Marsha F. Foregger; Thomas H. Foregger

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THE TREE-COVERING NUMBER OF A GRAPH

MARSHA F. FOREGGER and THOMAS H. FOREGGER, Murray Hill

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INTRODUCTION

The *point-arboricity* $\varrho(G)$ of a graph G is the minimum number of subsets in any partition of the point set of G such that each subset induces an acyclic subgraph. We let $\varrho'(G)$ denote the minimum number of subsets in any partition of the point set of G such that each subset induces a *connected* acyclic graph. Since a connected acyclic graph is a tree, we refer to $\varrho'(G)$ as the *tree-covering number* for G.

If $V_1, \ldots, V_n$ is a partition of the point set of G such that the subgraph $\langle V_i \rangle$ of G induced by V_i is connected and acyclic, $i = 1, \ldots, n$, we call $V_1, \ldots, V_n$ a *connected acyclic partition* for G (or simply a partition for G when there is no chance for confusion). Any connected acyclic partition for G is an acyclic partition, so $\varrho(G) \leqq \varrho'(G)$ for any graph G.

If H is a subgraph of G, it is always true that $\varrho(H) \leqq \varrho(G)$. This property does not hold for ϱ', as the graphs in Fig. 1 indicate. Consequently many theorems about point-arboricity whose proofs rely on induction on the number of points or edges in the graph do not easily carry over to results about the tree-covering number.

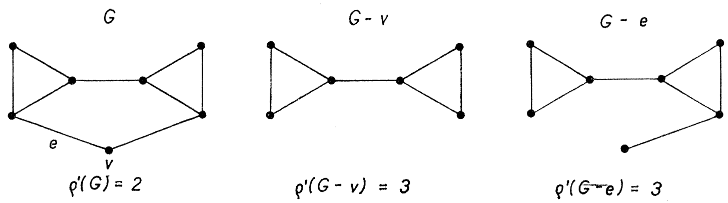

Fig. 1.

All definitions not given here may be found in [1].

SOME BOUNDS FOR $\varrho'(G)$

If $V_1, \ldots, V_n$ is a partition of the points of G such that each V_i contains at most two points, then each $\langle V_i \rangle$ is acyclic. Thus $\varrho(G) \leqq \lceil n/2 \rceil$ where G has order n and $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. However, it is not always possible to partition the points of G into $\lceil n/2 \rceil$ sets of size at most two such that each set induces a connected acyclic subgraph. For example, if G is the graph of order 10 formed by three K_4's which share a common point, then it is not possible to find five point-disjoint edges in G.

Nevertheless, the same upper bound of $\lceil n/2 \rceil$ also holds for $\varrho'(G)$ if G is connected. To prove this we need the following lemma about blocks.

Lemma 1. *Let G be a block with n points.*

a) *If $n \geqq 3$ there exists an edge v_1v_2 in G such that $G(v) = G - v_1 - v_2$ is connected.*

b) *If $n \geqq 4$ there exist disjoint edges u_1u_2 and v_1v_2 in G such that $G(u) = G - u_1 - u_2$ and $G(v) = G - v_1 - v_2$ are connected.*

The proof of this lemma uses the following theorem by Roberts [3].

Theorem (Roberts). *If G is a 2-connected graph which is not a single point, then we can write G as the union of subgraphs $G = P_0 \cup P_1 \cup \ldots \cup P_k$ where P_0 is an edge graph and for $i = 1, \ldots, k$, P_i is a simple path which avoids $B_{i-1} = P_0 \cup P_1 \cup \ldots \cup P_{i-1}$ except for its distinct endpoints which are contained in $V(B_{i-1})$.*

Roberts's theorem is proved by induction on the number of edges in G and relies on a theorem by Whitney.

Theorem (Whitney). *Let G be a 2-connected graph. Suppose K is a 2-connected proper subgraph of G containing at least one edge. Then we can write G as $H \cup L$, where H is a 2-connected proper subgraph of G containing K, and L is a simple path in G that avoids H except for its distinct endpoints which are in $V(H)$.*

For a proof of Whitney's theorem see [4] p. 19.

