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THE BICHROMATICITY OF A GRAPH

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F. HARARY, D. HSU and Z. MILLER in [1] define the bichromaticity of a bipartite graph as follows. Let B be a connected finite bipartite graph on the vertex sets U, V. (This means that the vertex set of B is the union of disjoint sets U, V and each edge of B joins a vertex of U with a vertex of V.) A bicomplete homomorphism of the graph B onto a complete bigraph $K_{r,s}$ is such a homomorphism ψ of B onto $K_{r,s}$ that for any two vertices x, y of B the equality $\psi(x) = \psi(y)$ implies that x and y are either both in U or both in V. The bichromaticity $\beta(B)$ of the graph B is the maximal number of vertices of a complete bipartite graph onto which B can be mapped by a bicomplete homomorphism.

Following [1], a bipartite graph will be shortly called a bigraph. If B is a bigraph on the sets U, V and $|U| \geqq |V|$, then U is called the majority of B and its cardinality is denoted by $\mu(B)$.

In [1] a problem was proposed to find $\beta(B \times K_2)$ in terms of $\beta(B)$. The authors noted that an exact formula determining $\beta(B \times K_2)$ as a function of $\beta(B)$ would yield a formula for $\beta(Q_n)$, where Q_n is the graph of the cube of dimension n. We shall show that $\beta(B \times K_2)$ is not uniquely determined by $\beta(B)$ and we shall give bounds for $\beta(B \times K_2)$ in terms of $\beta(B)$. First we prove a theorem which will serve us as a lemma.

Theorem 1. *Let B be a connected finite bigraph on the sets U, V. Let there exist a vertex $x \in U$ which is adjacent to all vertices of V and let the bigraph B_0 obtained from B by deleting u be connected. Then*

$$\beta(B) = \beta(B_0) + 1 .$$

Proof. Let B_0 be mapped by a bicomplete homomorphism ψ_0 onto the complete bigraph $K_{r,s}$ such that $r + s = \beta(B_0)$. Let $U - \{x\}$ (or V) be mapped by this homomorphism onto a set U_0 (or V_0) of the cardinality r (or s, respectively). Consider the graph $K_{r+1,s}$ obtained from $K_{r,s}$ by adding a vertex y and joining it by edges with all vertices of V_0. Then we define the mapping ψ of B onto $K_{r+1,s}$ such that $\psi(v) = \psi_0(v)$ for each $v \neq x$ and $\psi(x) = y$. This is evidently a bicomplete homomorphism of B onto $K_{r+1,s}$ and $\beta(B) \geqq \beta(B_0) + 1$. Now suppose $\beta(B) \geqq \beta(B_0) + 2$. Then there exists a bicomplete homomorphism ψ' of B onto a bigraph $K_{p,q}$ such that $p + q \geqq r + s + 2$. Let U (or V) be mapped by ψ' onto a set of the cardinality p (or q, respectively). By deleting $\psi'(x)$ from $K_{p,q}$ we obtain the graph $K_{p-1,q}$. The mapping ψ' maps an induced subgraph of B_0 onto this graph $K_{p-1,q}$, therefore also the whole graph B_0 can be mapped by a bicomplete homomorphism onto $K_{p-1,q}$. But $p - 1 + q \geqq r + s + 1$, which is a contradiction with the assumption that $\beta(B_0) = r + s$. Therefore the assertion of the theorem is true.

Now we prove a theorem on $\beta(B \times K_2)$.

Theorem 2. *Let B be a connected finite bigraph which can be mapped by a bicomplete homomorphism onto the complete bigraph $K_{r,s}$, where $r \leqq s$. Then $B \times K_2$ can be mapped by a bicomplete homomorphism onto the complete bigraph $K_{r,r+s}$. There exist bigraphs B with the property that B can be mapped by a bicomplete homomorphism onto $K_{r,s}$, where $r \leqq s$ and $\beta(B \times K_2) = 2r + s$.*

