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INTRODUCTION

This part of our theory of integration of vector valued functions with respect to operator valued measures consists of three supplementary sections. In § 1 we prove further properties of L_1-pseudonorms, thus continuing their study from Part II (see [4]). Particularly, we prove a diagonal convergence theorem for them, see Theorem 5. The main result of § 2 is Theorem 6 on the existence of infinite products of $L(\boldsymbol{X})$ valued measures. Here the assumption $c_0 \not\subset \boldsymbol{X}$ is essential. In § 3 we first prove a theorem of the form $\int_B \boldsymbol{f} \,\mathrm{d}(\boldsymbol{m}\varphi^{-1}) = \int_{\varphi^{-1}(B)} \boldsymbol{f}(\varphi(\cdot)) \,\mathrm{d}\boldsymbol{m}$, where φ is a measurable transformation, and then four theorems which in the scalar case reduce to assertions of the form $\int_E \boldsymbol{f} \,\mathrm{d}(\int_. \boldsymbol{g} \,\mathrm{d}\boldsymbol{n}) = \int_E \boldsymbol{g}\boldsymbol{f} \,\mathrm{d}\boldsymbol{n}$.

We shall use the notation and concepts of the previous parts of our theory. Particularly, T will be a non empty set, $\mathscr{P}$ a δ-ring of subsets of T, $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ and $\boldsymbol{Z}_1$ will be Banach spaces over the same scalars, and $L(\boldsymbol{X}, \boldsymbol{Y})$ the Banach space of bounded linear operators from $\boldsymbol{X}$ to $\boldsymbol{Y}$.

Remark. It is worth noting that our setting of the theory of integration covers also the case of integration of operator valued functions with respect to a countably additive vector measure, i.e., integration of $\mathscr{P}$-measurable functions $\boldsymbol{f}: T \to L(\boldsymbol{X}, \boldsymbol{Y})$ with respect to a countably additive vector measure $\boldsymbol{m} : \mathscr{P}_0 \to \boldsymbol{X}$. This is evident from the following considerations:

Let $\boldsymbol{m} : \mathscr{P}_0 \to \boldsymbol{X}$ be a countably additive vector measure, and let $\boldsymbol{f}: T \to L(\boldsymbol{X}, \boldsymbol{Y})$ be a $\mathscr{P}_0$-simple function of the form $\boldsymbol{f} = \sum_{i=1}^{r} \boldsymbol{U}_i \cdot \chi_{E_i}$, where $\boldsymbol{U}_i \in L(\boldsymbol{X}, \boldsymbol{Y})$, $E_i \in \mathscr{P}_0$ and $E_i \cap E_j = \emptyset$ for $i \neq j$, $i, j = 1, \ldots, r$. Then it is natural to define $\int_E \boldsymbol{f} \,\mathrm{d}\boldsymbol{m} = \sum_{i=1}^{r} \boldsymbol{U}_i \boldsymbol{m}(E \cap E_i)$ for $E \in \mathfrak{S}(\mathscr{P}_0)$. Being motivated by our definition of the semivariation, see p. 513 in Part I, we put

$$\boldsymbol{m}^{\wedge}(E) = \sup \left\{ \left| \int_E \boldsymbol{f} \,\mathrm{d}\boldsymbol{m} \right|, \; \boldsymbol{f}: T \to L(\boldsymbol{X}, \boldsymbol{Y}) \text{ is } \mathscr{P}_0\text{-simple and } \|\boldsymbol{f}\|_E \leqq 1 \right\}$$

for $E \in \mathfrak{S}(\mathscr{P}_0)$. Then

$$m^{\wedge}(E) = \sup \left\{ \left| \sum_{i=1}^{r} U_i m(E \cap E_i) \right| ,\ U_i \in L(X, Y),\ |U_i| \leqq 1,\ E_i \in \mathscr{P}_0,\ E_i \cap E_j = \emptyset \right.$$

$$\left. \text{for } i \neq j,\ i, j = 1, \ldots, r,\ r = 1, 2, \ldots \right\} =$$

$$= \sup \{ \sum_{i=1}^{r} |m(E \cap E_i)|,\ E_i \in \mathscr{P}_0,\ E_i \cap E_j = \emptyset \text{ for } i \neq j,\ i, j = 1, \ldots r,$$

$$r = 1, 2, \ldots \} = v(m, E)$$

for each $E \in \mathfrak{S}(\mathscr{P}_0)$ (using a corollary of the Hahn-Banach Theorem, see II.3.14 in [8], take $x_i^* \in X^*$ so that $|x_i^*| = 1$ and $x_i^* m(E \cap E_i) = |m(E \cap E_i)|$, and put $U_i = y \,.\, x_i^*$, where $y \in Y$ and $|y| = 1$).

Similarly $m^{\wedge}(f, E) = \int_E |f| \, dv(m, \cdot)$ for each $\mathscr{P}_0$-measurable function $f : T \to L(X, Y)$ and each $E \in \mathfrak{S}(\mathscr{P}_0)$, hence $\mathscr{L}_1 \mathscr{M}(m) = \mathscr{L}_1(m) = \mathscr{L}_1(v(m, \cdot), X)$, see Definitions 1 and 4 in Part II.

On the other hand, by the same Corollary II.3.14 in [8] of the Hahn-Banach Theorem, X has the natural imbedding $X \hookrightarrow L(L(X, Y), Y)$, $x \hookrightarrow \dot{x}$, defined by the equality $\dot{x} U = Ux$, $U \in L(X, Y)$. Hence $m : \mathscr{P}_0 \to X$ may be viewed as $\dot{m} : \mathscr{P}_0 \to L(L(X, Y), Y)$, and $\dot{m}$ is countably additive in the uniform operator topology by the countable additivity of m. Since $\int_E f \, d\dot{m} = \int_E f \, dm$ for each $\mathscr{P}_0$-simple function $f : T \to L(X, Y)$ and each $E \in \mathfrak{S}(\mathscr{P}_0)$, our original setting covers also this case.

1. FURTHER PROPERTIES OF L_1-PSEUDONORMS

This section may be considered a continuation of Part II. We note that from the definition of the L_1-pseudonorm, $m^{\wedge}(g, E) = \sup \{ |\int_E f \, dm|,\ f : T \to X \text{ is } \mathscr{P}\text{-simple},\ |f(t)| \leqq |g(t)| \text{ for each } t \in T \}$, it is clear that it depends only on $|g|$ and E. Hence the results about L_1-pseudonorms for vector valued functions remain valid for functions $g : T \to \langle 0, +\infty)$ (then $g \in \mathscr{L}_1(m)$ if and only if g is $\mathscr{P}$-measurable and $m^{\wedge}(g, \cdot)$ is continuous on $\mathfrak{S}(\mathscr{P})$). This was already noted in the paragraph before Theorem 2 in Part III. On the other hand, Lemmas 1 and 2 below have no meaning for X-valued functions in general.

We begin with the following extension of Theorem 16 from Part II.

Theorem 1. (Extended Vitali Convergence Theorem in $\mathscr{L}_1(m)$.) *Let $m : \mathscr{P} \to L(X, Y)$ be an operator valued measure countably additive in the strong operator topology with the finite semivariation $m^{\wedge}$ on $\mathscr{P}$, let $f_k \in \mathscr{L}_1(m)$, $k = 1, 2, \ldots$, and let $f_k(t) \to f(t) \in X$ m a.e. on T. Then the following assertions are equivalent*:

a) $f \in \mathscr{L}_1(m)$ *and* $m^{\wedge}(f_k, E) \to m^{\wedge}(f, E)$ *for each* $E \in \mathfrak{S}(\mathscr{P})$,

b) $m^{\wedge}(f_k, \cdot)$, $k = 1, 2, \ldots$, *are uniformly continuous on* $\mathfrak{S}(\mathscr{P})$, *and*

c) $f \in \mathscr{L}_1(\boldsymbol{m})$ *and* $\boldsymbol{m}^\wedge(f_k - f, T) \to 0$.

Proof. a) $\Rightarrow$ b) by the continuity of $\boldsymbol{m}^\wedge(f, \cdot)$ on $\mathfrak{S}(\mathscr{P})$ and by the monotonicity and continuity of each $\boldsymbol{m}^\wedge(f_k, \cdot)$, $k = 1, 2, \ldots$ on $\mathfrak{S}(\mathscr{P})$.

b) $\Rightarrow$ c) by Theorem 16 in Part II.

c) $\Rightarrow$ a), since $|\boldsymbol{m}^\wedge(f_k, E) - \boldsymbol{m}^\wedge(f, E)| \leqq \boldsymbol{m}^\wedge(f_k - f, E) \leqq \boldsymbol{m}^\wedge(f_k - f, T)$ for each $E \in \mathfrak{S}(\mathscr{P})$.

The next theorem and the example after Theorem 3 show that in $\mathscr{L}_1(\boldsymbol{m})$ the analog of the classical Monotone Convergence Theorem holds only if $\boldsymbol{Y}$ contains no subspace isomorphic to c_0.

Theorem 2. (Monotone Convergence Theorem in $\mathscr{L}_1(\boldsymbol{m})$.) *Let Y contain no subspace isomorphic to c_0, for example, let $\boldsymbol{Y}$ be a weakly complete Banach space, see pp.* 160 *and* 161 *in* [1], *and let $\boldsymbol{m} : \mathscr{P} \to L(\boldsymbol{X}, \boldsymbol{Y})$ be an operator valued measure countably additive in the strong operator topology with the finite semivariation $\boldsymbol{m}^\wedge$ on $\mathscr{P}$. Let further $f_k : T \to \boldsymbol{X}$, $k = 1, 2, \ldots$, be $\mathscr{P}$-measurable functions, let $f_k(t) \to f(t) \in \boldsymbol{X}$ $\boldsymbol{m}$ a.e. on T, and let $|f_k(t)| \nearrow |f(t)|$ $\boldsymbol{m}$ a.e. on T. Then the following assertions are equivalent*:

a) $\lim_{k \to \infty} \boldsymbol{m}^\wedge(f_k, T) < +\infty$,

b) $f \in \mathscr{L}_1(\boldsymbol{m})$, *and*

c) $f, f_k \in \mathscr{L}_1(\boldsymbol{m})$, $k = 1, 2, \ldots$, *and* $\boldsymbol{m}^\wedge(f_k - f, T) \to 0$.

Proof. a) $\Rightarrow$ b). According to Theorem 4 in Part II and the classical Monotone Convergence Theorem, see Theorem B in § 27 in [9],

$$+\infty > \lim_{k \to \infty} \boldsymbol{m}^\wedge(f_k, T) = \lim_{k \to \infty} \sup_{|y^*| \leqq 1} \int_T |f_k| \, \mathrm{d}v(y^*\boldsymbol{m}, \cdot) =$$

$$= \sup_{|y^*| \leqq 1} \lim_{k \to \infty} \int_T |f_k| \, \mathrm{d}v(y^*\boldsymbol{m}, \cdot) = \sup_{|y^*| \leqq 1} \int_T |f| \, \mathrm{d}v(y^*\boldsymbol{m}, \cdot) = \boldsymbol{m}^\wedge(f, T).$$

Since $\boldsymbol{Y}$ contains no subspace isomorphic to c_0, $f \in \mathscr{L}_1(\boldsymbol{m})$ by Theorem 5 in Part II.

b) $\Rightarrow$ c). Since $|f_k(t)| \leqq |f(t)|$ $\boldsymbol{m}$ a.e. on T for each $k = 1, 2, \ldots$, $f_k \in \mathscr{L}_1(\boldsymbol{m})$, $k = 1, 2, \ldots$, and the L_1-pseudonorms $\boldsymbol{m}^\wedge(f_k, \cdot)$, $k = 1, 2, \ldots$, are uniformly continuous on $\mathfrak{S}(\mathscr{P})$. Thus $\boldsymbol{m}^\wedge(f_k - f, T) \to 0$ by Theorem 1.

c) $\Rightarrow$ a). Since $f \in \mathscr{L}_1(\boldsymbol{m})$, $+\infty > \boldsymbol{m}^\wedge(f, T) \geqq \boldsymbol{m}^\wedge(f_k, T)$ for each $k = 1, 2, \ldots$.

