory of R-mod, consisting of all σ-closed R-modules. In this set-up R is σ-closed itself and a generator for (R, σ)-mod, since G is a generator for $\boldsymbol{C}$. If C is a noetherian object in $\boldsymbol{C}$, then $q_G(C)$ is noetherian in (R, σ)-mod. Now recall that a left R-module M is said to be σ-noetherian iff $Q_\sigma(M)$ is noetherian in (R, σ)-mod. This is equivalent to $\mathscr{K}(M, \sigma)$ being a noetherian lattice and also to saying that each non-empty set of submodules of M possesses a σ-maximal element, i.e. an element N of this set such that for each $N' \supset N$ in this set we have $N' \in \mathscr{L}(N; \sigma)$. This allows us to prove that if G is a noetherian generator for $\boldsymbol{C}$, then σ is a noetherian kernel functor in R-mod. Under the same condition, every direct sum of σ-closed left R-modules is σ-closed, i.e. direct sums of objects of $\boldsymbol{C}$ (or, equivalently, of (R, σ)-mod) may be calculated in R-mod. Note that this is always true for intersections of σ-closed left R-modules, i.e. objects in $\boldsymbol{C}$.

(1.7.) Let $\operatorname{Spec}(R, \sigma)$ stand for the set of all σ-closed prime ideals of R and $\operatorname{Spec}_\sigma(R)$ the set of all prime ideals in R, lying in $\mathscr{K}(\sigma)$. Then we have:

(1.8.) Lemma. *If R is σ-closed, then* $\operatorname{Spec}(R, \sigma) = \operatorname{Spec}_\sigma(R)$.

Proof. If $P \in \operatorname{Spec}_\sigma(R)$, then P is σ-closed, hence $P \in \operatorname{Spec}(R, \sigma)$. Indeed, consider the following exact diagram with $I \in \mathscr{L}(\sigma)$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \xrightarrow{i} & R & \xrightarrow{\pi} & R/I & \longrightarrow & 0 \\ & & \varphi\downarrow & & \downarrow\varphi' & & \downarrow\varphi'' & & \\ 0 & \longrightarrow & P & \xrightarrow[j]{} & R & \xrightarrow[\tau]{} & R/P & \longrightarrow & 0 \end{array}$$

Since R is σ-closed, $j\varphi$ factorizes through i, i.e. we may find $\varphi' : R \to R$ such that $\varphi' i = j\varphi$. Since φ' extends φ, clearly $\tau\varphi'$ factorizes through π, i.e. $\tau\varphi' = \varphi''\pi$ for some $\varphi'' : R/I \to R/P$. But $I \in \mathscr{L}(\sigma)$ and $P \in \mathscr{K}(\sigma)$, hence $\varphi'' = o$, implying that φ' factorizes through the kernel of τ, i.e. through P. This proves that P is σ-closed.

Conversely, if $P \in \operatorname{Spec}(R, \sigma)$, then in the exact sequence

$$0 \to P \to R \to R/P \to 0,$$

P is σ-closed and R is σ-torsion free, which is well-known (cf. [6, 13]) to imply that R/P is σ-torsion free. But this amounts to say that $P \in \mathscr{K}(\sigma)$, showing that $P \in \operatorname{Spec}_\sigma(R)$. ■

(1.9.) Lemma. *If R is left σ-noetherian and if $P \in \operatorname{Spec}_\sigma(R)$, then R/P is a left Goldie ring.*

Proof. Since $P \in \operatorname{Spec}_\sigma(R)$, it is clear that $Q_\sigma(R/P)$ is an essential extension of R/P, so, to show that R/P is left Goldie, it suffices to check that $Q_\sigma(R/P)$ is a left Goldie ring. First, if $S \subset Q_\sigma(R/P)$ and $r \in Q_\sigma(\operatorname{Ann} S)$, the annihilator being taken in $Q_\sigma(R/P)$, then for some $L \in \mathscr{L}(\sigma)$ we have $Lr \subset \operatorname{Ann} S$, i.e. $Lrs = o$, This implies $rS \subset \sigma Q_\sigma(R/P) = o$, thus $r \in \operatorname{Ann} S$, proving that $\operatorname{Ann} S = Q_\sigma(\operatorname{Ann} S)$. Since R is left σ-noetherian, we have that $Q_\sigma(R)$ is a noetherian object in (R, σ)-mod and since the sequence