Proof. Let B be a bigraph on the sets U, V. The graph $B \times K_2$ can be described as follows: Take two copies B' and B'' of the graph B and an isomorphic mapping φ of B' onto B'' and join each vertex x of B' with its image $\varphi(x)$ by an edge. The sets corresponding to U, V in B' will be denoted by U', V' and in B'' by U'', V''. Without loss of generality suppose that in the bicomplete homomorphism ψ which maps B' onto $K_{r,s}$ the set U is mapped onto a set U_0 of the cardinality r and V is mapped onto a set V_0 of the cardinality s. Let K', K'' be two copies of $K_{r,s}$. We map B' onto K' and B'' onto K'' by a bicomplete homomorphism corresponding to ψ. The images of U', U'', V', V'' will be consequently U'_0, U''_0, V'_0, V''_0; we have $|U'_0| = |U''_0| = r$, $|V'_0| = |V''_0| = s$. Now we choose a surjection $\varphi : V''_0 \to U'_0$ and we identify each $x \in U''_0$ with its image $\varphi(x)$. Thus we obtain a complete bigraph $K_{r,r+s}$ onto which $B \times K_2$ is mapped.

Now let B be a complete bigraph $K_{2,s}$, where $s > 2$. We shall prove that $\beta(K_{2,s} \times K_2) = s + 4$. We use the notation introduced above; we have $|U| = |U'| = |U''| = 2$, $|V| = |V'| = |V''| = s$. Let $K_{p,q}$ be a complete bigraph onto which $K_{2,s} \times K_2$ can be mapped by a bicomplete homomorphism and such that $p + q = \beta(K_{2,s} \times K_2)$. If each vertex $x \in U' \cup U''$ has the property that $\psi(x) \neq \psi(y)$ for each $y \neq x$, where ψ is the bicomplete homomorphism of $K_{2,s} \times K_2$ onto $K_{p,q}$, then neither the subgraph of $K_{p,q}$ induced by $\psi(U' \cup U'')$ nor $K_{p,q}$ are complete bigraphs which is a contradiction. Therefore there exists at least one vertex $x \in U' \cup U''$ to which a vertex y exists such that $x \neq y$ and $\psi(x) = \psi(y)$. Without loss of generality let $x \in U'$. Then $y \in U' \cup V''$. Let G_0 be the graph obtained from $K_{2,s}$ by identifying x and y; the graph G_0 can be mapped by a bichromatic homomorphism onto $K_{p,q}$ and $\beta(G_0) = \beta(K_{2,s})$. If $y \in U'$, then the vertex obtained by identifying x

and y is adjacent to all vertices of $U'' \cup V'$ in G_0. Let G_1 be the bigraph obtained from G_0 by deleting this vertex; by Theorem 1 we have $\beta(G_1) = \beta(G_0) - 1$. Let $z \in U''$. The graph G_1 is a bigraph on the set V'', $U'' \cup V'$ and z is adjacent to all vertices of V'' in G_1. Let G_2 be the bigraph obtained from G_1 by deleting z; we have $\beta(G_2) = \beta(G_1) - 1 = \beta(G_0) - 2$. But G_2 is a tree with the majority $(U'' - \{z\}) \cup V'$ of the cardinality $\mu(G_2) = s + 1$, therefore by Theorem 1 from [1] we have $\beta(G_2) = s + 2$ and this implies $\beta(G) = \beta(G_0) = s + 4$. Now suppose that the images of vertices of U' in ψ are different. If the images of vertices of U'' are equal, then we proceed analogously as in the preceding case. Therefore suppose also that the images of vertices of U'' in ψ are different. Let $U' = \{u'_1, u'_2\}$, $U'' = \{u''_1, u''_2\}$ and let u'_1 be adjacent with u''_1 and u'_2 with u''_2 in $K_{2,s} \times K_2$. Then one of the following four cases must occur:

(a) There exist vertices y_1, y_2 of V'' such that $y_1 \neq y_2$ and $\psi(u'_1) = \psi(y_1)$, $\psi(u'_2) = \psi(y_2)$.

(a′) There exist vertices z_1, z_2 of V' such that $z_1 \neq z_2$ and $\psi(u''_1) = \psi(z_1)$, $\psi(u''_2) = \psi(z_2)$.

(b) There exist vertices $y_1 \in V''$, $z_2 \in V'$ such that $\psi(u'_1) = \psi(y_1)$, $\psi(u''_2) = \psi(z_2)$.