Remark. In Theorems 1 and 2, $\mathscr{L}_1(\boldsymbol{m})$ is the space of all $\boldsymbol{m}$-essentially $\mathscr{P}$-measurable functions, see Definition 2 in Part III, with the continuous L_1-pseudonorms on $\mathfrak{S}(\mathscr{P})$.

We now give an application of Theorem 2.

Theorem 3. *Let T be a locally compact Hausdorff topological space, let $\boldsymbol{Y}$ contain no subspace isomorphic to c_0, and let $\boldsymbol{U} : C_0(T) \to \boldsymbol{Y}$ be a bounded linear operator.*

Let further $f_k \in C_0(T)$, $k = 1, 2, \ldots$, *be such that* $f_k(t) \to f(t) \in X =$ *the scalars of* Y, *let* $|f_k(t)| \nearrow |f(t)|$, *and let* $\sup_k |U(f_k \cdot g)| < +\infty$ *for each* $g \in C_0(T)$. *For* $k = 1, 2, \ldots$ *define* $U_k : C_0(T) \to Y$ *by the equality* $U_k g = U(f_k \cdot g)$, $g \in C_0(T)$. *Then* U_k, $k = 1, 2, \ldots$, *is a Cauchy sequence in* $L(C_0(T), Y)$.

Proof. According to the well-known representation theorem, see [8, VI.7.3] and [7, Theorem 2], U has a unique representation $Uf = \int_T f \,\mathrm{d}\mu$, where $\mu : \mathfrak{S}(\mathscr{B}_0) \to$ $\to Y$ is a countably additive vector measure ($\mathfrak{S}(\mathscr{B}_0)$ is the σ-ring of Baire measurable subsets of T), and $|U| = \|\mu\| (T)$. By Theorem 6 in Part II,

$$|U_k| = \sup_{\|g\|_T \leqq 1, g \in C_0(T)} |U_k g| = \sup_{\|g\|_T \leqq 1, g \in C_0(T)} \left| \int_T f_k g \,\mathrm{d}\mu \right| = \hat{\mu}(f_k, T) \,.$$

By assumption, $U_k g$, $k = 1, 2, \ldots$ is a bounded sequence in Y for each $g \in C_0(T)$, hence $\sup_k \hat{\mu}(f_k, T) = \sup_k |U_k| < +\infty$ by the Uniform Boundedness Principle, see [8, II.1.11 and II.3.21]. Since Y contains no subspace isomorphic to c_0, $|U_k - U_n| =$ $= \hat{\mu}(f_k - f_n, T) \to 0$ as $k, n \to \infty$ by Theorem 2.

Let us note that the function f in the preceding theorem is integrable with respect to μ, $U_0 = \lim_{k \to \infty} U_k$ (in $L(C_0(T), Y)$) has the representation $U_0 g = \int_T g \,\mathrm{d}(\int_. f \,\mathrm{d}\mu) =$ $= \int_T fg \,\mathrm{d}\mu$, $g \in C_0(T)$, and $|U_0| = \hat{\mu}(f, T)$.

The following simple example shows that the assumption $c_0 \not\subset Y$ is essential for the validity of Theorems 2 and 3.

Example. Let $T = \{1, 2, \ldots\}$ with the discrete topology, let $\mathscr{P} = 2^T$, and let $\mu : \mathscr{P} \to c_0$ be determined by countable additivity from the values $\mu(\{t\}) =$ $= (0, \ldots, 0, t^{-1}, 0, \ldots)$. Then T is a locally compact Hausdorff topological space, and $\mu : \mathscr{P} \to c_0$ is a countably additive vector Baire measure. Let $f_k : C_0(T) = c_0$, $k = 1, 2, \ldots$, be defined as follows: $f_k(t) = t$ if $t \leqq k$ and $f_k(t) = 0$ if $t > k$. Then $f_k(t) \nearrow f(t) = t$ for each $t \in T$, $\hat{\mu}(f_k, T) = 1$ for each $k = 1, 2, \ldots$, but f is clearly not integrable with respect to μ, and $|U_n - U_k| = \hat{\mu}(f_n - f_k, T) = 1$ if $n \neq k$, $n, k = 1, 2, \ldots$.

Let $\mu : \mathscr{P} \to \langle 0, +\infty)$ be a countably additive measure, let $g : T \to \langle 0, +\infty)$ be a $\mathscr{P}$-measurable function, and let $\int_T g \,\mathrm{d}\mu < +\infty$. Then, by the additivity of the integral, clearly for each $\varepsilon > 0$ there is a positive integer N_ε such that whenever $f_1, \ldots, f_{N_\varepsilon}$ are $\mathscr{P}$-measurable and $\sum_{n=1}^{N} |f_n| \leqq g$, then $\int_T |f_n| \,\mathrm{d}\mu < \varepsilon$ for at least one $n \in \{1, \ldots, N_\varepsilon\}$. (Take $N_\varepsilon \geqq [\varepsilon^{-1} \cdot \int_T g \,\mathrm{d}\mu] + 1$.) This fact may be considered as a strengthening of the Lebesgue Dominated Convergence Theorem, see [9, § 26, Theorem D]. Now we show that this strengthening holds also in $\mathscr{L}_1(m)$.

Theorem 4. *Let* $m : \mathscr{P} \to L(X, Y)$ *be an operator valued measure countably additive in the strong operator topology and let* $g \in \mathscr{L}_1(m)$. *Then for each* $\varepsilon > 0$ *there is* *a positive integer* N_ε *such that whenever* $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_{N_\varepsilon} : T \to \boldsymbol{X}$ *are* $\mathscr{P}$*-measurable and* $\sum_{n=1}^{N_\varepsilon} |\boldsymbol{f}_n| \leqq |\boldsymbol{g}|$, *then* $\boldsymbol{m}^\wedge(\boldsymbol{f}_n, T) < \varepsilon$ *for at least one* $n \in \{1, \ldots, N_\varepsilon\}$.

Proof. Let $\varepsilon > 0$. For $k = 1, 2, \ldots$ put $G_k = \{t, t \in T, k^{-1} \leqq |\boldsymbol{g}(t)| \leqq k\}$. Then $G_k \in \mathfrak{S}(\mathscr{P})$ for each $k = 1, 2, \ldots$ and $G_k \nearrow G = \{t, t \in T, \boldsymbol{g}(t) \neq 0\}$. Hence, by the continuity of the L_1-pseudonorm $\boldsymbol{m}^\wedge(\boldsymbol{g}, \cdot)$ on $\mathfrak{S}(\mathscr{P})$, there is a k_0 such that $\boldsymbol{m}^\wedge(\boldsymbol{g}, G - G_{k_0}) < 2^{-1} \,.\, \varepsilon$. Since $k_0^{-1} \,.\, \boldsymbol{m}^\wedge(G_{k_0} \cap E) \leqq \boldsymbol{m}^\wedge(\boldsymbol{g}, G_{k_0} \cap E) \leqq k_0 \,.\, \boldsymbol{m}^\wedge(G_{k_0} \cap E)$ for each $E \in \mathfrak{S}(\mathscr{P})$, see Theorem 1 in Part II, the semivariation $\boldsymbol{m}^\wedge$ is continuous on the σ-ring $G_{k_0} \cap \mathfrak{S}(\mathscr{P})$. Thus, according to the $*$-Theorem in Section 1.1 in Part I, there is a countably additive measure $\lambda : G_{k_0} \cap \mathfrak{S}(\mathscr{P}) \to \langle 0, +\infty)$ such that $\lambda(G_{k_0} \cap E) \to 0 \Rightarrow \boldsymbol{m}^\wedge(G_{k_0} \cap E) \to 0$, $E \in \mathfrak{S}(\mathscr{P})$. Take $\delta > 0$ so that $E \in \mathfrak{S}(\mathscr{P})$ and $\lambda(G_{k_0} \cap E) < \delta \Rightarrow \boldsymbol{m}^\wedge(G_{k_0} \cap E) < (3k_0)^{-1}\, \varepsilon$, put $a = (3(1 + \boldsymbol{m}^\wedge(G_{k_0}))^{-1} \,.\, \varepsilon$, and take $b > 0$ so that $b \,.\, a^{-1} < \delta$. Since $\int_{G_{k_0}} |\boldsymbol{g}| \, d\lambda \leqq k_0 \,.\, \lambda(G_{k_0}) < +\infty$, there is a positive integer N_b such that whenever $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_{N_b} : T \to \boldsymbol{X}$ are $\mathscr{P}$-measurable and $\sum_{n=1}^{N_b} |\boldsymbol{f}_n| \leqq |\boldsymbol{g}|$, then $\int_{G_{k_0}} |\boldsymbol{f}_n| \, d\lambda < b$ for at least one $n \in \{1, \ldots, N_b\}$. We assert that such N_b may be taken to be the required N_ε. Indeed, if $\int_{G_{k_0}} |\boldsymbol{f}_n| \, d\lambda < b$, then $\lambda(\{t : t \in G_{k_0}, |\boldsymbol{f}_n(t)| \geqq a\}) \leqq b \,.\, a^{-1} < \delta$, hence $\boldsymbol{m}^\wedge(\{t : t \in G_{k_0}, |\boldsymbol{f}_n(t)| \geqq a\}) < \varepsilon \,.\, (3k_0)^{-1}$. Thus

$$\boldsymbol{m}^\wedge(\boldsymbol{f}_n, T) \leqq \boldsymbol{m}^\wedge(\boldsymbol{f}_n, G - G_{k_0}) + \boldsymbol{m}^\wedge(\boldsymbol{f}_n, G_{k_0}) \leqq \boldsymbol{m}^\wedge(\boldsymbol{g}, G - G_{k_0}) + \\ + a \,.\, \boldsymbol{m}^\wedge(\{t : t \in G_{k_0}, |\boldsymbol{f}_n(t)| < a\}) + k_0 \,.\, \boldsymbol{m}^\wedge(\{t : t \in G_{k_0}, |\boldsymbol{f}_n(t)| \geqq a\}) < \varepsilon\,,$$

which we wanted to show. The theorem is proved.

The first assertion of the next lemma immediately follows from Theorem 4 in Part II and the classical Fatou's lemma, see [9, § 27, Theorem F], while the second assertion is immediate from the definition of the L_1-pseudonorm, see Definition 1 in Part II.

Lemma 1. (Fatou's lemma for L_1-pseudonorms.) 1) *Let* $f_n : T \to \langle 0, +\infty)$, $n = 1, 2, \ldots$, *be* $\mathscr{P}$*-measurable functions, and let* $\boldsymbol{m} : \mathscr{P} \to L(\boldsymbol{X}, \boldsymbol{Y})$ *be an operator valued measure countably additive in the strong operator topology. Then*

$$\boldsymbol{m}^\wedge(\liminf_n f_n, E) \leqq \liminf_n \boldsymbol{m}^\wedge(f_n, E)$$

for each $E \in \mathfrak{S}(\mathscr{P})$.

2) *Let* $\boldsymbol{m}, \boldsymbol{m}_n : \mathscr{P} \to L(\boldsymbol{X}, \boldsymbol{Y})$, $n = 1, 2, \ldots$, *be operator valued measures countably additive in the strong operator topology, and let* $\boldsymbol{m}^\wedge(g, T) \to \boldsymbol{m}^\wedge(g, T)$ *for each* $\mathscr{P}$*-simple function* $g : T \to \langle 0, +\infty)$. *Then*

$$\boldsymbol{m}^\wedge(f, E) \leqq \liminf_n \boldsymbol{m}^\wedge{}_n(f, E)$$

for each $\mathscr{P}$*-measurable function* $f : T \to \langle 0, +\infty)$ *and for each* $E \in \mathfrak{S}(\mathscr{P})$.