$$0 \to Q_\sigma(P) \to Q_\sigma(R) \to Q_\sigma(R/P) \to 0$$

is exact in (R, σ)-mod, $Q_\sigma(R/P)$ is noetherian too. In view of the foregoing remarks, this shows that $Q_\sigma(R/P)$ satisfies the ascending chain condition on left annihilators. Similarly, if $I_1 \oplus \ldots \oplus I_n \oplus \ldots$ is a direct sum of left $Q_\sigma(R/P)$-ideals, then so is $Q_\sigma(I_1) \oplus \ldots \oplus Q_\sigma(I_n) \oplus \ldots$. Since $Q_\sigma(R/P)$ is noetherian in (R, σ)-mod we may find n_0 such that for all $n \geqq n_0$ we have $Q_\sigma(I_n) = o$. But then I_n is σ-torsion and in view of $P \in \mathscr{K}(\sigma)$, this yields that $I_n = o$ for all $n \geqq n_0$. Thus we have shown that $Q_\sigma(R/P)$ has finite rank, which finishes the proof. ■

(1.10.) Corollary. *Let σ and τ be idempotent kernel functors in* R-mod *such that R is left σ-noetherian, then a prime ideal $P \in \operatorname{Spec}_\sigma(R)$ lies in $\mathscr{K}(\tau)$ if and only if P is not contained in $\mathscr{L}(\tau)$.*

Proof. If $P \in \mathscr{K}(\tau)$, then obviously $P \notin \mathscr{L}(\tau)$. Conversely, if R/P is not τ-torsion free, we may find a left ideal I which strictly contains P and such that I/P is τ-torsion. The foregoing lemma now implies that R/P is a left Goldie ring, hence a left order in a simple ring, implying that R/P satisfies the descending chain condition on left annihilators. Since P is prime, the annihilator of I/P in R/P is zero, yielding a finite set of elements $x_1, \ldots, x_n \in I/P$ such that $\bigcap \operatorname{Ann}(x_i) = \operatorname{Ann}(I/P) = o$. But then

$$R/P \subset (Rx_1/P) \oplus \ldots \oplus (Rx_n/P) \subset (I/P) \oplus \ldots \oplus (I/P),$$

i.e. R/P is τ-torsion and $P \in \mathscr{C}(\tau)$. ■

2. ASSOCIATED PRIMES

(2.1.) Let us recall some ideas introduced in [15]. For any object C of $\mathbf{C}$ and any subset $\Lambda \subset q_G(C)$ we define the annihilator of Λ in G by

$$\mathbf{Ann}_G(\Lambda) = \{\varphi \in q_G(G);\ \forall \alpha \in \Lambda\ \ \alpha\varphi = o\}.$$

In particular we put $\mathbf{Ann}_G(C) = \mathbf{Ann}_G(q_G(C))$ and call in the *external annihilator* of C in G. Similarly the internal annihilator of C in G is defined by

$$\mathrm{Ann}_G(C) = \bigcap\{\mathrm{Ker}\,\alpha;\ \alpha \in q_G(C)\}$$

and one easily checks that $\mathbf{Ann}_G(C) = q_G(\mathrm{Ann}_G(C))$.

(2.2.) The set $\mathbf{Ass}_G(C)$ is defined to consist of all those two-sided ideals P of $R = q_G(G)$ with the property that there is a nonzero R-submodule Γ of $q_G(C)$ such that for each nonzero R-submodule Λ of Γ we have $P = \mathbf{Ann}_G(\Lambda)$. One may verify that for each C in $\mathbf{C}$ we have $\mathbf{Ass}_G(C) \subset \mathrm{Spec}(R)$.

(2.3.) Lemma. *Let σ be an idempotent kernel functor in R-mod such that $\sigma M = o$ for some left R-module M, then* $\mathrm{Ass}_R(M) \subset \mathrm{Spec}_\sigma(R)$.