Proof of Lemma 1. a) For $n \geqq 3$ the concepts of block and 2-connected graph are identical so by Roberts's theorem we can write $G = P_0 \cup \ldots \cup P_k$ where P_0 is an edge and P_i is a path avoiding $B_{i-1} = P_0 \cup \ldots \cup P_{i-1}$ except for its distinct endpoints which lie in B_{i-1}, $i = 1, \ldots, k$. It is easy to see that each B_i is a block. Since P_1 has length at least 2 there exists a largest $j \geqq 1$ such that P_j has length at least 2. Then for $i > j$, P_i is an edge between two points of B_{j-1}. Let $V(P_j) = \{v_1, v_2, \ldots, v_m\}$ where $v_1, v_m \in V(B_{j-1})$, $v_i \notin V(B_{j-1})$ for $2 \leqq i \leqq m - 1$, and $m \geqq 3$. Let $G(v) = G - v_1 - v_2$. Since B_{j-1} is a block, $B_{j-1} - v_1$ is connected. $B_j - v_1 - v_2$ consists of $B_{j-1} - v_1$ plus a path joined to $B_{j-1} - v_1$ at v_m, so it is also connected. Since $G(v)$ is just $B_j - v_1 - v_2$ plus additional edges, $G(v)$ is also connected.

b) The proof will be by induction on n. If $n = 4$, G is K_4, K_4 minus an edge, or C_4 and the result holds by inspection. Now assume that if H is any block on m points with $4 \leqq m < n$, there exist two disjoint edges such that removing either one from H leaves a connected graph. Let G be a block of order $n \geqq 5$. By part a) there exists an edge v_1v_2 such that $G(v)$ is connected. Moreover if the path P_j used in the proof of part a) has length at least 3 we can let $u_1 = v_m$, $u_2 = v_{m-1}$. By the same reasoning as above $G(u)$ is connected and we have the desired result. Thus we may assume that P_j has length 2. Then G is $B_{j-1} + v_1v_2 + v_2v_3$ plus possibly other edges, so we may regard G as a block H together with the point v_2 joined to H at v_1, v_3, and possibly other points.

Since H has order $n - 1$ the induction hypothesis implies the existence of disjoint edges x_1x_2 and y_1y_2 such that $H(x) = H - x_1 - x_2$ and $H(y) = H - y_1 - y_2$ are connected. If $\{x_1, x_2\} = \{v_1, v_3\}$ the edges v_1v_2 and y_1y_2 in G have the desired properties. If $x_1 = v_1$ but $x_2 \neq v_3$ use the edges x_1x_2 and v_2v_3. Similar arguments apply when $\{y_1, y_2\} = \{v_1, v_3\}$ and $y_1 = v_1$ but $y_2 \neq v_3$. Thus we may suppose x_1, x_2, y_1, y_2, v_1, and v_3 are all distinct, and the edges x_1x_2 and y_1y_2 have the desired property. This completes the proof.

Theorem 2. *If G is a connected graph of order n then $\varrho'(G) \leqq \lceil n/2 \rceil$.*

Proof. We use induction on n. If n is 2 or 3 the result is easily checked. Suppose that if H is any connected graph of order $m < n$ then $\varrho'(H) \leqq \lceil m/2 \rceil$, and let G be a connected graph of order $n \geqq 4$. If G is a block then Lemma 1a implies the existence of an edge v_1v_2 such that $G(v) = G - v_1 - v_2$ is connected. If $V_1, \ldots, V_k$ is a connected acyclic partition for $G(v)$, $\{v_1, v_2\}$, $V_1, \ldots, V_k$ is a partition for G. Since the induction hypothesis applies to $G(v)$ we have

$$\varrho'(G) \leqq 1 + \varrho'(G(v)) \leqq 1 + \left\lceil \frac{n-2}{2} \right\rceil = \left\lceil \frac{n}{2} \right\rceil.$$