We shall use the following three assumptions:

(a_1): *let $f_k : T \to X$ or $f_k : T \to \langle 0, +\infty)$, $k = 1, 2, \ldots$, be $\mathscr{P}$-measurable functions and let $f_k(t) \to f(t) \in X$ (or $\langle 0, +\infty)$) for each $t \in T$;*

(a_2): *let $m, m_n : \mathscr{P} \to L(X, Y)$, $n = 1, 2, \ldots$, be operator valued measures countably additive in the strong operator topology, and let $m^\wedge{}_n(g, T) \to m^\wedge(g, T) < < +\infty$ for each $\mathscr{P}$-simple function $g : T \to X$ (or $\to \langle 0, +\infty)$), and*

(a_2u): *(a_2) and $m^\wedge{}_n(g, E) \to m^\wedge(g, E)$ uniformly with respect to $E \in \mathscr{P}$ for each $\mathscr{P}$-simple function $g : T \to X$ (or $\langle 0, +\infty)$).*

Lemma 2. *Let $\mathscr{P} = \mathfrak{S}(\mathscr{P})$, let $f, f_k : T \to \langle 0, +\infty)$, $k = 1\ 2, \ldots$, be $\mathscr{P}$-measurable functions, let $\|f\|_T < +\infty$, and let $\|f_k - f\|_T \to 0$. Suppose further that (a_2) holds. Then $\lim_{k,n\to\infty} m^\wedge{}_n(f_k, E) = m^\wedge(f, E)$ for each $E \in \mathfrak{S}(\mathscr{P})$. This limit is uniform with respect to $E \in \mathfrak{S}(\mathscr{P})$ provided (a_2u) holds.*

Proof. Since $m^\wedge{}_n(E) \to m^\wedge(E) < +\infty$ for each $E \in \mathscr{P} = \mathfrak{S}(\mathscr{P})$, there is an n_0 such that $\sup_{n \geqq n_0} m^\wedge{}_n(T) < +\infty$. Since $f : T \to \langle 0, +\infty)$ is a bounded $\mathscr{P} = \mathfrak{S}(\mathscr{P})$-measurable function, by Theorem B in § 20 in [9] there is a sequence of $\mathscr{P}$-simple functions $g_j : T \to \langle 0, +\infty)$, $j = 1, 2, \ldots$, such that $\|g_j - f\|_T \to 0$. Now the assertions of the lemma are immediate from the following inequalities:

$$|m^\wedge{}_n(f_k, E) - m^\wedge(f, E)| \leqq |m^\wedge{}_n(f_k, E) - m^\wedge{}_n(f, E)| + |m^\wedge{}_n(f, E) - m^\wedge(f, E)| \leqq$$
$$\leqq m^\wedge{}_n(f_k - f, E) + |m^\wedge{}_n(f, E) - m^\wedge{}_n(g_j, E)| + |m^\wedge{}_n(g_j, E) - m^\wedge(f, E)| \leqq$$
$$\leqq m^\wedge{}_n(f_k - f, E) + m^\wedge{}_n(f - g_j, E) + |m^\wedge{}_n(g_j, E) - m^\wedge(g_j, E)| + m^\wedge(g_j - f, E) \leqq$$
$$\leqq \|f_k - f\|_T \cdot \sup_{i \geqq n} m^\wedge{}_i(T) + \|f - g_j\|_T \cdot \sup_{i \geqq n} m^\wedge{}_i(T) +$$
$$+ |m^\wedge(g_j, E) - m^\wedge{}_n(g_j, E)| + \|f - g_j\|_T\, m^\wedge(T)\,.$$

Using this lemma we have:

Theorem 5. (Diagonal Convergence Theorem for L_1-pseudonorms.) *Suppose (a_1) and (a_2), and let $f_n \in \mathscr{L}_1(m_n)$ for each $n = 0, 1, 2, \ldots$, where $m_0 = m$ and $f_0 \approx f$. Then the following conditions are equivalent:*

a) *$m^\wedge{}_n(f_n, E) \to m^\wedge(f, E)$ for each $E \in \mathfrak{S}(\mathscr{P})$, and*

b) *the L_1-pseudonorms $m^\wedge{}_n(f_n, \cdot)$, $n = 1, 2, \ldots$, are uniformly continuous on $\mathfrak{S}(\mathscr{P})$.*

If they hold, and if (a_2u) holds, then $m^\wedge{}_n(f_n, E) \to m^\wedge(f, E)$ uniformly with respect to $E \in \mathfrak{S}(\mathscr{P})$.

Proof. a) $\Rightarrow$ b) by the monotonocity and continuity of $m^\wedge(f, \cdot)$ and of $m^\wedge{}_n(f_n, \cdot)$, $n = 1, 2, \ldots$.

Suppose b). For $E \in \mathfrak{S}(\mathscr{P})$ put

$$\mu(E) = \sum_{n=0}^{\infty} \frac{1}{2^n} \cdot \frac{m^\wedge{}_n(f_n, E)}{1 + m^\wedge{}_n(f_n, T)}\,.$$

Then $\mu : \mathfrak{S}(\mathscr{P}) \to \langle 0, 2)$ is monotone, subadditive, continuous, and $N \in \mathfrak{S}(\mathscr{P})$, $\mu(N) = 0 \Rightarrow \boldsymbol{m}^\wedge{}_n(\boldsymbol{f}_n, N) = 0$ for each $n = 0, 1, \ldots$. Put $F = \bigcup_{n=1}^{\infty} \{t : t \in T, \boldsymbol{f}_n(t) \neq 0\} \in$ $\in \mathfrak{S}(\mathscr{P})$. Then by (a_1) and the Egoroff-Lusin Theorem, see Section 1.4 in Part I, there are $N \in \mathfrak{S}(\mathscr{P})$, and $F_j \in \mathscr{P}$, $j = 1, 2, \ldots$, such that $\mu(N) = 0$, $F_j \nearrow F - N$, and on each F_j, $j = 1, 2, \ldots$, the sequence $\boldsymbol{f}_k$, $k = 1, 2, \ldots$, converges uniformly to $\boldsymbol{f}$. Since clearly

$$\left|\boldsymbol{m}^\wedge{}_n(\boldsymbol{f}_n, E) - \boldsymbol{m}^\wedge{}_k(\boldsymbol{f}_k, E)\right| \leqq \boldsymbol{m}^\wedge{}_n(\boldsymbol{f}_n, E - F_j) + \boldsymbol{m}^\wedge{}_k(\boldsymbol{f}_k, E - F_j) +$$
$$+ \left|\boldsymbol{m}^\wedge{}_n(\boldsymbol{f}_n, E \cap F_j) - \boldsymbol{m}^\wedge{}_k(\boldsymbol{f}_k, E \cap F_j)\right|$$

for each $E \in \mathfrak{S}(\mathscr{P})$ and each $j, k, n = 1, 2, \ldots$, from b) and Lemma 2 we immediately see that $\lim_{n\to\infty} \boldsymbol{m}^\wedge{}_n(\boldsymbol{f}_n, E) = \nu(E) < +\infty$ exists for each $E \in \mathfrak{S}(\mathscr{P})$. and that this limit is uniform with respect to $E \in \mathfrak{S}(\mathscr{P})$ provided $(a_2 u)$ holds. Obviously $\nu : \mathfrak{S}(\mathscr{P}) \to$ $\to \langle 0, +\infty)$ is monotone, subadditive, continuous by b), and $\nu(E \cap F_j) \nearrow \nu(E)$ for each $E \in \mathfrak{S}(\mathscr{P})$. According to Lemma 2, $\nu(E \cap F_j) = \lim_{n\to\infty} \boldsymbol{m}^\wedge{}_n(\boldsymbol{f}_n, E \cap F_j) =$ $= \boldsymbol{m}^\wedge(\boldsymbol{f}, E \cap F_j)$ for each $E \in \mathfrak{S}(\mathscr{P})$ and each $j = 1, 2, \ldots$. But $\boldsymbol{m}^\wedge(\boldsymbol{f}, E \cap F_j) \nearrow$ $\nearrow \boldsymbol{m}^\wedge(\boldsymbol{f}, E - N)$ for each $E \in \mathfrak{S}(\mathscr{P})$ by Theorem 4 in Part II. Hence $\nu(E) =$ $= \boldsymbol{m}^\wedge(\boldsymbol{f}, E - N) = \boldsymbol{m}^\wedge(\boldsymbol{f}, E)$ for each $E \in \mathfrak{S}(\mathscr{P})$. The theorem is proved.

Remark to Theorem 5. If there is a countably additive measure $\lambda : \mathfrak{S}(\mathscr{P}) \to$ $\to \langle 0, +\infty)$ such that $N \in \mathfrak{S}(\mathscr{P})$, $\lambda(N) = 0 \Rightarrow \boldsymbol{m}^\wedge(\boldsymbol{f}, N) = 0$, then, replacing in the definition of μ above $\boldsymbol{m}^\wedge{}_0(\boldsymbol{f}_0, E)/(1 + \boldsymbol{m}^\wedge{}_0(\boldsymbol{f}_0, T))$ by $\lambda(E)/(1 + \lambda(T))$, we immediately see (the equation $\boldsymbol{m}^\wedge(\boldsymbol{f}, E - N) = \boldsymbol{m}^\wedge(\boldsymbol{f}, E)$ remains valid) that b) implies a) regardless of the assumption $\boldsymbol{f} \in \mathscr{L}_1(\boldsymbol{m})$ (in fact, then b) through a) implies that $\boldsymbol{m}^\wedge(\boldsymbol{f}, \cdot)$ is continuous on $\mathfrak{S}(\mathscr{P})$, hence that $\boldsymbol{f} \in \mathscr{L}_1(\boldsymbol{m})$; $\boldsymbol{m}^\wedge(\boldsymbol{f}, T) < +\infty$ by Corollary of Theorem 5 in Part II.). According to Theorem 13 in Part III such a measure λ always exists in the following cases: 1) $\boldsymbol{X}$ is separable, 2) $\boldsymbol{Y}$ has a countable norming set, particularly, if $\boldsymbol{Y}$ is separable or a dual of a separable Banach space, 3) if $\mathfrak{S}(\mathscr{P}_2) \supset \mathscr{P}$, for example, if $\boldsymbol{m} : \mathscr{P} \to L(\boldsymbol{X}, \boldsymbol{Y})$ is countably additive in the uniform operator topology. In fact, it is enough to have only a monotone, subadditive and continuous $\lambda : \mathfrak{S}(\mathscr{P}) \to \langle 0, +\infty)$ such that $N \in \mathfrak{S}(\mathscr{P})$, $\lambda(N) = 0 \Rightarrow \boldsymbol{m}^\wedge(\boldsymbol{f}, N) = 0$.