Proof. Let $P \in \mathrm{Ass}_R(M)$, then for some $m \neq o$ we have $P = \mathrm{Ann}(Rm)$. If $r \in R$ is such that $\bar{r} \in \sigma(R/P)$, then for some $L \in \mathscr{C}(\sigma)$ we find $Lr \subset P = \mathrm{Ann}(Rm)$, hence $LrRm = o$. But then $rRm \subset \sigma M = o$, i.e. $r \in \mathrm{Ann}(Rm) = P$, thus proving that $\bar{r} = \bar{o}$ and $P \in \mathrm{Spec}(R)$. ∎

(2.4.) Corollary. *If R is σ-closed and $\sigma M = o$ then* $\mathrm{Ass}_R(M) \subset \mathrm{Spec}(R, \sigma)$. ∎

(2.5.) If σ is the idempotent kernel functor in R-mod associated to the Gabriel-Popescu-embedding for $\mathbf{C}$, then the foregoing shows that for each object C in $\mathbf{C}$, we have $\mathbf{Ass}_G(C) \subset \mathrm{Spec}(R, \sigma)$. Objects in $\mathrm{Spec}(R, \sigma)$ correspond to objects in $\mathbf{C}$, subobjects of G, which form the *internal prime spectrum* of G, denoted by $\mathrm{Spec}(G, \mathbf{C})$, while to the subset $\mathbf{Ass}_G(C)$ of $\mathrm{Spec}(R, \sigma)$ corresponds a subset $\mathrm{Ass}_G(C)$ of $\mathrm{Spec}(G, \mathbf{C})$, the *internal associated spectrum* of C in G. In order to study the internal decomposition theory in $\mathbf{C}$ in function of $\mathrm{Spec}(G, \mathbf{C})$ and internal associated primes, it thus suffices to study the external situation, i.e. we may work in (R, σ)-mod, where R is noetherian in (R, σ)-mod, or again: where R is σ-noetherian and σ-closed.

(2.6.) Lemma. *Let σ be an idempotent kernel functor in* R-mod *such that R is left σ-noetherian and torsion free, then for any left R-module M we have*

$$\mathrm{Ass}_\sigma(M/\sigma M) = \mathrm{Ass}_\sigma(Q_\sigma(M)) = \mathrm{Ass}_R(Q_\sigma(M)) = \mathrm{Ass}_R(M/\sigma M) \neq \emptyset,$$

where $\mathrm{Ass}_\sigma(N) = \mathrm{Ass}_R(N) \cap \mathscr{K}(\sigma)$, *for each* $N \in R$-mod.

Proof. We may clearly assume M to be σ-torsion free. Consider the following set

$$\Omega = \{\mathrm{Ann}(N);\ o \neq N < M\}.$$

We claim that Ω possesses a maximal element. If not, we may find a strictly ascending chain

$$\mathrm{Ann}(N_1) \subsetneq \mathrm{Ann}(N_2) \subsetneq \dots \subsetneq \mathrm{Ann}(N_p) \subsetneq \dots$$

where the N_i are nonzero submodules of M. Since R is σ-torsion free, this yields a strictly ascending chain

$$Q_\sigma(\mathrm{Ann}(N_1)) \subsetneqq Q_\sigma(\mathrm{Ann}(N_2)) \subsetneqq \dots \subsetneqq Q_\sigma(\mathrm{Ann}(N_p)) \subsetneqq \dots$$