Now suppose G is not a block. Then there exists a cut-point v of G such that all blocks of G containing v with at most one exception are endblocks (see [1], p. 29). Let $B_1, \ldots, B_k$ be the endblocks of G containing v, with $k \geqq 1$. If $|V(B_i)| \geqq 4$ for some i then Lemma 1b implies the existence of disjoint edges u_1u_2 and v_1v_2 in B_i such that removing either one from B_i leaves a connected subgraph of B_i. Without loss of generality u_1u_2 does not contain the point v, so $G(u) = G - u_1 - u_2$ is connected. Applying the induction hypothesis to $G(u)$ we again have

$$\varrho'(G) \leqq 1 + \varrho'(G(u)) \leqq 1 + \left\lceil \frac{n-2}{2} \right\rceil = \left\lceil \frac{n}{2} \right\rceil.$$

If $|V(B_i)| = 3$ for some i then $B_i = K_3$ and there exists an edge u_1u_2 in $B_i - v$ whose removal leaves a connected subgraph of B_i, namely $\langle v \rangle$. Then $G(u)$ is again connected and use of the induction hypothesis on $G(u)$ gives the desired bound.

Thus we may assume $|V(B_i)| = 2$ for $i = 1, \ldots, k$ so that $\langle \bigcup_{i=1}^{k} V(B_i) \rangle$ is acyclic. Let $H = G - \bigcup_{i=1}^{k} V(B_i) + v$. Then $H - v$ is connected of order $n - k - 1$ so by induction

$$\varrho'(G) \leqq 1 + \varrho'(H - v) \leqq 1 + \left\lceil \frac{n - k - 1}{2} \right\rceil = \left\lceil \frac{n - k + 1}{2} \right\rceil \leqq \left\lceil \frac{n}{2} \right\rceil$$

since $k \geqq 1$. This completes the proof.

The graphs for which the inequality of this theorem is an equality are characterized in the final section of this paper.

Corollary 3. *If G is a graph on n points with k components then*

$$\varrho'(G) \leqq \left\lceil \frac{n + k - 1}{2} \right\rceil .$$

Proof. Let $C_1, \ldots, C_k$ be the components of G and let $n_i = |V(C_i)|$, $i = 1, \ldots, k$. Then for each i, $\varrho'(C_i) \leqq \lceil n_i/2 \rceil$ by Theorem 2, so

$$\varrho'(G) = \sum_{i=1}^{k} \varrho'(C_i) \leqq \sum_{i=1}^{k} \left\lceil \frac{n_i}{2} \right\rceil \leqq \sum_{i=1}^{k} \frac{n_i + 1}{2} = \left\lceil \frac{n + k}{2} \right\rceil .$$

Since $\varrho'(G)$ is an integer,

$$\varrho'(G) \leqq \left\lceil \frac{n + k - 1}{2} \right\rceil .$$

If G is a connected graph of order n, then $\varrho(G)$ and $\varrho'(G)$ are both bounded by $\lceil n/2 \rceil$. However, the ratio $\varrho'(G)/\varrho(G)$ can be arbitrarily large. To see this consider the graph G consisting of k copies of K_m all sharing a common point v, where $m \geqq 4$ is even. Let V_i consist of a pair of points from each $K_m - v$ for $i = 1, \ldots, (m - 2)/2$. The remaining points induce an acyclic graph, so $\varrho(G) = (m - 2)/2 + 1 = m/2$. It is not hard to see that

$$\varrho'(G) = k\left(\frac{m - 2}{2}\right) + 1 = \frac{k(m - 2) + 2}{2} .$$

Thus

$$\frac{\varrho'(G)}{\varrho(G)} = \frac{k(m - 2) + 2}{m}$$

which becomes arbitrarily large as k increases.