Using Lemma 1 and Corollary of Theorem 5 in Part II we immediately obtain:

Corollary 1. *Suppose* (a_1) *and* (a_2) *and let the* L_1*-pseudonorms* $\boldsymbol{m}^\wedge{}_n(\boldsymbol{f}_k, \cdot)$, $n, k =$ $= 1, 2, \ldots$, *be uniformly continuous on* $\mathfrak{S}(\mathscr{P})$. *Then* $\boldsymbol{f}_k \in \mathscr{L}_1(\boldsymbol{m}_n)$ *for each* $k, n =$ $= 0, 1, 2, \ldots$, *where* $\boldsymbol{f}_0 = \boldsymbol{f}$ and $\boldsymbol{m}_0 = \boldsymbol{m}$, *and*

$$\lim_{k,n\to\infty} \boldsymbol{m}^\wedge{}_n(\boldsymbol{f}_k, E) = \lim_{n\to\infty} \lim_{k\to\infty} \boldsymbol{m}^\wedge{}_n(\boldsymbol{f}_k, E) = \lim_{n\to\infty} \boldsymbol{m}^\wedge{}_n(\boldsymbol{f}, E) =$$
$$= \boldsymbol{m}^\wedge(\boldsymbol{f}, E) = \lim_{k\to\infty} \boldsymbol{m}^\wedge(\boldsymbol{f}_k, E) = \lim_{k\to\infty} \lim_{n\to\infty} \boldsymbol{m}^\wedge{}_n(\boldsymbol{f}_k, E)$$

for each $E \in \mathfrak{S}(\mathscr{P})$. *If* (a_2u) *holds, then* $\lim_{k,n\to\infty} \boldsymbol{m}^\wedge_n(f_k, E) = \boldsymbol{m}^\wedge(f, E)$ *uniformly with respect to* $E \in \mathfrak{S}(\mathscr{P})$.

Corollary 2. *Suppose* (a_1) *and let* $\boldsymbol{m}^\wedge_n : \mathscr{P} \to L(X, Y)$, $n = 1, 2, \ldots$, *be operator valued measures countably additive in the strong operator topology, and let* $\boldsymbol{m}^\wedge_n(E) \to 0$ *for each* $E \in \mathscr{P}$. *Suppose further that the* L_1*-pseudonorms* $\boldsymbol{m}^\wedge_n(f_k, \cdot)$, $k, n = 1, 2, \ldots$, *are uniformly continuous on* $\mathfrak{S}(\mathscr{P})$. *Then* $\boldsymbol{m}^\wedge_n(f, \cdot)$, $n = 1, 2, \ldots$, *are uniformly continuous on* $\mathfrak{S}(\mathscr{P})$ *and*

$$\lim_{n\to\infty} (\sup_k \boldsymbol{m}^\wedge_n(f_k, T) + \boldsymbol{m}^\wedge_n(f, T)) = 0 .$$

Proof. Since $\boldsymbol{m}^\wedge_n(f, E) \leqq \liminf_k \boldsymbol{m}^\wedge_n(f_k, E)$ for each $n = 1, 2, \ldots$ and each $E \in \mathfrak{S}(\mathscr{P})$ by Lemma 1, $\boldsymbol{m}^\wedge_n(f, \cdot)$, $n = 1, 2, \ldots$, are uniformly continuous on $\mathfrak{S}(\mathscr{P})$. Hence $\boldsymbol{m}^\wedge_n(f, T) \to 0$ by Theorem 5 (take $E = \{t;\, t \in T,\ f(t) \neq 0\} \in \mathfrak{S}(\mathscr{P})$). If $\boldsymbol{m}^\wedge_n(f_k, T) \to 0$ as $n \to \infty$ not uniformly with respect to $k = 1, 2, \ldots$, then either $\boldsymbol{m}^\wedge_n(f_{k_0}, T) \nrightarrow 0$ for some k_0, or there is a subsequence $k_n \to \infty$, $n = 1, 2, \ldots$, such that $\boldsymbol{m}^\wedge_n(f_{k_n}, T) \nrightarrow 0$. In both cases we have a contradiction with Theorem 5.

2. ON INFINITE PRODUCTS OF OPERATOR VALUED MEASURES

Suppose that, for each $i = 1, 2, \ldots$, T_i is a non empty set, $\mathscr{P}_i$ is a σ-algebra of subsets of T_i and $\boldsymbol{m}_i : \mathscr{P}_i \to L(X, X) = L(X)$ is an operator valued measure countably additive in the strong operator topology, such that $\boldsymbol{m}_i(T_i) = \boldsymbol{I}$ = the identity in $L(X)$ and $\boldsymbol{m}^\wedge_i(T_i) = 1$. Then clearly $|\boldsymbol{m}_i(A)| \leqq 1$ for each $A \in \mathscr{P}_i$, $i = 1, 2, \ldots$. Let us recall, see § 38 in [9], that a measurable rectangle in $T = \bigtimes_{i=1}^{\infty} T_i$ is a subset of T of the form $E = \bigtimes_{i=1}^{\infty} A_i$, where $A_i \in \mathscr{P}_i$ for each $i = 1, 2, \ldots$ and $A_i = T_i$ for all but a finite number of values of i. It is easy to see that the class of all finite unions of pairwise disjoint measurable rectangles forms an algebra of subsets of T, which we denote by $\mathscr{R}$. By $\mathscr{P} = \bigotimes_{i=1}^{\infty} \mathscr{P}_i$ we denote the smallest σ-ring containing $\mathscr{R}$, i.e., $\mathscr{P} = \mathfrak{S}(\mathscr{R})$. For any measurable rectangle $E = \bigtimes_{i=1}^{\infty} A_i$ define $\boldsymbol{m}(E) = \boldsymbol{m}_1(A_1) \ldots \boldsymbol{m}_i(A_i) \ldots \ldots \in L(X)$. Then it is clear that $\boldsymbol{m}$ has a unique additive extension $\boldsymbol{m} : \mathscr{R} \to L(X)$.

Definition 1. We say that the *product of the measures* $\boldsymbol{m}_i$, $i = 1, 2, \ldots$, exists on $\mathscr{P} = \bigotimes_{i=1}^{\infty} \mathscr{P}_i$, if there is an operator valued measure $\boldsymbol{m} = \bigotimes_{i=1}^{\infty} \boldsymbol{m}_i : \bigotimes_{i=1}^{\infty} \mathscr{P}_i \to L(X)$ countably additive in the strong operator topology, which extends $\boldsymbol{m} : \mathscr{R} \to L(X)$; this measure is necessarily unique.

The uniqueness of $\boldsymbol{m} : \mathfrak{S}(\mathscr{R}) \to L(\boldsymbol{X})$ follows immediately from the Hahn-Banach Theorem, see [8, § II.3], and from the uniqueness of the extension of a scalar measure from a ring to the generated σ-ring, see Theorem A in § 13 in [9].

For $n = 1, 2, \ldots$ put $T^{(n)} = \mathop{\times}\limits_{i=n+1}^{\infty} T_i$ and let $\mathscr{R}^{(n)}$, $\boldsymbol{m}^{(n)}$, etc. be the analogs of $\mathscr{R}$, $\boldsymbol{m}$, etc. in $T^{(n)}$.

Define the $\mathscr{R}$-semivariation $\boldsymbol{m}\hat{}_{\mathscr{R}}$ of $\boldsymbol{m} : \mathscr{R} \to L(\boldsymbol{X})$ by the equality

$$\boldsymbol{m}\hat{}_{\mathscr{R}}(E) = \sup \{ | \sum_{j=1}^{r} \boldsymbol{m}(E_j)\, \boldsymbol{x}_j |;\ E_j \in \mathscr{R},\ E_i \cap E_j = \emptyset \text{ for } i \neq j,\ \boldsymbol{x}_j \in \boldsymbol{X},$$
$$|\boldsymbol{x}_j| \leqq 1,\ j = 1, \ldots, r,\ r = 1, 2, \ldots \}.$$

If $E \in \mathscr{R}$ is of the form $E = A \times T^{(n)}$ with $A \in \bigotimes_{i=1}^{n} \mathscr{P}_i$, then clearly

$$\boldsymbol{m}(A)\, x = \int_{T_1} \ldots \int_{T_n} \chi_A(t_1, \ldots, t_n)\, \boldsymbol{x}\, \mathrm{d}\boldsymbol{m}_n \ldots \mathrm{d}\boldsymbol{m}_1 \tag{1}$$

for each $\boldsymbol{x} \in \boldsymbol{X}$, see Theorem 1 in Part III. Since each $E \in \mathscr{R}$ is of the above form, from the definition of $\boldsymbol{m}\hat{}_{\mathscr{R}}$ and from (1) we immediately obtain that $1 = \boldsymbol{m}\hat{}_1(T_1) \leqq \leqq \boldsymbol{m}\hat{}_{\mathscr{R}}(T) \leqq 1$, i.e., $\boldsymbol{m}\hat{}_{\mathscr{R}}(T) = 1$. Using this fact, similarly as Lemma 1 in Part III, we have:

Lemma 3. *For each $\boldsymbol{x} \in \boldsymbol{X}$ let there exist a countably additive vector measure $\mu_{\boldsymbol{x}} : \bigotimes_{i=1}^{\infty} \mathscr{P}_i \to \boldsymbol{X}$ such that $\mu_{\boldsymbol{x}}(E) = \boldsymbol{m}(E)\, \boldsymbol{x}$ for each measurable rectangle E. Then the product measure $\bigotimes_{i=1}^{\infty} \boldsymbol{m}_i : \bigotimes_{i=1}^{\infty} \mathscr{P}_i \to L(\boldsymbol{X})$ exists.*

From the proof of the next lemma it is clear that its assertion remains valid for an arbitrary ring $\mathscr{R}$ and $\mathscr{P} = \mathfrak{S}(\mathscr{R})$.

Lemma 4. *Let the product measure $\boldsymbol{m} : \mathscr{P} \to L(\boldsymbol{X})$ exist. Then $\boldsymbol{m}\hat{}(E) = \boldsymbol{m}\hat{}_{\mathscr{R}}(E)$ for each $E \in \mathscr{R}$, particularly $\boldsymbol{m}\hat{}(T) = \boldsymbol{m}\hat{}_{\mathscr{R}}(T) = 1$.*

Proof. From the definitions and the Hahn-Banach Theorem we immediately obtain that $\boldsymbol{m}\hat{}(E) = \sup_{|\boldsymbol{x}^*| \leqq 1} v(\boldsymbol{x}^*\boldsymbol{m}, E)$ for each $E \in \mathscr{P}$ and $\boldsymbol{m}\hat{}_{\mathscr{R}}(E) = \sup_{|\boldsymbol{x}^*| \leqq 1} v_{\mathscr{R}}(\boldsymbol{x}^*\boldsymbol{m}, E)$ for each $E \in \mathscr{R}$. Since $\boldsymbol{m}\hat{}_{\mathscr{R}}(T) = 1$ and since by assumption $\boldsymbol{m}$ is countably additive in the strong operator topology, $v_{\mathscr{R}}(\boldsymbol{x}^*\boldsymbol{m}, \cdot)$ is countably additive and bounded by 1 for each $\boldsymbol{x}^* \in \boldsymbol{X}^*$ with $|\boldsymbol{x}^*| \leqq 1$. Thus $v(\boldsymbol{x}^*\boldsymbol{m}, E) = v_{\mathscr{R}}(\boldsymbol{x}^*\boldsymbol{m}, E)$ for each $E \in \mathscr{R}$ and each $\boldsymbol{x}^* \in \boldsymbol{X}^*$, see § 5 in Chapter I in [2]. Hence $\boldsymbol{m}\hat{}(E) = \boldsymbol{m}\hat{}_{\mathscr{R}}(E)$ for each $E \in \mathscr{R}$.

A measurable n-cylinder in T is a subset of T of the form $E = A \times T^{(n)}$, where $A \in \bigotimes_{i=1}^{n} \mathscr{P}_i$, see § 38 in [9]. The class of all measurable cylinders in T forms an algebra of subsets of T which we denote by $\mathscr{F}$. Elements of $\mathscr{F}$ are called finite dimensional measurable subsets of T. Clearly $\mathscr{R} \subset \mathscr{F} \subset \mathscr{P}$. If for each $n = 2, 3, \ldots$ the product measures $\bigotimes_{i=1}^{n} \boldsymbol{m}_i : \bigotimes_{i=1}^{n} \mathscr{P}_i \to L(\boldsymbol{X})$ exists, then clearly $\boldsymbol{m} : \mathscr{R} \to L(\boldsymbol{X})$ has a unique additive extension $\boldsymbol{m} : \mathscr{F} \to L(\boldsymbol{X})$ such that $\boldsymbol{m}$ restricted to $\bigotimes_{i=1}^{n} \mathscr{P}_i$ is equal to $\bigotimes_{i=1}^{n} \boldsymbol{m}_i$.