of σ-closed left ideals of $Q_\sigma(R)$. Indeed, if $\mathrm{Ann}(N') \subsetneqq \mathrm{Ann}(N'')$ then $Q_\sigma(\mathrm{Ann}(N')) \subsetneqq$ $\subsetneqq Q_\sigma(\mathrm{Ann}(N''))$ for any two R-submodules of M, for if the converse holds, then for any $r \in \mathrm{Ann}(N'') \subset Q_\sigma(\mathrm{Ann}(N'')) = Q_\sigma(\mathrm{Ann}(N'))$, we may find $L \in \mathscr{L}(\sigma)$ such that $Lr \subset \mathrm{Ann}(N')$, i.e. $LrN' = o$. This yields $rN'' \subset \sigma M = o$, i.e. $r \in \mathrm{Ann}(N')$ contradicting $\mathrm{Ann}(N') \neq \mathrm{Ann}(N'')$. Thus the chain of left ideals defined above is strictly ascending, contradicting the fact that R is σ-noetherian. This proves that Ω possesses a maximal element P, and one may now proceed as in the classical situation to show that $P \in \mathrm{Ass}_R(M)$, hence $\mathrm{Ass}_R(M) \neq \emptyset$. It has been pointed out in (2.3) that under the above conditions $\mathrm{Ass}_R(M) \subset \mathrm{Spec}_\sigma(R)$, i.e. $\mathrm{Ass}_R(M) = \mathrm{Ass}_\sigma(M)$ and $\mathrm{Ass}_R(Q_\sigma(M)) = \mathrm{Ass}_\sigma(Q_\sigma(M))$. To finish the proof, it thus suffices to prove that $\mathrm{Ass}_R(M) = \mathrm{Ass}_R(Q_\sigma(M))$, for any M which is assumed to be σ-torsion free. The inclusion $\mathrm{Ass}_R(M) \subset \mathrm{Ass}_R(Q_\sigma(M))$ being obvious, let us check the converse inclusion as follows. Take $P \in \mathrm{Ass}_R(Q_\sigma(M))$, then we may find $o \neq M_1 < Q_\sigma(M)$ such that for all $o \neq N_1 < M_1$ we have $P = \mathrm{Ann}(N_1)$. In particular, if we take $M_1' = M_1 \cap M \neq o$, then for all $o \neq N_1' < M_1'$, we find $P = \mathrm{Ann}(N_1')$, showing that $P \in \mathrm{Ass}_R(M)$. ■

(2.7.) Corollary. *If G is a noetherian generator for* **C**, *then for each C in* **C** *we have* $\mathrm{Ass}_G(C) \neq \emptyset$. ■

(2.8.) The elementary properties of the associated spectrum may be given as follows, cf. [15]:

(2.8.1.) if $0 \to M' \to M \to M'' \to 0$ is an exact sequence in (R, σ)-mod then $\mathrm{Ass}_\sigma(M') \subset \mathrm{Ass}(M) \subset \mathrm{Ass}_\sigma(M') \cup \mathrm{Ass}_\sigma(M'')$.

(2.8.2.) $\mathrm{Ass}(\oplus M_i) = \mathrm{Ass}(Q_\sigma(\oplus M_i)) = \mathrm{Ass}(\oplus Q_\sigma(M_i)) = \bigcup \mathrm{Ass}(M_i) =$ $= \bigcup \mathrm{Ass}(Q_\sigma(M_i))$ for each family of σ-torsion free left R-modules.

(2.8.3.) if M is σ-torsion free and if $E(-)$ resp $E^\sigma(-)$ denote injective hulls in R-mod resp. (R, σ)-mod, then $\mathrm{Ass}(M) = \mathrm{Ass}(E(M)) = \mathrm{Ass}(E^\sigma(Q_\sigma(M))) =$ $= \mathrm{Ass}(E(Q_\sigma(M)))$.

(2.8.4.) if M is a nonzero coirreducible σ-closed R-module, then $\mathrm{Ass}(M) =$ $= \mathrm{Ass}_\sigma(M)$ consists of exactly one member, which will then be denoted by $\mathrm{ass}(M)$.

(2.8.5.) for the foregoing follows easily that for any σ-closed R-module M of finite rank in (R, σ)-mod, the associated spectrum $\mathrm{Ass}(M)$ is finite.

(3.1.) Recall that in a lattice K an element $x \in K$ is said to be *irreducible* if $x = y \wedge z$ implies $x = y$ or $x = z$. In a noetherian lattice each element is a finite infimum of irreducible elements. In particular, we may apply this to $\mathscr{K}(M, \sigma)$, if M is a σ-noetherian left R-module. Let us note the following: if N is an irreducible element of the lattice $\mathscr{K}(M, \sigma)$, then N is an irreducible submodule of M. Indeed, if for each submodule T of M we denote by $\tilde{T}$ the unique submodule of M containing T such that $\tilde{T}/T = \sigma(M/T)$, then for any decomposition $N = P \cap Q$ we find $N = \tilde{N} = (P \cap Q)^{\sim} = \tilde{P} \cap \tilde{Q}$, and since $\tilde{P}, \tilde{Q} \in \mathscr{K}(M, \sigma)$, this yields $N = \tilde{P}$ or $N = \tilde{Q}$, hence $N = P$ or $N = Q$, yielding the assertion. For more details, cf. [1].