Given any graph G, its complement $\bar{G}$ has $V(\bar{G}) = V(G)$ and $uv \in E(\bar{G})$ if and only if $uv \notin E(G)$. Nordhaus and Gaddum [2] found bounds on the chromatic number χ of a graph and its complement:

Theorem (Nordhaus and Gaddum). *Let G be a graph of order n. Then*

$$2\sqrt{(n)} \leqq \chi(G) + \chi(\bar{G}) \leqq n + 1\,, \quad n \leqq \chi(G)\,\chi(\bar{G}) \leqq (n + 1)^2/4\,.$$

We prove a similar theorem for ϱ'.

Theorem 4. *Let G be a graph of order n. Then*

$$\sqrt{(n)} \leqq \varrho'(G) + \varrho'(\bar{G}) \leqq \lceil 3n/2 \rceil\,, \quad n/4 \leqq \varrho'(G)\,\varrho'(\bar{G}) \leqq n(n+1)/2\,.$$

Proof. Since $\varrho'(H) \geqq \varrho(H) \geqq \chi(H)/2$ for any graph H, the lower bounds in each case follow from the theorem of Nordhaus and Gaddum. Observe that for any graph G, at least one of G and $\bar{G}$ must be connected, so without loss assume G is connected. Then $\varrho'(G) \leqq \lceil n/2 \rceil$ and clearly $\varrho'(\bar{G}) \leqq n$, so

$$\varrho'(G) + \varrho'(\bar{G}) \leqq n + \lceil n/2 \rceil = \lceil 3n/2 \rceil\,,$$
$$\varrho'(G)\,.\,\varrho'(\bar{G}) \leqq n\lceil n/2 \rceil \leqq n(n+1)/2\,.$$

The upper bounds in Theorem 4 are best possible, since equality holds when $G = K_n$ with n odd. If G is a path $v_1, v_2, \ldots, v_{4k}$, then any 5 points of $\bar{G}$ induce a cyclic subgraph. Since the sets $\{v_1, \ldots, v_4\}, \{v_5, \ldots, v_8\}, \ldots, \{v_{4k-3}, \ldots, v_{4k}\}$ form a connected acyclic partition for $\bar{G}$, we have $\varrho'(\bar{G}) = k$. Thus $\varrho'(G)\,\varrho'(\bar{G}) = n/4$ and the lower bound in the second inequality is best possible.

The lower bound in the first inequality is also best possible as the following example shows. Take $n = 16k^2$ and let G' be the complete $2k$-partite graph $K(8k, \ldots, 8k)$. Let U_j, $j = 1, 2, \ldots, 2k$ be the basic partitioning sets for G', so that each $\langle U_j \rangle$ consists of $8k$ isolated points. Let $U_j = \{v_{1j}, v_{2j}, \ldots, v_{8k,j}\}$. Form G by adding to G' the edges $v_{i,j}v_{i+1,j}$, $1 \leqq i < 8k$, $1 \leqq j \leqq 2k$ (so that each U_j induces

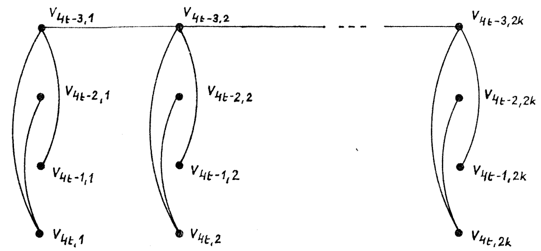

Fig. 2. Subgraph of $\bar{G}$ induced by V_t.

a path in G) and deleting from G' the edges $v_{i,j}v_{i,j+1}$, $1 \leqq j < 2k$, $1 \leqq i < 8k$, $i \equiv 1 \bmod 4$. The sets $U_1, \ldots, U_{2k}$ form a connected acyclic partition for G, so $\varrho'(G) \leqq 2k$. Let $V_t = \{v_{i,j} \mid 4t - 3 \leqq i \leqq 4t, 1 \leqq j \leqq 2k\}$ for $t = 1, \ldots, 2k$. Then the subgraph of $\bar{G}$ induced by each V_t has the form indicated in Fig. 2, so the sets $V_1, \ldots, \ldots, V_{2k}$ form a partition form $\bar{G}$. Thus $\varrho'(\bar{G}) \leqq 2k$ so that $\varrho'(G) + \varrho'(\bar{G}) \leqq 4k = \sqrt{n}$.