Lemma 5. *Let the semivariation $\boldsymbol{m}^\wedge{}_i$ be continuous on $\mathscr{P}_i$ for each $i = 1, 2, \ldots$. Then for each $n = 2, 3, \ldots$ the product measure $\bigotimes_{i=1}^{n} \boldsymbol{m}_i : \bigotimes_{i=1}^{n} \mathscr{P}_i \to L(\boldsymbol{X})$ exists, and the $\mathscr{F}$-semivariation $\boldsymbol{m}^\wedge{}_{\mathscr{F}}$ of $\boldsymbol{m} : \mathscr{F} \to L(\boldsymbol{X})$ is continuous on $\mathscr{F}$. Particularly $\boldsymbol{m} : \mathscr{F} \to L(\boldsymbol{X})$ is countably additive in the norm topology of $L(\boldsymbol{X})$.*

Proof. The first assertion of the lemma immediately follows from Theorem 3 in Part III. Suppose $\boldsymbol{m}^\wedge{}_{\mathscr{F}}$ is not continuous on $\mathscr{F}$. Then there is an $\varepsilon > 0$ and a sequence $E_k \in \mathscr{F}$, $k = 1, 2, \ldots$, such that $E_k \searrow \emptyset$ and $\boldsymbol{m}^\wedge{}_{\mathscr{F}}(E_k) > \varepsilon$ for each $k = 1, 2, \ldots$.

According to Theorem 1 in Part III we have the equality

$$\boldsymbol{m}(E)\, \boldsymbol{x} = \int_{T_1} \Big(\bigotimes_{i=2}^{\infty} \boldsymbol{m}_i\Big)(E^{t_1})\, \boldsymbol{x}\, \mathrm{d}\boldsymbol{m}_1$$

for each $E \in \mathscr{F}$ and each $\boldsymbol{x} \in \boldsymbol{X}$. Hence

$$\varepsilon < \boldsymbol{m}^\wedge{}_{\mathscr{F}}(E_k) \leqq \boldsymbol{m}^\wedge{}_1\Big(\widehat{\Big(\bigotimes_{i=2}^{\infty} \boldsymbol{m}_i\Big)}_{\mathscr{F}^{(1)}}(\mathrm{E}_k^{t_1}), T_1\Big)$$

for each $k = 1, 2, \ldots$.

We now show that for each $E \in \mathscr{F}$ the function $t_1 \to \widehat{\big(\bigotimes_{i=2}^{\infty} \boldsymbol{m}_i\big)}_{\mathscr{F}^{(1)}}(E^{t_1}, T_1)$, $t_1 \in T_1$, is $\mathscr{P}_1$-measurable. Let $E \in \mathscr{F}$ be of the form $E = A \times T^{(n_0)}$ where $A \in \bigotimes_{i=1}^{n_0} \mathscr{P}_i$, let $t_1 \in T_1$, and let a positive integer m be fixed. According to the definition of the $\mathscr{F}^{(1)}$-semivariation take $B_j \in \mathscr{F}^{(1)}$, $B_j \subset A^{t_1}$, $B_i \cap B_j = \emptyset$ for $i \neq j$ and $\boldsymbol{x}_j \in \boldsymbol{X}$, $|\boldsymbol{x}_j| \leqq 1$, $i, j = 1, \ldots, r$, so that

$$\widehat{\Big(\bigotimes_{i=2}^{\infty} \boldsymbol{m}_i\Big)}_{\mathscr{F}^{(1)}}(A^{t_1}) \leqq \Big|\sum_{j=1}^{r} \Big(\bigotimes_{i=2}^{\infty} \boldsymbol{m}_i\Big)(B_j)\, \boldsymbol{x}_j\Big| + \frac{1}{m}.$$

Since each B_j, $j = 1, \ldots, r$, is finite dimensional, there is an $n_1 \geqq n_0$ such that $B_j = C_j \times T^{(n_1)}$ with $C_j \in \bigotimes_{i=2}^{n_1} \mathscr{P}_i$ for each $j = 1, \ldots, r$. But then

$$\Big|\sum_{j=1}^{r} \Big(\bigotimes_{i=2}^{\infty} \boldsymbol{m}_i\Big)(B_j)\, \boldsymbol{x}_j\Big| \leqq \widehat{\Big(\bigotimes_{i=2}^{n_1} \boldsymbol{m}_i\Big)}(A^{t_1}) \leqq \widehat{\Big(\bigotimes_{i=2}^{\infty} \boldsymbol{m}_i\Big)}_{\mathscr{F}^{(1)}}(A^{t_1}).$$

Since the sequence $(\widehat{\bigotimes_{i=2}^{k} m_i})(A^{t_1})$, $k = 2, 3, \ldots$, is nondecreasing,

$$(\widehat{\bigotimes_{i=2}^{\infty} m_i})_{\mathscr{F}^{(1)}}(A^{t_1}) = \lim_{k\to\infty} (\widehat{\bigotimes_{i=2}^{k} m_i})(A^{t_1}) .$$

Since $t_1 \in T_1$ was arbitrary, this equality holds for each $t_1 \in T_1$. According to Theorem 6 in Part III the functions $t_1 \to (\widehat{\bigotimes_{i=2}^{k} m_i})(A^{t_1})$, $t_1 \in T_1$, are $\mathscr{P}_1$-measurable for each $k = 2, 3, \ldots$, hence their limit is also $\mathscr{P}_1$-measurable, which we wanted to show. Put

$$F_k = \left\{t_1, t_1 \in T_1, (\widehat{\bigotimes_{i=2}^{\infty} m_i})_{\mathscr{F}^{(1)}}(E_k^{t_1}) > \frac{\varepsilon}{2}\right\}.$$

Then $F_k \in \mathscr{P}_1$ by the above proved measurability, and $\varepsilon < m^\wedge{}_{\mathscr{F}}(E_k) \leqq m^\wedge{}_1^{(1)}(F_k) + \frac{1}{2}\varepsilon$ for each $k = 1, 2, \ldots$. Since F_k, $k = 1, 2, \ldots$, is a decreasing sequence of sets from $\mathscr{P}_1$ and since $m^\wedge{}_1$ is continuous on $\mathscr{P}_1$, there exists at least one point $\bar{t}_1 \in T_1$ such that $m^{\wedge(1)}(A_k^{\bar{t}_1}) \geqq \frac{1}{2}\varepsilon$ for each $k = 1, 2, \ldots$. Now similarly as in the scalar case, see the proof of Theorem B in § 38 in [9], we obtain a contradiction with $E_k \searrow \emptyset$. The lemma is proved.

Theorem 6. *Let X contain no subspace isomorphic to c_0, particularly let X be a weakly complete Banach space, see pp.* 160 *and* 161 *in* [1]. *Then the product measure $m : \mathscr{P} \to L(X)$ exists and its semivariation $m^\wedge$ is continuous on $\mathscr{P}$, hence m is countably additive in the norm topology of $L(X)$.*

Proof. According to $*$-Theorem in Section 1.1 in Part I each semivariation $m^\wedge{}_i$, $i = 1, 2, \ldots$, is continuous on $\mathscr{P}_i$. Hence by Lemma 5 the product measure $\bigotimes_{i=1}^{n} m_i : \bigotimes_{i=1}^{n} \mathscr{P}_i \to L(X)$ exists for each $n = 2, 3, \ldots$, and $m : \mathscr{F} \to L(X)$ is countably additive in the norm topology of $L(X)$. Thus $m(\cdot)\, x : \mathscr{F} \to X$ is a countably additive and bounded vector measure for each $x \in X$. Since X contains no subspace isomorphic to c_0, by Theorem on Extension on pp. 178–179 in [10] there is a countably additive extension $m(\cdot)\, x : \mathfrak{S}(\mathscr{F}) = \bigotimes_{i=1}^{\infty} \mathscr{P}_i \to X$ for each $x \in X$. Now the product measure $m : \mathscr{P} \to L(X)$ exists by Lemma 3. Since $m^\wedge(T) = 1$ by Lemma 4 and since $c_0 \not\subset X$, the semivariation $m^\wedge$ is continuous on $\mathscr{P}$ by $*$-Theorem in Section 1.1 in Part I. Thus the theorem is proved.

Remark. We note that similarly as in the scalar case, see Exercise 2 after § 38 in [9], the results of Lemmas 3, 4, 5 and of Theorem 6 remain valid if $\omega = \{1, 2, \ldots\}$ is replaced by any ordinal number α.

The next theorem, an addendum to Part III, is a generalization of the classical result of Scholium 5.3 on p. 132 in [11].

Theorem 7. *Let X be a Banach algebra with the unit e and let T and S be locally compact Hausdorff topological spaces. Let further $U : C_0(T) \to X$ be a weakly compact linear operator, and let $V : C_0(S, X) \to X$ be a bounded linear operator such that:*

a) *$V : C_0(S) \to X$ is weakly compact ($C_0(S)$ is naturally imbedded in $C_0(S, X)$ by the correspondence $f \to e . f$, $f \in C_0(S)$),*

b) *$V(x . g) = x . V(g)$ for each $x \in X$ and each $g \in C_0(S)$, and*

c) *the semivariation $l^\wedge$ of the representing measure $l : \mathscr{B}_0(S) \to L(X)$ of V, $Vg = \int_S g \, \mathrm{d}l$, $g \in C_0(S, X)$, which by* a), b), *and Theorem 2 in* [7] *exists, is continuous on $\mathscr{B}_0(S)$.*

(*Note that if $V : C_0(S, X) \to X$ is unconditionally converging, particularly weakly compact, then by Theorem* 3 *in* [7], a) *and* c) *are automatically fulfilled.*)

Then there is a unique weakly compact linear operator $W : C_0(T \times S) \to X$ such that $W(f . g) = Vg . Uf$ for each $f \in C_0(T)$ and each $g \in C_0(S)$. At the same time $|W| \leqq |U| . |V|$ ($|V|$ is considered in $L(C_0(S, X), X)$).

Proof. Let $m : \mathscr{B}_0(T) \to X$ be the representing measure of $U : C_0(T) \to X$, see (A) before Theorem 2 in [7]. Then the product measure $l \otimes m : \mathscr{B}_0(T \times S) \to X$ exists by c) and Theorem 3 in Part III, and $\|l \otimes m\| (T \times S) \leqq \|m\| (T) . l^\wedge(S) = |U| . |V|$ by Theorem 2 in Part III and Theorem 2 in [7]. For $h \in C_0(T \times S)$ put $Wh = \int_{T \times S} h \, \mathrm{d}(l \otimes m)$. Then $W : C_0(T \times S) \to X$ is a weakly compact linear operator, and $|W| = \|l \otimes m\| (T \times S) \leqq |U| . |V|$ by(A) before Theorem 2 in [7]. According to Theorem 5 in Part III, $\int_{T \times S} h \, \mathrm{d}(l \otimes m) = \int_S \int_T h(\cdot, s) \, \mathrm{d}m \, \mathrm{d}l$ for each $h \in C_0(T \times S)$, hence $W(h . g) = Vg . Uf$ for each $f \in C_0(T)$ and each $g \in C_0(S)$. The uniqueness of W immediately follows from the fact that the set of all finite linear combinations of functions of the form $f . g$ with $f \in C_0(T)$ and $g \in C_0(S)$ is dense in $C_0(T \times S)$ by the Stone-Weierstrass Theorem, see Lemma 5.2.2 on *p.* 132 in [11].