(3.2.) Let us call a submodule N of M *tertiary* if Ass (M/N) consists of precisely one member. If Ass $(M/N) = \{P\}$, then we call N a *P-tertiary* submodule of M and we write $P =$ ass (M/N). This definition is in accordance with the ideas introduced in [14, 15]. In particular, for two σ-closed left R-modules $N \subset M$ we know that N is P-tertiary in M in the sense that $\text{Ass}_\sigma (Q_\sigma(M/N)) = \{P\}$, i.e. in (R, σ)-mod if and only if N is P-tertiary in M, as just defined, i.e. in R-mod.

(3.3.) Lemma. *If R is σ-noetherian and σ-torsion free and if N is irreducible in $\mathscr{K}(M, \sigma)$, then N is a tertiary submodule of M.*

Proof. Since N is irreducible in $\mathscr{K}(M, \sigma)$ the foregoing remarks show that N is irreducible in M, hence that M/N is coirreducible. This implies that any nonzero $L_1, L_2 \subset M/N$ have nonzero intersection, hence that $\Omega = \{\text{Ann}(L); o \neq L < M/N\}$ has at most one maximal member, in view of $\text{Ann}(L_1) + \text{Ann}(L_2) \subset \text{Ann}(L_1 \cap L_2)$. But since Ass (M/N) consists exactly of the maximal members of Ω and is nonempty, as we have seen in (2.6), this shows that N is tertiary in M. ∎

(3.4.) Proposition. *Let M be a σ-noetherian left R-module, where σ is a noetherian idempotent kernel functor such that R is σ-torsion free, then every $N \in \mathscr{K}(M, \sigma)$ has an essentially unique reduced tertiary decomposition*

$$N = Q_1 \cap \ldots \cap Q_n$$

with $Q_i \in \mathscr{K}(M, \sigma)$.

Proof. Since M is σ-noetherian, the lattice $\mathscr{K}(M, \sigma)$ is noetherian, hence N possesses an essentially unique reduced decomposition in irreducible elements of $\mathscr{K}(M, \sigma)$

$$N = Q_1 \cap \ldots \cap Q_n .$$

The foregoing lemma yields that these Q_i, being irreducible in $\mathscr{K}(M, \sigma)$ are tertiary submodules of M, proving the assertion. ∎

(3.5.) It is well-known that under the above conditions, we have $\mathrm{Ass}(M/N) = \bigcup \mathrm{Ass}(M/Q_i)$ and $\mathrm{Ass}(Q_i/N) = \mathrm{Ass}(M/N) - \mathrm{Ass}(M/Q_i)$ for each $1 \leqq i \leqq n$ exactly as in the usual situation. Moreover, the foregoing proposition also yields.

(3.5.1.) Let M be a σ-noetherian left R-module, where σ is a noetherian idempotent kernel functor such that R is σ-torsion free, then for every left R-submodule N of M we may find a finite family of tertiary submodules of M, say $\{Q_i\}$, with $Q_i \in \mathscr{K}(M, \sigma)$ such that $\mathrm{Ass}_\sigma(M/N) = \bigcup \mathrm{Ass}(M/Q_i)$ and if we denote $\bigcup Q_i$ by N_1, then $N \subset N_1$ and $N \in \mathscr{L}(N_1, \sigma)$.

It suffices to take $N_1 = \tilde{N}$ and to apply the foregoing proposition.