(b′) There exist vertices $y_2 \in V''$, $z_2 \in V'$ such that $\psi(u'_2) = \psi(y_2)$, $\psi(u''_2) = \psi(z_2)$.

If the case (a) occurs, let G_0 be the graph obtained from $K_{2,s} \times K_2$ by identifying u'_1 with y_1 and u'_2 with y_2. We have then $\beta(G_0) = p + q$. The graph G_0 is a bigraph on the sets V'', $U'' \cup V'$ and the vertices u''_1, u''_2 are adjacent with all vertices of V'' in G_0. Let G_1 be the graph obtained from G_0 by deleting u''_1 and u''_2; by Theorem 1 we have $\beta(G_1) = \beta(G_0) - 2$. Each vertex $V'' - \{y_1, y_2\}$ has degree 1 in G_1 and the subgraph of G_1 induced by $\{y_1, y_2\} \cup U'' \cup V'$ is isomorphic to $K_{2,s}$. Evidently, if G_1 is mapped onto a complete bigraph by a bicomplete homomorphism, then either all vertices of $U'' \cup V'$ are mapped onto the same vertex, or each vertex of $V'' - \{y_1, y_2\}$ is mapped onto the same vertex as u'_1 or u'_2. In the first case the mentioned complete bigraph is $K_{1,s}$, in the second case $K_{2,s}$. Therefore $\beta(G_1) = s + 2$ and $\beta(G_0) = \beta(K_{2,s} \times K_2) = s + 4$. The case (a′) is analogous. Let the case (b) occur and let G_0 be the bigraph obtained from $K_{2,s} \times K_2$ by identifying u'_1 with y_1 and u''_1 with z_1. The vertices u'_2, u''_2 fulfil the condition of Theorem 1; let G_1 be the graph obtained from G_0 by deleting them. We have $\beta(G_1) = \beta(G_0) - 2$. The graph G_1 is a tree with the majority of the cardinality s, therefore $\beta(G_1) = s + 1$ and $\beta(K_{2,s} \times K_2) = \beta(G_0) = s + 3$. This is impossible, since by the above proved results $\beta(K_{2,s} \times K_2) \geqq s + 4$. Analogously in the case (b′). Therefore one of the cases (a), (a′) occurs and $\beta(K_{2,s} \times K_2) = s + 4$. As $r = 2$, this is $2r + s$.

Theorem 3. *Let B be a connected finite bigraph on the sets U, V which can be mapped by a bicomplete homomorphism onto a complete bigraph $K_{r,s}$, where $r \leqq s$. Let there exist an automorphism of B which maps U onto V and V onto U. Then $B \times K_2$ can be mapped by a bicomplete homomorphism onto a complete bigraph $K_{r,2s}$.*

Proof. We use the same notation as in the proof of Theorem 2. Note that if U can be mapped onto U_0 and V onto V_0 by a bicomplete homomorphism, then also U can be mapped onto V_0 and V onto U_0 (superposing the original homomorphism with the mentioned automorphism). Then also the images of U', U'', V', V'' can be consequently U_0', V_0'', V_0', U_0''. By an analogous procedure as in the proof of Theorem 2 we obtain a complete bigraph $K_{r,2s}$.

Now we can give bounds for $\beta(B \times K_2)$.

Theorem 4. *Let B be a connected finite bigraph with at least three vertices. Then*

$$\beta(B) + 2 \leqq \beta(B \times K_2) \leqq 2\,\beta(B),$$

and these bounds cannot be improved.