3. INTEGRATION BY SUBSTITUTION

First we generalize the classical result on integrals with respect to a transformed measure, see [9, Theorem C in § 39]. Let $\mathscr{Q}$ be a δ-ring of subsets of a non empty set S, and let $\varphi : T \to S$ be a $(\mathscr{P}, \mathscr{Q})$-measurable transformation, i.e., let $\varphi^{-1}(\mathscr{Q}) \subset \mathscr{P}$. For $B \in \mathscr{Q}$ put $(m\varphi^{-1})(B) = m(\varphi^{-1}(B))$. Then clearly $m\varphi^{-1} : \mathscr{Q} \to L(X, Y)$ is an operator valued measure countably additive in the strong operator topology with the finite semivariation $\widehat{m\varphi^{-1}}$ on $\mathscr{Q}$, and $\int_B f \mathrm{d}(m\varphi^{-1}) = \int_{\varphi^{-1}(B)} f(\varphi(\cdot)) \, \mathrm{d}m$ for each $\mathscr{Q}$-simple function $f : S \to X$ and for each $B \in \mathfrak{S}(\mathscr{Q})$. Moreover, we have:

Theorem 8. *Let $f : S \to X$ be a $\mathcal{Q}$-measurable function and let $\varphi : T \to S$ be a $(\mathcal{P}, \mathcal{Q})$-measurable transformation. Then $f(\varphi(\cdot)) : T \to X$ is $\varphi^{-1}(\mathcal{Q})$-measurable. Further, f is integrable with respect to $m\varphi^{-1} : \mathcal{Q} \to L(X, Y)$ if and only if $f(\varphi(\cdot))$ is integrable with respect to $m : \varphi^{-1}(\mathcal{Q}) \to L(X, Y)$. In that case*

$$(1) \qquad \int_B f \, \mathrm{d}(m\varphi^{-1}) = \int_{\varphi^{-1}(B)} f(\varphi(\cdot)) \, \mathrm{d}m$$

for each $B \in \mathfrak{S}(\mathcal{Q})$.

Proof. Let $f_n : S \to X$, $n = 1, 2, \ldots$, be a sequence of $\mathcal{Q}$-simple functions such that $f_n(s) \to f(s)$ for each $s \in S$. Then $f_n(\varphi(\cdot))$, $n = 1, 2, \ldots$, are clearly $\varphi^{-1}(\mathcal{Q})$-simple functions, and $f_n(\varphi(t)) \to f(\varphi(t))$ for each $t \in T$. Hence, by definition, $f(\varphi(\cdot)) : T \to X$ is $\varphi^{-1}(\mathcal{Q})$-measurable.

Let f be integrable with respect to $m\varphi^{-1}$. Then, according to Theorem 7 in Part I, we may suppose that $\int_B f_n \, \mathrm{d}(m\varphi^{-1}) \to \int_B f \, \mathrm{d}(m\varphi^{-1})$ for each $B \in \mathfrak{S}(\mathcal{Q})$. Since $\int_B f_n \, \mathrm{d}(m\varphi^{-1}) = \int_{\varphi^{-1}(B)} f_n(\varphi(\cdot)) \, \mathrm{d}m$ for each $n = 1, 2, \ldots$ and $B \in \mathfrak{S}(\mathcal{Q})$, $f(\varphi(\cdot))$ is integrable with respect to $m : \varphi^{-1}(\mathcal{Q}) \to L(X, Y)$ and $\int_{\varphi^{-1}(B)} f(\varphi(\cdot)) \, \mathrm{d}m = \int_B f \, \mathrm{d}(m\varphi^{-1})$ for each $B \in \mathfrak{S}(\mathcal{Q})$ by Theorem 7 in Part I.

Let now $f(\varphi(\cdot))$ be integrable with respect to $m : \varphi^{-1}(\mathcal{Q}) \to L(X, Y)$, and let us have the sequence f_n, $n = 1, 2, \ldots$, from above. For $n = 1, 2, \ldots$ and $B \in \mathfrak{S}(\mathcal{Q})$ put

$$\mu_n(B) = \sup_{\substack{D \in \mathfrak{S}(\mathcal{Q}) \\ D \subset B}} \left| \int_D f_n \, \mathrm{d}(m\varphi^{-1}) \right|, \text{ and let } \mu(B) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\mu_n(B)}{1 + \mu_n(S)}, \quad B \in \mathfrak{S}(\mathcal{Q}).$$

Then $\mu : \mathfrak{S}(\mathcal{Q}) \to \langle 0, 1)$ is monotone, subadditive, continuous, and $N \in \mathfrak{S}(\mathcal{Q})$, $\mu(N) = 0 \Rightarrow \mu_n(N) = 0$ for each $n = 1, 2, \ldots$. Put $F = \bigcup_{n=1}^{\infty} \{s,\ s \in S,\ f_n(s) \neq 0\}$. Then $F \in \mathfrak{S}(\mathcal{Q})$, and by the Egoroff-Lusin Theorem, see Section 1.4 in Part I, there is an $N \in \mathfrak{S}(\mathcal{Q})$, $N \subset F$, and a sequence $F_k \in \mathcal{Q}$, $k = 1, 2, \ldots$, such that $\mu(N) = 0$, $F_k \nearrow F - N$, and on each F_k, $k = 1, 2, \ldots$, the sequence f_n, $n = 1, 2, \ldots$, converges uniformly to f. Thus $f \,.\, \chi_N$, and $f \,.\, \chi_{F_k}$, $k = 1, 2, \ldots$, are integrable with respect to $m\varphi^{-1}$, see Theorem 9 in Part I. Hence

$$\int_{\varphi^{-1}(B \cap N)} f(\varphi(\cdot)) \, \mathrm{d}m = \int_B f \,.\, \chi_N \, \mathrm{d}(m\varphi^{-1}) = 0$$

for each $B \in \mathfrak{S}(\mathcal{Q})$ and

$$\int_B f \,.\, \chi_{F_k} \, \mathrm{d}(m\varphi^{-1}) = \int_{\varphi^{-1}(B)} f(\varphi(\cdot)) \,.\, \chi_{\varphi^{-1}(F_k)} \, \mathrm{d}m$$

for each $k = 1, 2, \ldots$ and each $B \in \mathfrak{S}(\mathcal{Q})$ by the paragraph before. Since by assumption $f(\varphi(\cdot))$ is integrable with respect to $m : \varphi^{-1}(\mathcal{Q}) \to L(X, Y)$,

$$\int_B f \cdot \chi_{F_k} \, \mathrm{d}(m\varphi^{-1}) = \int_{\varphi^{-1}(B)} f(\varphi(\cdot)) \cdot \chi_{\varphi^{-1}(F_k)} \, \mathrm{d}m \to$$

$$\to \int_{\varphi^{-1}(B)} f(\varphi(\cdot)) \cdot \chi_{\varphi^{-1}(F-N)} \, \mathrm{d}m = \int_{\varphi^{-1}(B)} f(\varphi(\cdot)) \, \mathrm{d}m$$

for each $B \in \mathfrak{S}(\mathcal{Q})$ by the countable additivity of the integral $\int f(\varphi(\cdot)) \, \mathrm{d}m$ on $\varphi^{-1}(\mathcal{Q})$. Since $f \cdot \chi_{F_k} \to f \cdot \chi_{F-N}$, $f \cdot \chi_{F-N}$ is integrable with respect to $m\varphi^{-1}$ and

$$\int_B f \cdot \chi_{F-N} \, \mathrm{d}(m\varphi^{-1}) = \int_{\varphi^{-1}(B)} f(\varphi(\cdot)) \, \mathrm{d}m$$

for each $B \in \mathfrak{S}(\mathcal{Q})$ according to Theorem 16 in Part I. Hence $f = f \cdot \chi_{F-N} + f \cdot \chi_N$ is integrable with respect to $m\varphi^{-1}$, and (1) holds. The theorem is proved.

In what follows, $f : T \to X$ is a given $\mathcal{P}$-measurable function, $F = \{t;\, t \in T,\, f(t) \neq 0\}$, $\mathcal{P}_f = \mathcal{P} \cap F$, and $m : \mathcal{P} \to L(X, Y)$ is an operator valued measure countably additive in the strong operator topology with the finite semivariation $m^\wedge$ on $\mathcal{P}_f$.

Theorem 9. *Let $n : \mathcal{P}_f \to L(Z, Y)$ be an operator valued measure countably additive in the strong operator topology with the finite semivariation $n^\wedge$ on $\mathcal{P}_f$, let $g : T \to L(X, Z)$ be such that $gx \cdot \chi_E$ is essentially integrable with respect to n, see Definition 2 in Part III, for each $x \in X$ and each $E \in \mathcal{P}_f$, and let $m(E)\,x = \int_E gx \cdot \chi_E \, \mathrm{d}n = \int_E g(\cdot)\, x \, \mathrm{d}n$ for each $x \in X$ and each $E \in \mathcal{P}_f$. Then $m^\wedge(E) \leqq n^\wedge(g, E)$ for each $E \in \mathfrak{S}(\mathcal{P})$, and the following conditions are equivalent:*

a) *f is integrable with respect to m, and*

b) *$gf((gf)(t) = g(t) f(t))$ is integrable with respect to n. If they hold, then*

$$\int_E f \, \mathrm{d}m = \int_E gf \, \mathrm{d}n \quad \text{for each} \quad E \in \mathfrak{S}(\mathcal{P}) \, .$$

Proof. The inequality $m^\wedge(E) \leqq n^\wedge(g, E)$, $E \in \mathfrak{S}(\mathcal{P})$, immediately follows from Corollary of Theorem 2 in Part II.

a) $\Rightarrow$ b) and (1). Suppose a), and according to Theorem 7 in Part I take a sequence of $\mathcal{P}_f$-simple functions $f_n : T \to X$, $n = 1, 2, \ldots$, such that $f_n(t) \to f(t)$ for each $t \in T$ and $\int_E f_n \, \mathrm{d}m \to \int_E f \, \mathrm{d}m$ for each $E \in \mathfrak{S}(\mathcal{P}_f)$, see Theorem 7 in Part I. Then clearly $(gf_n)(t) \to (gf)(t)$ for each $t \in T$ and $\int_E gf_n \, \mathrm{d}n = \int_E f_n \, \mathrm{d}m \to \int_E f \, \mathrm{d}m$ for each $E \in \mathfrak{S}(\mathcal{P})$. Hence b) and (1) immediately follow from Theorem 16 in Part I.

b) $\Rightarrow$ a). Suppose b), and let $f_n : T \to X$, $n = 1, 2, \ldots$, be a sequence of $\mathcal{P}_f$-simple functions such that $f_n(t) \to f(t)$ for each $t \in T$. Then clearly $(gf_n)(t) \to (gf)(t)$ for each $t \in T$ and $\int_E f_n \, \mathrm{d}m = \int_E gf_n \, \mathrm{d}n$ for each $n = 1, 2, \ldots$ and each $E \in \mathfrak{S}(\mathcal{P})$. For $E \in \mathfrak{S}(\mathcal{P})$ put

$$\mu(E) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\mu_n(E)}{1 + \mu_n(T)}, \quad \text{where} \quad \mu_n(E) = \underset{F \in \mathfrak{S}(\mathscr{P}), F \subset E}{\text{supremum}} \left| \int_F f_n \, \mathrm{d}\boldsymbol{m} \right|, \quad n = 1, 2, \ldots .$$