(3.6.) A subobject C' of an object C of $\boldsymbol{C}$ is called *tertiary* if $\mathbf{Ass}_G(C)$ or equivalently, if $\mathrm{Ass}_G(C)$ contains exactly one member, which will then be denoted by $\mathbf{ass}_G(C)$ resp. $\mathrm{ass}_G(C)$. In this case we will also say that C' is *p-tertiary* resp. *P-tertiary*, where $p = \mathrm{ass}_G(C)$ resp. $P = \mathbf{ass}_G(C)$. Similarly, if $P \in \mathrm{Spec}_\sigma(R)$ and $M \in (R, \sigma)$-mod, then a submodule N of M is said to be *P-tertiary* if $\mathrm{Ass}(Q_\sigma(M/M')) = \{P\}$. *Tertiary decomposition* with respect to these notions is defined as usually.

(3.7.) Theorem. *Let C be a noetherian object in the noetherian category $\boldsymbol{C}$, then each subobject C' of C possesses an essentially unique, reduced tertiary decomposition.*

Proof. We may as well work in (R, σ)-mod, with R closed for the noetherian idempotent kernel functor σ. The objects C and C' then correspond to σ-closed left R-modules M and N. Let us call $i : (R, \sigma)\text{-mod} \to R\text{-mod}$ the canonical inclusion. Since we now have to distinguish between quotients in R-mod and (R, σ)-mod, we should note that $i(M/N) = iQ_\sigma(iM/iN)$. Since $N \subset M$ clearly $iN \subset iM$, hence $iM/iN \subset i(M/N)$, implying that $iN \in \mathscr{K}(iM, \sigma)$. By the foregoing this yields that iN possesses a tertiary decomposition

$$iN = Q_1 \cap \ldots \cap Q_n$$

with $Q_i \in \mathscr{K}(iM, \sigma)$ and $\mathrm{Ass}_R(iM/Q_i) = \{P_i\}$. Thus

$$N = Q_\sigma(Q_1) \cap \ldots \cap Q_\sigma(Q_n)$$

is a decomposition for N. To show that it is a tertiary decomposition, it suffices to prove that $\mathrm{Ass}_\sigma(M/Q_\sigma(Q_i))$ contains only one member. Now, since $Q_i \in \mathscr{K}(iM, \sigma)$, clearly iM/Q_i is σ-torsion free, hence we may apply (2.6) to derive that $\mathrm{Ass}_R(iM/Q_i) = \mathrm{Ass}_\sigma(Q_\sigma(iM/Q_i)) = \mathrm{Ass}_\sigma(M/Q_\sigma(Q_i))$, proving the assertion. Finally, let us check that the decomposition is reduced, unicity being obvious. If $N = Q_\sigma(Q_2) \cap \ldots \cap Q_\sigma(Q_n)$, then $iN = Q_2 \cap \ldots \cap Q_n$. Indeed, since $Q_i \in \mathscr{K}(iM, \sigma)$ and $M \in (R, \sigma)$-mod, it is easy to see that Q_i is σ-closed too. But then the new decomposition of iN contradicts the fact that the initial tertiary decomposition was reduced. This finishes the proof. ∎

(3.8.) Corollary. *If $C' = C_1 \cap \ldots \cap C_n$ is reduced tertiary decomposition of C' in C then* $\mathrm{Ass}_G(C/C') = \bigcup \mathrm{Ass}_G(C/C_i)$.

Proof. Apply (3.5) and the foregoing. ∎

4. PRIMARY DECOMPOSITION

(4.1.) To each injective object E of $\boldsymbol{C}$ we may associate an idempotent kernel functor $\varkappa_E$ which is maximal in the set of all idempotent kernel functors $\varkappa$ in $\boldsymbol{C}$ such that E is $\varkappa$-torsion free. More precisely, $\varkappa_E$ is defined by putting for each C in $\boldsymbol{C}$

$$\varkappa_E(C) = \bigcap \{\mathrm{Ker}(f);\ f \in \mathrm{Hom}_{\boldsymbol{C}}(C, E)\}\ .$$

One easily shows that each idempotent kernel functor in $\boldsymbol{C}$ is of the form $\varkappa_E$ for some injective object E in $\boldsymbol{C}$. In particular, a *prime kernel* functor in $\boldsymbol{C}$ is a kernel functor $\varkappa_E$ with $E = E(S)$, the injective hull of a support for κ_E. Recall that a *support* for a kernel functor μ is an object S such that $\mu S = o$ and $\mu(S/S') = S/S'$ for each nonzero subobject S' of S. In [15] we have proved that every prime kernel functor μ has an essentially unique support which is μ-injective.