Proof. Let the notation be the same as in the proof of Theorem 2. If we identify all vertices of U'' and all vertices of V'', we obtain a graph isomorphic to the graph obtained from B by adding two vertices and an edge which joins them, joining one of them with all vertices of U and the other with all vertices of V. By Theorem 1 the bichromaticity of this graph is $\beta(B) + 2$ and thus the lower bound is obtained. The case $B \cong K_{2,s}$ investigated in the proof of Theorem 2 shows that this bound cannot be improved. Let $K_{p,q}$ be a complete bigraph with $p + q = \beta(B \times K_2)$ onto which $B \times K_2$ can be mapped by a bicomplete homomorphism ψ. The vertex set of this graph is the union of the vertex sets of $\psi(B')$ and $\psi(B'')$. Each of these sets has the cardinality at most $\beta(B)$, therefore $\beta(B \times K_2) \leqq 2\,\beta(B)$. Consider the path P_n with n vertices, n even. By Corollary 1a from [1] we have $\beta(P_n) = [\frac{1}{2}(n + 3)] = \frac{1}{2}n + 1$. Let the vertices of this path be $u_1, \ldots, u_n$ and the edges u_iu_{i+1} for $i = 1, \ldots, n - 1$. Now $P_n \times K_2$ is the graph with the vertex set $\{u_1', \ldots, u_n', u_1'', \ldots, u_n''\}$ and with the edges $u_i'u_{i+1}', u_i''u_{i+1}''$ for $i = 1, \ldots, n - 1$ and $u_i'u_i''$ for $i = 1, \ldots, n$. This is a bigraph on the sets $U = \{u_i' \mid i \equiv 1 \pmod 2\} \cup \{u_i'' \mid i \equiv 0 \pmod 2\}$, $V = \{u_i' \mid i \equiv 0 \pmod 2\} \cup \{u_i'' \mid i \equiv 1 \pmod 2\}$. There exists a bicomplete homomorphism ψ onto a complete bigraph $K_{2,n}$ such that $\psi(x) = \psi(y)$ if and only if either $x = y$ or $\{x, y\} \subset \{u_i' \mid i \equiv 1 \pmod 2\}$ or $\{x, y\} \subset \{u_i'' \mid i \equiv 0 \pmod 2\}$. Therefore $\beta(P_n \times K_2) = n + 2 = 2\,\beta(P_n)$ and the upper bound cannot be improved.

Theorem 5. *Let B be a connected finite bigraph on the sets U, V. Let there exist an automorphism of B which maps U onto V and V onto U. Then*

$$\tfrac{3}{2}\mu(B) \leqq \beta(B \times K_2) \leqq 2\beta(B).$$

This is an immediate consequence of Theorems 3 and 4.

As we see, the number $\beta(B \times K_2)$ is not uniquely determined by the number $\beta(B)$. Therefore our results cannot yield a formula for the bichromaticity of the graph of the cube of dimension n as a function of n. We shall give only partial results.

Theorem 6. *Let Q_n denote the graph of the cube of dimension n. Then*

$$\beta(Q_1) = 2\,, \quad \beta(Q_2) = 4\,, \quad \beta(Q_3) = 6\,, \quad \beta(Q_n) \geqq 2^{n-1} + 4 \textit{ for } n \geqq 4\,.$$

Proof. The computation of $\beta(Q_1)$, $\beta(Q_2)$ and $\beta(Q_3)$ is left to the reader. Now consider Q_4 as a bigraph on the sets U, V. To each vertex x of Q_4 there exists a unique vertex $\bar{x}$ corresponding to the vertex of the cube which is opposite to the vertex of the cube corresponding to x. If $x \in U$, then also $\bar{x} \in U$. By identifying each pair $\{x, \bar{x}\}$ for $x \in U$ we obtain the complete bigraph $K_{4,8}$. Using Theorem 3 we prove by induction that Q_n for each $n \geqq 4$ can be mapped by a bicomplete homomorphism onto $K_{4,s}$, where $s = 2^{n-1}$. Therefore $\beta(Q_n) \geqq 2^{n-1} + 4$ for each $n \geqq 4$.

Now we shall add some more results on the bichromaticity of a graph.

Theorem 7. *Let B be a finite bigraph obtained from the complete bigraph $K_{n,n}$, where $n \geqq 3$, by deleting edges of a complete matching of $K_{n,n}$. Then*