Then $\mu : \mathfrak{S}(\mathscr{P}) \to \langle 0, 1)$ is monotone, subadditive, continuous, and $N \in \mathfrak{S}(\mathscr{P})$, $\mu(N) = 0 \Rightarrow \mu_n(N) = 0$ for each $n = 1, 2, \ldots$. Hence by the Egoroff-Lusin Theorem, see Section 1.4 in Part I, which remains valid for such μ and for essentially measurable functions, there is a set $N \in \mathfrak{S}(\mathscr{P}_f)$ with $\mu(N) = 0$ and a sequence $H_k \in \mathscr{P}_f$, $k = 1, 2, \ldots$, such that $H_k \nearrow F - N$, and on each H_k, $k = 1, 2, \ldots$, the sequence f_n, $n = 1, 2, \ldots$, converges uniformly to $\boldsymbol{f}$ and the sequence $\boldsymbol{g}f_n$, $n = 1, 2, \ldots$, converges uniformly to $\boldsymbol{g}f$. Since $\boldsymbol{m}^\wedge(H_k) + \boldsymbol{n}^\wedge(H_k) < +\infty$ by assumption, by Theorems 14 and 16 from Part I we have

$$\int_{E \cap H_k} f \, \mathrm{d}\boldsymbol{m} = \lim_{n \to \infty} \int_{E \cap H_k} f_n \, \mathrm{d}\boldsymbol{m} = \lim_{n \to \infty} \int_{E \cap H_k} \boldsymbol{g} f_n \, \mathrm{d}\boldsymbol{n} = \int_{E \cap H_k} \boldsymbol{g} f \, \mathrm{d}\boldsymbol{n}$$

for each $E \in \mathfrak{S}(\mathscr{P})$ and each $k = 1, 2, \ldots$. Since $\int_{E \cap N} f_n \, \mathrm{d}\boldsymbol{m} = \int_{E \cap N} \boldsymbol{g} f_n \, \mathrm{d}\boldsymbol{n} = 0$ for each $n = 1, 2, \ldots$ and each $E \in \mathfrak{S}(\mathscr{P})$, $\int_{E \cap N} f \, \mathrm{d}\boldsymbol{m} = \int_{E \cap N} \boldsymbol{g} f \, \mathrm{d}\boldsymbol{n} = 0$ for each $E \in \mathfrak{S}(\mathscr{P})$. Since $(f \cdot \chi_{H_k \cup N})(t) \to f(t)$ for each $t \in T$ and since

$$\int_{E \cap (H_k \cup N)} f \, \mathrm{d}\boldsymbol{m} = \int_{E \cap (H_k \cup N)} \boldsymbol{g} f \, \mathrm{d}\boldsymbol{n} \to \int_E \boldsymbol{g} f \, \mathrm{d}\boldsymbol{n}$$

for each $E \in \mathfrak{S}(\mathscr{P})$ by the integrability of $\boldsymbol{g}f$ with respect to $\boldsymbol{n}$, $\boldsymbol{f}$ is integrable with respect to $\boldsymbol{m}$ and (1) holds in virtue of Theorem 16 in Part I. The theorem is proved.

When $\boldsymbol{X}$ is the space of real or complex numbers, then the next corollary is a classical result on the integration by substitution, see [9, § 32, Theorem A].

Corollary. *Let $\boldsymbol{X} = \boldsymbol{Y}$ be a Banach algebra and let $\boldsymbol{n} : \mathscr{P}_f \to L(\boldsymbol{X})$ be an operator valued measure countably additive in the strong operator topology with the finite semivariation $\boldsymbol{n}^\wedge$ on $\mathscr{P}_f$. Let further $\boldsymbol{g} : T \to \boldsymbol{X}$ be such that $\boldsymbol{g} \cdot \chi_E$ is essentially integrable with respect to $\boldsymbol{n}$ for each $E \in \mathscr{P}_f$, and let $\boldsymbol{m}(E) = \int_E \boldsymbol{g} \, \mathrm{d}\boldsymbol{n}$ for each $E \in \mathscr{P}_f$. Then $\boldsymbol{m}^\wedge(E) \leqq \boldsymbol{n}^\wedge(\boldsymbol{g}, E)$ for each $E \in \mathfrak{S}(\mathscr{P}_f)$, and the following conditions are equivalent:*

a) *$\boldsymbol{f}$ is integrable with respect to $\boldsymbol{m}$, and*

b) *$\boldsymbol{g}f$ is essentially integrable with respect to $\boldsymbol{n}$.*

If they hold, then

$$\int_E f \, \mathrm{d}\boldsymbol{m} = \int_E f \, \mathrm{d}\left(\int_\bullet \boldsymbol{g} \, \mathrm{d}\boldsymbol{n}\right) = \int_E \boldsymbol{g} f \, \mathrm{d}\boldsymbol{n} \quad \textit{for each} \quad E \in \mathfrak{S}(\mathscr{P}) .$$

Proof. Theorem 7 in Part I immediately yields that $\boldsymbol{g}(\cdot)\, \boldsymbol{x} \cdot \chi_E$ is essentially integrable with respect to $\boldsymbol{n}$ and $\boldsymbol{m}(E)\, \boldsymbol{x} = \int_E \boldsymbol{g}(\cdot)\, \boldsymbol{x} \, \mathrm{d}\boldsymbol{n}$ for each $\boldsymbol{x} \in \boldsymbol{X}$ and each $E \in \mathscr{P}_f$. Now the corollary directly follows from the theorem.

Similarly as Theorem 9 one can prove

Theorem 10. *Let $X \subset L(Z_1, Z)$ and let $\boldsymbol{n} : \mathscr{P}_f \to L(Z, Y)$ be an operator valued measure countably additive in the strong operator topology with the finite semi-variation $\boldsymbol{n}^\wedge$ on $\mathscr{P}_f$. Let further $\boldsymbol{g} : T \to Z_1$ be such that $x\boldsymbol{g}(\cdot) . \chi_E$ is essentially integrable with respect to $\boldsymbol{n}$ for each $x \in X$ and each $E \in \mathscr{P}_f$, and let $\boldsymbol{m}(E)\, x = \int_E x\boldsymbol{g}(\cdot)\, \mathrm{d}\boldsymbol{n}$ for each $E \in \mathscr{P}_f$ and each $x \in X$. Then $\boldsymbol{m}^\wedge(E) \leqq \boldsymbol{n}^\wedge(\boldsymbol{g}, E)$ for each $E \in \mathfrak{S}(\mathscr{P}_f)$, and the following conditions are equivalent:*

a) *$\boldsymbol{f}$ is integrable with respect to $\boldsymbol{m}$, and*

b) *$\boldsymbol{fg}$ is essentially integrable with respect to $\boldsymbol{n}$.*

If they hold, then

$$\int_E \boldsymbol{f}\, \mathrm{d}\boldsymbol{m} = \int_E \boldsymbol{fg}\, \mathrm{d}\boldsymbol{n} \quad \text{for each} \quad E \in \mathfrak{S}(\mathscr{P})\,.$$

Theorem 11. *Let $\boldsymbol{n} : \mathscr{P}_f \to L(X, L(Z, Y))$ be an operator valued measure countably additive in the strong operator topology, let*

$$\boldsymbol{n}^{\wedge\wedge}(E) = \sup \{ \big| \sum_{i=1}^{r} \sum_{j=1}^{s} \boldsymbol{n}(E_{i,j})\, x_i z_j \big|,\ x_i \in X,\ z_j \in Z,\ |x_i| \leqq 1\,,$$
$$|z_j| \leqq 1,\ E_{i,j} \in E \cap \mathscr{P}_f,\ E_{i,j} \cap E_{i_1,j_1} = \emptyset \text{ for } (i,j) \neq (i_1, j_1)\,,$$
$$i, i_1 = 1, \ldots, r,\ j, j_1 = 1, \ldots, s,\ r, s = 1, 2, \ldots \} < +\infty \quad \text{for each} \quad E \in \mathscr{P}_f\,,$$

and let $\boldsymbol{f} . \chi_E$ be integrable with respect to $\boldsymbol{n}$ for each $E \in \mathscr{P}_f$. Let further $\boldsymbol{g} : T \to Z$ be $\mathscr{P}$-measurable, let $\boldsymbol{g} . \chi_E$ be essentially integrable with respect to $\boldsymbol{n}(\cdot)\, x : \mathscr{P}_f \to L(Z, Y)$ for each $E \in \mathscr{P}_f$ and each $x \in X$, and let $\boldsymbol{m}(E)\, x = \int_E \boldsymbol{g}\, \mathrm{d}(\boldsymbol{n}(\cdot)\, x)$ for each $E \in \mathscr{P}_f$ and each $x \in X$. Then $\boldsymbol{f}$ is integrable with respect to $\boldsymbol{m}$ if and only if $\boldsymbol{g}$ is integrable with respect to $\int_\bullet \boldsymbol{f}\, \mathrm{d}\boldsymbol{n} : \mathscr{P}_f \to L(Z, Y)$. In that case

$$\int_E \boldsymbol{f}\, \mathrm{d}\boldsymbol{m} = \int_E \boldsymbol{g}\, \mathrm{d}\left(\int_\bullet \boldsymbol{f}\, \mathrm{d}\boldsymbol{n} \right) \quad \text{for each} \quad E \in \mathfrak{S}(\mathscr{P})\,.$$

Remark. We do not suppose that the semivariation $\boldsymbol{l}^\wedge$ of $\boldsymbol{l}$, $\boldsymbol{l}(E) = \int_E \boldsymbol{f}\, \mathrm{d}\boldsymbol{n}$, $E \in \mathscr{P}_f$, is finite on $\mathscr{P}_f$. Nevertheless, since $\boldsymbol{n}^{\wedge\wedge}$ is finite on $\mathscr{P}_f$, we show that $\boldsymbol{l}^\wedge$ is finite on the δ-ring $\mathscr{P}'_f = \bigcup_{k=1}^{\infty} \mathscr{P}_f \cap \{t \colon t \in T,\ |\boldsymbol{f}(t)| \leqq k\}$. Since clearly $\mathfrak{S}(\mathscr{P}'_f) = \mathfrak{S}(\mathscr{P}_f)$, we may suppose without loss of generality that $\boldsymbol{l}^\wedge$ is finite on $\mathscr{P}_f$. To see that $\boldsymbol{l}^\wedge$ is finite on $\mathscr{P}'_f$, let $E' \in \mathscr{P}'_f$ be of the form $E' = E \cap \{t \colon t \in T,\ |\boldsymbol{f}(t)| \leqq k\}$, where $E \in \mathscr{P}_f$. Take a sequence $\boldsymbol{f}_n$, $n = 1, 2, \ldots$, of $\mathscr{P}_f$-simple functions such that $\boldsymbol{f}_n(t) \to \boldsymbol{f}(t)$ and $|\boldsymbol{f}_n(t)| \nearrow |\boldsymbol{f}(t)|$ for each $t \in T$, see Section 1.2 in Part I. By assumption, $\boldsymbol{f} . \chi_E$ is integrable with respect to $\boldsymbol{n}$, hence by the proof of Theorem 15 in Part I there is a subsequence $\boldsymbol{f}_{n_k}$, $k = 1, 2, \ldots$, a set $N \in \mathfrak{S}(\mathscr{P}_f)$ and a sequence $F_k \in \mathscr{P}_f$, $k = 1, 2, \ldots$, such that $(F_k \cup N) \cap E \nearrow E$ and $\int_G \boldsymbol{f}_{n_k} . \chi_{F_k \cup N}\, \mathrm{d}\boldsymbol{n} \to \int_G \boldsymbol{f}\, \mathrm{d}\boldsymbol{n}$ uniformly with respect to $G \in \mathscr{P}_f \cap E$. Hence $\boldsymbol{l}^\wedge(E') \leqq k \,.\, \boldsymbol{n}^{\wedge\wedge}(E) < +\infty$.