(4.2.) A subobject I of G is said to be *critical* if G/I is a support for $\varkappa_{E(G/I)}$. One may prove that this amounts to say that for some idempotent kernel functor $\varkappa$ in $\boldsymbol{C}$ we have that G/I is $\varkappa$-torsion free and I is maximal as such. In a locally noetherian category $\boldsymbol{C}$ in which I is a critical subobject of G we know that $E(G/I)$ is an indecomposable injective object. The following result is an easy consequence of properties mentioned in [15].

(4.3.) Proposition. *In a locally noetherian Grothendieck category $\boldsymbol{C}$ with noetherian generator G the set of all prime kernel functors $\mathscr{P}(\boldsymbol{C})$ and the set of isomorphism classes of indecomposable injectives $\mathscr{E}(\boldsymbol{C})$ are isomorphic.*

Proof. We will mimic a similar result proved for modules in [15]. If E is an indecomposable injective object in $\boldsymbol{C}$, then we will denote by $\hat{E}$ its isomorphism class. Define the following map:

$$\Delta : \mathscr{R}(\boldsymbol{C}) \to \mathscr{E}(\boldsymbol{C}) : \mu \mapsto E(S\mu)^\wedge$$

where $S\mu$ is a support for μ. We will show that it is bijective. Let E be an indecomposable injective object in $\boldsymbol{C}$. Since G is a generator for $\boldsymbol{C}$, we may find a nonzero morphism $\varphi : G \to E$. Let $I_1 = \mathrm{Ker}\ \varphi$, then G/I_1 injects into E, and since E is indecomposable, this implies I_1 to be irreducible or G/I_1 to be coirreducible. Moreover $\hat{E} = E(G/I_1)^\wedge$, by (1.4). Now, if $\mathrm{Hom}_{\boldsymbol{C}}(G/L_1, E) \neq o$ for some $I_1 \subset L_1$, then we may find $\varphi : G/L_1 \to E$, $\varphi \neq o$. If $\mathrm{Ker}\ \varphi = I_2/L_1$, then $I_1 \subset L_1 \subset I_2$, and $G/I_2 = (G/L_1)/(I_2/L_1)$ injects into E, i.e. $E(G/I_1)^\wedge = E(G/I_2)^\wedge$. Thus we may find an ascending chain $I_1 \subset I_2 \subset \dots$ of strict subobjects of G, such that $E(G/I_1)^\wedge = E(G/I_2)^\wedge = \dots = \hat{E}$. Since G is noetherian this chain has a maximal element, say I, such that $\hat{E} = \hat{E}(G/I)$. If we take $L \supset I$ such that $\mathrm{Hom}_{\boldsymbol{C}}(G/L, E(G(I)) \neq o$, then, as above, we may find $o \neq \varphi : G/L \to E(G/I)$, and if $\mathrm{Ker}\, \varphi = K/L$ with $K \supsetneqq L$, then $K \supsetneqq I$ and G/K injects into $E(G/I)$. But since then $E(G/K)^\wedge = E(G/I)^\wedge = E$, this contradicts the maximality of I. Let $\mu = \varkappa_E$, then this shows that G/I is $\varkappa$-torsion free and I is maximal as such, i.e. I is μ-critical and μ is prime. Let S be a support for μ, then $E \in E(S)^\wedge$, proving the surjectivity of Δ.

On the other hand, if μ and μ' are prime kernel functors with supports S and S', and if $E(S)^\wedge = E(S')^\wedge$, then $\varkappa_{E(S)} = \varkappa_{E(S')}$, i.e. $\mu = \mu'$, proving the injectivity of Δ. This finishes the proof. ■

(4.4.) It follows that each prime kernel functor in $\boldsymbol{C}$ is of the form $\mu = \varkappa_E$ for some indecomposable injective object of $\boldsymbol{C}$ which is essentially uniquely determined by μ. For an arbitrary object C in $\boldsymbol{C}$, with injective hull $E = E(C)$, we will write as usually $\varkappa_C = \varkappa_E$. This allows us to define an object C to be *Goldman-primary*, if $\varkappa_C$ is prime and C is stable, i.e. for each $o \neq C' < C$ we have $\varkappa_{C'} = \varkappa_C$. Mimicing the proof of a similar result in [10] it is now easy to prove the following.