$$\beta(B) = [\tfrac{3}{2}n]\,.$$

Proof. Let ψ be a bicomplete homomorphism of B onto $K_{r,s}$ for some r and s. Consider a bijection $\gamma : U \to V$ such that for each $x \in U$ the vertex $\gamma(x)$ is the unique vertex of V non-adjacent to x. If $x \in U$, then there exists $y \in U$ such that either $\psi(y) = \psi(x)$ or $\psi\,\gamma(y) = \psi\,\gamma(x)$; otherwise $\psi(x)$ and $\psi\,\gamma(x)$ would not be adjacent. We can define a mapping $\delta : U \to U$ so that if $x \in U$, then we put $\delta(x)$ equal to a vertex y with the property that $\psi(y) = \psi(x)$, or $\psi\,\gamma(y) = \psi\,\gamma(x)$. This mapping defines a graph H on the vertex set U such that two vertices x and y are adjacent if and only if $y = \delta(x)$ or $x = \delta(y)$. If x and y are adjacent in H, then the four vertices $x, y, \gamma(x), \gamma(y)$ are mapped by ψ onto at most three vertices. Therefore the difference between the number of vertices of B and $r + s$ is equal at least to the number of edges of H. The graph H must be a graph without isolated vertices. Hence the minimal number of edges of H is $\frac{1}{2}n$ for n even and $\frac{1}{2}(n + 1)$ for n odd. We have proved that $\beta(B) \leqq 2n - \frac{1}{2}n = \frac{3}{2}n$ for n even and $\beta(G) \leqq 2n - \frac{1}{2}(n + 1) = = \frac{3}{2}n - \frac{1}{2} = [\frac{3}{2}n]$ for n odd. The equality can be proved by showing the corresponding bicomplete homomorphism ψ. We choose a partition $\mathcal{P}$ of U such that if n is even, then each class of $\mathcal{P}$ consists of two elements, and if n is odd, then one class of $\mathcal{P}$ consists of three elements and each other class consists of two elements. The number of classes of $\mathcal{P}$ is $[\frac{1}{2}n]$. Now we can define a bicomplete homomorphism ψ of B onto $K_{r,s}$, where $r = [\frac{1}{2}n]$, $s = n$ so that $\psi(x) = \psi(y)$ if and only if either $x = y$ or x and y belong to the same class of $\mathcal{P}$.

Theorem 8. *Let B be a finite bigraph obtained from the complete bigraph $K_{n,n}$ for even $n \geqq 4$ by deleting all edges of a Hamiltonian circuit of $K_{n,n}$. Then*

$$\beta(B) = \tfrac{3}{2}n\,.$$

Proof. As the graph B from this theorem is a spanning subgraph of the graph B from Theorem 7, its bichromaticity cannot be greater than $\frac{3}{2}n$. Therefore it suffices to describe a bicomplete homomorphism ψ of B onto $K_{r,s}$, where $r = \frac{1}{2}n$, $s = n$. Let C be the mentioned Hamiltonian circuit of $K_{n,n}$. To each vertex $u \in U$ there exists exactly one vertex $\bar{u} \in U'$ which is opposite to u in C. The neighbourhoods of u and $\bar{u}$ in C are disjoint, therefore we may define the bicomplete homomorphism ψ of B onto $K_{r,s}$ so that $\psi(x) = \psi(y)$ if and only if either $x = y$ or $x \in U$, $y \in U$, $\bar{x} = y$.

In the end we shall give a result on infinite bigraphs. For infinite connected bigraphs the bichromaticity can be defined analogously as for finite ones.

Theorem 9. *The bichromaticity of an infinite connected bigraph is equal to the cardinality of its vertex set.*

Proof. Let B be an infinite connected bigraph on the sets U, V. The bichromaticity of B evidently cannot be greater than the cardinality of its vertex set. Without loss of generality let $|U| \geqq |V|$. As $U \cup V$ is infinite, also U is infinite and $|U| = |U \cup V|$. If ψ is a bicomplete homomorphism of B such that $\psi(x) = \psi(y)$ if and only if either $x = y$ or $\{x, y\} \in V$, then ψ maps B onto a star with the vertex set of the same cardinality as the vertex set of B and the assertion is proved.

This theorem shows that the considerations on the bichromaticity of an infinite bigraph are trivial. Nonetheless, it might be interesting to study the pairs $\{r, s\}$ with the property that B can be mapped by a bicomplete homomorphism onto $K_{r,s}$.

Reference

[1] *Harary, F., Hsu, D., Miller, Z.:* The bichromaticity of a tree. In: Theory and Applications of Graphs, ed. by Y. Alavi and D. R. Lick, Springer Verlag Berlin—Heidelberg—New York 1978, pp. 236—246.

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