Proof. For $k = 1, 2, \ldots$, put $G_k = \{t: t \in T, |g(t)| \leqq k\} \cap F \in \mathfrak{S}(\mathscr{P}_f)$. Let $f_n : T \to X$, $n = 1, 2, \ldots$, be $\mathscr{P}_f$-simple functions and let $f_n(t) \to f(t)$ for each $t \in T$. For $n = 1, 2, \ldots$ and $E \in \mathfrak{S}(\mathscr{P}_f)$ put

$$\mu_n(E) = \sup_{H \in \mathfrak{S}(\mathscr{P}_f), H \subset E} \left| \int_H f_n \, \mathrm{d}m \right| + \sup_{H \in \mathfrak{S}(\mathscr{P}_f), H \subset E} \left| \int_H f_n \, \mathrm{d}n \right|$$

and let

$$\mu(E) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\mu_n(E)}{1 + \mu_n(T)},$$

$E \in \mathfrak{S}(\mathscr{P}_f)$. Then $\mu : \mathfrak{S}(\mathscr{P}_f) \to \langle 0, 1)$ is monotone, subadditive, continuous, and $N \in \mathfrak{S}(\mathscr{P}_f)$, $\mu(N) = 0$ implies that $f \cdot \chi_N$ is integrable with respect to m and n and $|\int_N f \, \mathrm{d}m| + |\int_N f \, \mathrm{d}n| = 0$. Thus by the Egoroff-Lusin Theorem, see Section 1.4 in Part I, there is a set $N \in \mathfrak{S}(\mathscr{P}_f)$ and a sequence $F_k \in \mathscr{P}_f$, $k = 1, 2, \ldots$, such that $\mu(N) = 0$, $F_k \nearrow F - N$, and on each F_k, $k = 1, 2, \ldots$, the sequence f_n, $n = 1, 2, \ldots$, converges uniformly to f. Hence, in virtue of Theorem 9 in Part I, $f \cdot \chi_{G_k \cap F_k}$ is integrable with respect to m and n, and

$$\int_{E \cap G_k \cap F_k} f_n \, \mathrm{d}m \to \int_{E \cap G_k \cap F_k} f \, \mathrm{d}m$$

and

$$l_n(E \cap G_k \cap F_k) = \int_{E \cap G_k \cap F_k} f_n \, \mathrm{d}n \to \int_{E \cap G_k \cap F_k} f \, \mathrm{d}n = l(E \cap G_k \cap F_k)$$

both uniformly with respect to $E \in \mathfrak{S}(\mathscr{P})$ for each $k = 1, 2, \ldots$.

Let $k \in \{1, 2, \ldots\}$ be fixed. Since $n^{\wedge\wedge}(G_k \cap F_k) < +\infty$,

$$\widehat{(l_n - l_i)}(G_k \cap F_k) = \sup \left\{ \left| \left(\sum_{j=1}^{s} \int_{E_j} (f_n - f_i) \, \mathrm{d}n \right) z_j \right|, \quad z_j \in Z, \quad |z_j| \leqq 1, \right.$$

$$\left. E_j \in \mathscr{P}_f \cap G_k \cap F_k, \; E_j \cap E_r = \emptyset \text{ for } j \neq r, \; j, r = 1, \ldots, s; \; s = 1, 2, \ldots \right\} \leqq$$

$$\leqq \|f_n - f_i\|_{G_k \cap F_k} \cdot n^{\wedge\wedge}(G_k \cap F_k) \to 0 \quad \text{as} \quad n, i \to \infty .$$

But $l_n(E \cap G_k \cap F_k) \to l(E \cap G_k \cap F_k)$, hence $\widehat{(l_n - l)}(G_k \cap F_k) \to 0$. Thus

$$\widehat{(l_n - l)}(g, G_k \cap F_k) \leqq \|g\|_{G_k \cap F_k} \cdot \widehat{(l_n - l)}(G_k \cap F_k) \leqq k \cdot \widehat{(l_n - l)}(G_k \cap F_k) \to 0 .$$

Hence

$$\int_{E \cap G_k \cap F_k} g \, \mathrm{d}l_n \to \int_{E \cap G_k \cap F_k} g \, \mathrm{d}l ,$$

and therefore

$$\int_{E \cap G_k \cap F_k} f \, \mathrm{d}m = \int_{E \cap G_k \cap F_k} g \, \mathrm{d}l$$

for each $E \in \mathfrak{S}(\mathscr{P}_f)$. Put $f'_k = f \cdot \chi_{G_k \cap F_k}$ and $g'_k = g \cdot \chi_{G_k \cap F_k}$. Then f'_k is integrable with respect to m and g'_k is integrable with respect to l. Since $f'_k(t) \to f(t)$ m a.e. and $g'_k(t) \to g(t)$ l a.e., Theorem 16 in Part I implies that f is integrable with respect to m and $\int_E f'_k \, dm \to \int_E f \, dm$ for each $E \in \mathfrak{S}(\mathscr{P}) \Leftrightarrow g$ is integrable with respect to l and $\int_E g'_k \, dl \to \int_E g \, dl$ for each $E \in \mathfrak{S}(\mathscr{P})$. The theorem is proved.

Our last theorem is

Theorem 12. *Let $Z \subset L(X, Z)$ and let $n : \mathscr{P}_f \to L(L(X, Z), L(X, Y))$ be an operator valued measure countably additive in the strong operator topology with the finite semivariation $n^\wedge$ on $\mathscr{P}$. Let further $g : T \to L(X, Z)$ be $\mathscr{P}_f$-measurable, let $g \cdot \chi_E$ be integrable with respect to n for each $E \in \mathscr{P}_f$ and let $m(E) = \int_E g \, dn$ for each $E \in \mathscr{P}_f$. Then f is integrable with respect to m if and only if gf is integrable with respect to n. In that case*

$$\int_E f \, dm = \int_E f \, d\left(\int_\bullet g \, dn\right) = \int_E gf \, dn$$

for each $E \in \mathfrak{S}(\mathscr{P})$.

Proof. Similarly as in Theorem 11 we may suppose that the semivariation $l^\wedge$ of l, $l(E) = \int_E g \, dn$, $E \in \mathscr{P}_f$, is finite on $\mathscr{P}_f$, see Remark after Theorem 11 above.

Let $f_n : T \to X$ and $g_n : T \to L(X, Z)$, $n = 1, 2, \ldots$, be $\mathscr{P}_f$-simple functions such that $f_n(t) \to f(t)$ and $g_n(t) \to g(t)$ for each $t \in T$. Then $g_n f_n : T \to Z$, $n = 1, 2, \ldots$, are $\mathscr{P}_f$-simple, and $(g_n f_n)(t) \to (gf)(t)$ for each $t \in T$. For $E \in \mathfrak{S}(\mathscr{P})$ and $n = 1, 2, \ldots$ put

$$\mu_n(E) = \sup_{H \in \mathfrak{S}(\mathscr{P}), H \subset E} \left| \int_H f_n \, dm \right| + \sup_{H \in \mathfrak{S}(\mathscr{P}), H \subset E} \left| \int_H g_n \, dn \right| + \sup_{H \in \mathfrak{S}(\mathscr{P}), H \subset E} \left| \int_H g_n f_n \, dm \right|,$$

and let

$$\mu(E) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\mu_n(E)}{1 + \mu_n(T)},$$

$E \in \mathfrak{S}(\mathscr{P})$. Then $\mu : \mathfrak{S}(\mathscr{P}) \to \langle 0, 1)$ is monotone, subadditive, continuous, and $N \in \mathfrak{S}(\mathscr{P})$, $\mu(N) = 0 \Rightarrow \mu_n(N) = 0$ for each $n = 1, 2, \ldots$. Hence by the Egoroff-Lusin Theorem, see Section 1.4 in Part I, there is a set $N \in \mathfrak{S}(\mathscr{P})$, $N \subset F$, with $\mu(N) = 0$, and a sequence $F_k \in \mathscr{P}_f$, $k = 1, 2, \ldots$, such that $F_k \nearrow F - N$, and on each F_k, $k = 1, 2, \ldots$, the sequence f_n, $n = 1, 2, \ldots$, converges uniformly to f and simultaneously the sequence g_n, $n = 1, 2, \ldots$, converges uniformly to g. Since all f_n and g_n, $n = 1, 2, \ldots$, being $\mathscr{P}$-simple functions, are bounded, $\sup_n \|f_n\|_{F_k} + \sup_n \|g_n\|_{F_k} < +\infty$ for each $k = 1, 2, \ldots$. Hence $g_n f_n$, $n = 1, 2, \ldots$, converges uniformly to gf on each F_k, $k = 1, 2, \ldots$. Since $m^\wedge(F_k) + n^\wedge(F_k) < +\infty$ for each $k = 1, 2, \ldots$, in virtue of Theorem 9 in Part I, $f \cdot \chi_{F_k}$ and $gf \cdot \chi_{F_k}$, $k = 1, 2, \ldots$, are integrable with respect to m and n, respectively,

$$\int_E f_n \cdot \chi_{F_k} \, \mathrm{d}\boldsymbol{m} \to \int_E f \cdot \chi_{F_k} \, \mathrm{d}\boldsymbol{m} , \quad \int_E g_n \cdot \chi_{F_k} \, \mathrm{d}\boldsymbol{n} \to \int_E g \cdot \chi_{F_k} \, \mathrm{d}\boldsymbol{n} ,$$

and

$$\int_E g_n f_n \cdot \chi_{F_k} \, \mathrm{d}\boldsymbol{n} \to \int_E gf \cdot \chi_{F_k} \, \mathrm{d}\boldsymbol{n} \quad \text{as} \quad n \to \infty ,$$

uniformly with respect to $E \in \mathfrak{S}(\mathscr{P})$, for each $k = 1, 2, \ldots$.

Since $(f \cdot \chi_{F_k \cup N})(t) \to f(t)$ and $(gf \cdot \chi_{F_k \cup N})(t) \to (gf)(t)$ for each $t \in T$, according to Theorem 16 in Part I it is enough to prove that $\int_E f \cdot \chi_{F_k} \, \mathrm{d}\boldsymbol{m} = \int_E gf \cdot \chi_{F_k} \, \mathrm{d}\boldsymbol{n}$ for each $E \in \mathfrak{S}(\mathscr{P})$ and each $k = 1, 2, \ldots$.

Let $k \in \{1, 2, \ldots\}$ be fixed, and for $E \in \mathfrak{S}(\mathscr{P})$ put $\boldsymbol{m}_n(E) = \int_E g_n \cdot \chi_{F_k} \, \mathrm{d}\boldsymbol{n}$. Then $\boldsymbol{m}_n(E) \to \int_E g \cdot \chi_{F_k} \, \mathrm{d}\boldsymbol{n} = \boldsymbol{m}(E \cap F_k)$ for each $E \in \mathfrak{S}(\mathscr{P})$, $\sup_n \boldsymbol{m}\hat{}_n(E) \leqq \sup_n \|g_n\|_{F_k} \cdot$ $\cdot \, \boldsymbol{m}\hat{}(F_k) < +\infty$, for each $E \in \mathfrak{S}(\mathscr{P})$, and

$$\int_E f_n \cdot \chi_{F_k} \, \mathrm{d}\boldsymbol{m}_n = \int_E g_n f_n \cdot \chi_{F_k} \, \mathrm{d}\boldsymbol{n} \to \int_E gf \cdot \chi_{F_k} \, \mathrm{d}\boldsymbol{n}$$

for each $E \in \mathfrak{S}(\mathscr{P})$. Since $(f_n \cdot \chi_{F_k})(t) \to (f \cdot \chi_{F_k})(t)$ for each $t \in T$, $\int_E f \cdot \chi_{F_k} \, \mathrm{d}\boldsymbol{m} =$ $= \int_E gf \cdot \chi_{F_k} \, \mathrm{d}\boldsymbol{n}$ for each $E \in \mathfrak{S}(\mathscr{P})$ by Theorem 1 in Part IV. The theorem is proved.

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