(4.5.) Proposition. *A finitely generated object C in a locally noetherian category $\boldsymbol{C}$ is Goldman-primary if and only if its injective hull is isotypic.* ■

(4.6.) If the injective object E in the locally noetherian category $\boldsymbol{C}$ splits as $E = E_1 \oplus \dots \oplus E_n$ into a finite direct sum of undecomposable injectives, then if we write μ_i for $\varkappa_{E_i}$, this yields a decomposition

$$\varkappa_E = \mu_1 \wedge \dots \wedge \mu_n$$

of $\varkappa_E$ into prime kernel functors. For $E = E(C)$, this results into Goldmans primary decomposition, cf. [6] for a precise outline of the module case. In order to link primary and tertiary decompositions, one has to consider the map $\Phi_G : \mathscr{E}(\boldsymbol{C}) \to \mathrm{Spec}\,(G, \boldsymbol{C})$, which to each class of indecomposable injectives represented by E associates its associated internal prime $\mathrm{ass}_G\,(E)$. It is well known in the noetherian case Φ_G is surjective.

(4.7.) Theorem. *Let G be a noetherian generator of the Grothendieck category $\boldsymbol{C}$, then the following properties of G are equivalent*:

(4.7.1.) *the map $\Phi_G : \mathscr{E}(\boldsymbol{C}) \to \mathrm{Spec}\,(G, \boldsymbol{C})$ is bijective;*

(4.7.2.) *every G-tertiary object in $\boldsymbol{C}$ is isotypic;*

(4.7.3.) *for every $p \in \mathrm{Spec}\,(G, \boldsymbol{C})$ and every essential subobject I of G/p there exists a nonzero subobject I' of I such that $q_G(I')$ is a twosided $q_G(G)$-module.*

Proof. cf. [15]. ■

(4.8.) A generator G satisfying one of the above properties is said to be *fully bounded*. Similarly, $\boldsymbol{C}$ is called a *fully bounded category* if it possesses a fully bounded generator. Properties of fully bounded Grothendieck categories have been studied in [14, 15]. Clearly, if G is fully bounded, then (4.7.2) implies that every finitely generated G-tertiary object is Goldman primary, in view of (4.5). More generally we may prove:

(4.9.) Theorem. *For any noetherian generator G of the Grothendieck category $\boldsymbol{C}$, the following statements are equivalent*:
(4.9.1.) *G is fully bounded*;
(4.9.2.) *every finitely generated G-tertiary object C is Goldman primary.*

Proof. (1) $\Rightarrow$ (2) has just been noted, so, let us prove (2) $\Rightarrow$ (1), by showing that the map Φ_G defined above is bijective. If not, we may find two nonisomorphic indecomposable injectives E_1, E_2 in $\boldsymbol{C}$ such that

$$\operatorname{ass}_G E_1 = \operatorname{ass}_G E_2 = p \in \operatorname{Spec}(G, \boldsymbol{C}) .$$

As in the proof of (4.3), we may exhibit critical subobjects I_i of G such that $E_i = E(G/I_i)$. But then I_1 and I_2 are tertiary subobjects of G with the same associated internal prime p, hence $I_1 \cap I_2$ is p-tertiary in G too. But then the assumption says that $G/I_1 \cap I_2$ is Goldman primary G being noetherian in $\boldsymbol{C}$. Now, (4.5) states that $E(G/I_1 \cap I_2)$ should be isotypic, while on the other hand

$$E(G/I_1 \cap I_2) \cong E(G/I_1) \oplus E(G/I_2) \cong E_1 \oplus E_2 .$$

This contradiction proves the result. ■

(4.10.) This result shows that for fully bounded Grothendieck categories tertiary and Goldman primary decompositions are essentially identical. Using this, one easily generalizes the results of [10] to arbitrary fully bounded categories, while most properties of Loewy objects stated in [1] for commutative Groethendieck categories also hold in this situation. Details are left to the reader.

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NEWS AND NOTICES

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