THE CASE OF EQUALITY IN THEOREM 2

In this section we characterize those connected graphs for which the tree-covering number is as large as possible. Before stating the theorem, we present some notation.

We use $K_n(u)$ to denote any graph obtained from K_n by the deletion of at most

$n - 3$ edges incident with u. Although $K_n(u)$ refers to a family of graphs, we will treat the family as a single graph for simplicity, since we will not be concerned with the number of deleted edges.

Given n_1 and n_2 odd integers, let $A = K_{n_1}$ and $B = K_{n_2}$, and take two additional points x_1 and x_2. Form the graph $G(n_1, n_2)$ of even order $n_1 + n_2 + 2$ by joining x_1 and x_2 to all points of A and all points of B, and possibly to each other. Since we will not be concerned with whether or not x_1 and x_2 are joined, we use the same notation for both graphs.

Given $n_1, \ldots, n_k$ odd integers $\geqq 3$, we form the "daisy" graph $D(n_1, \ldots, n_k)$ of even order $n_1 + \ldots + n_k - k$ as follows. Let $x_1, x_2, \ldots, x_k, x_1$ be a k-cycle. Use x_1, x_2 and $n_1 - 2$ more points distinct from $x_1, \ldots, x_k$ to form a K_{n_1}. Similarly, construct a K_{n_i} on x_i and x_{i+1} for $i = 2, \ldots, k$ (read subscripts modulo k). The resulting graph is $D(n_1, \ldots, n_k)$; in this case the notation refers to a single graph.

Figure 3 illustrates the above notation.

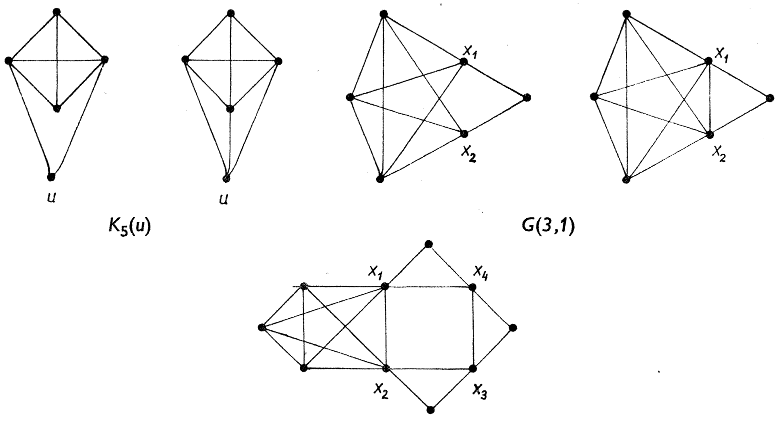

Fig. 3.

If a block has even order, we will refer to it as an even block. Similarly an odd block is a block of odd order. We now characterize those graphs with tree-covering numbers as large as possible.

Theorem 5. *Let G be connected of order n. Then $\varrho'(G) = \lceil n/2 \rceil$ if and only if*

1) *All odd blocks of G are complete;*

2) *G has at most one even block;*

and

3) *if G has an even block, that block is K_m, $K_m(u)$, $G(n_1, n_2)$ or $D(n_1, \ldots, n_k)$.*

The proof essentially consists of a case by case analysis of the possibilities; it is available from the authors upon request.

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Authors' address: Bell Laboratories, Murray Hill, New Jersey 07974, U.S.A.