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Jiří Čížek

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ON TAUBERIAN CONSTANTS FOR THE (D, λ) SUMMABILITY

JIŘÍ ČÍŽEK, Plzeň

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1. INTRODUCTION

1.1. A Tauberian theorem is such a theorem which says that if any series of complex numbers $\sum_{n=1}^{\infty} a_n$ is summable by a certain method and, moreover, satisfies a certain condition (usually on the terms a_n), then it is convergent. This condition is called a *Tauberian condition* (cf. [4], § 7.1).

In 1944 H. Hadwiger [3] proved that there exists a constant M_1 such that

$$\limsup_{n\to\infty} \left| \sum_{k=1}^{\infty} a_k x_n^k - \sum_{k=1}^{n} a_k \right| \leqq M_1 \limsup_{n\to\infty} n|a_n| ,$$

where $x_n = 1 - 1/n$ and $\sum_{k=1}^{\infty} a_k$ is an arbitrary series of complex numbers. This constant was called a *Tauberian constant*, because the well-known first Tauber's theorem is a consequence of Hadwiger's result. Further, WINTNER [9] proved existence of a constant M_2 such that for an arbitrary series of complex numbers $\sum_{k=1}^{\infty} a_k$ and $x_n = 1 - 1/n$,

$$\limsup_{n\to\infty} \left| \sum_{k=1}^{\infty} a_k x_n^k - \sum_{k=1}^{n} a_k \right| \leqq M_2 \limsup_{n\to\infty} \frac{1}{n} \left| \sum_{k=1}^{n} k a_k \right| .$$

This fact, by the way, implies the second Tauber's theorem for the Abel summability method. HARTMAN [5] found that the least (it means the best) values of these constants are $M_1 = \int_0^1 (1 - e^{-u}) u^{-1} du + \int_1^{\infty} e^{-u} u^{-1} du$ and $M_2 = M_1 + 2e^{-1}$.

1.2. We say that a series of complex numbers $\sum_{k=1}^{\infty} a_k$ *is summable by the* (D, λ) *summability method to a sum s*, if the series $f(t) = \sum_{k=1}^{\infty} a_k \exp(-\lambda_k t)$ converges for

$t > 0$ and $\lim_{t \to 0+} f(t) = s$. We assume that the sequence $\{\lambda_n\}_{n=1}^{\infty}$ is an increasing sequence of positive numbers and $\lim_{n \to \infty} \lambda_n = \infty$. It is easily seen that the Abel summability method is a special case of the (D, λ) summability. It suffices to put $\lambda_n = n$ for all $n \in \mathrm{N}$ (cf. [10]).

This work concerns (in comparison with the older results) a more general (D, λ) summability. Let $\{\lambda_n\}_{n=1}^{\infty}$ be an increasing sequence of positive numbers such that $\lim_{n \to \infty} \lambda_n = \infty$ and $\{t_n\}_{n=1}^{\infty}$ an arbitrary sequence of positive numbers, $\lim_{n \to \infty} t_n = 0$. We shall examine for which sequences $\{\lambda_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ there exists a constant M such that for every series of complex numbers $\sum_{k=1}^{\infty} a_k$ with

$$A_1 = \limsup_{n \to \infty} \frac{\lambda_n |a_n|}{\lambda_n - \lambda_{n-1}} < \infty .$$

the inequality

$$\limsup_{n \to \infty} |s_n - f(t_n)| \leqq M A_1$$

holds, where $s_n = \sum_{k=1}^{n} a_k$, $n \in \mathrm{N}$, and $f(t) = \sum_{k=1}^{\infty} a_k \exp(-\lambda_k t)$ for $t > 0$. We call this constant the *Tauberian constant for the* (D, λ) *summability method under the condition* $A_1 < \infty$ *belonging to the sequence* $\{t_n\}_{n=1}^{\infty}$. We solve the same problem under the Tauberian condition

$$A_2 = \limsup_{n \to \infty} \frac{1}{\lambda_n} \left| \sum_{k=1}^{n} \lambda_k a_k \right| < \infty .$$

$$A_3^{(p)} = \limsup_{n \to \infty} \left(\frac{1}{\lambda_n} \sum_{k=1}^{n} \frac{\lambda_k^p |a_k|^p}{(\lambda_k - \lambda_{k-1})^{p-1}} \right)^{1/p},$$

where $p \geqq 1$. (We put $\lambda_0 = 0$ if necessary.)

The present work is based on two general theorems of I. J. MADDOX [8]. (Cf. further Theorem 1, Theorem 2.) It is shown how the conditions of these theorems can be simplified for the special case of the (D, λ) summability. Our results which concern the conditions $A_1 < \infty$ and $A_2 < \infty$ are presented in the third section. (Cf. Theorem 3, Theorem 4.) As to the condition $A_3^{(p)} < \infty$, one theorem was proved in 1971 by K. A. JUKES. (Cf. [7], p. 754, Corollary 1.) We shall give here a simple necessary and sufficient condition (see further condition (*)) for the existence of a Tauberian constant under the condition $A_3^{(p)} < \infty$, $p \geqq 1$. We avoid (cf. Theorem 5) the application of Jukes' theorem. In fact, we obtain a little more than the straightforward application of his theorem yields, because we have no such restriction on $\{\lambda_{n+1}/\lambda_n\}_{n=1}^{\infty}$ as him (cf. [7], p. 750, (11) and (11)'). On the other hand, Jukes considers more generally $p > 0$.

2. TWO GENERAL THEOREMS OF I. J. MADDOX

2.1. We denote the set of all natural numbers by N. For an infinite matrix $B = (b_{nk})$, $k \in \mathrm{N}$, $n \in \mathrm{N}$, and a series $\sum_{k=1}^{\infty} a_k$ we write

$$B_n(a) = \sum_{k=1}^{\infty} b_{nk} a_k\,,$$

$$\Delta(b_{nk}) = b_{nk} - b_{n,k+1}$$

for all $k \in \mathrm{N}$, $n \in \mathrm{N}$.

We shall use these general theorems of I. J. Maddox [8] to our purposes:

Theorem 1. $B_n(a)$, $C_n(a)$ *exist for each* $n \in \mathrm{N}$ *and there is a constant* M_1 *such that*

$$\limsup_{n\to\infty} |B_n(a) - C_n(a)| \leqq M_1 A_1$$

whenever $A_1 < \infty$ *if and only if*

(b)′ $\lim_{n\to\infty} (b_{nk} - c_{nk}) = 0$ *for each* $k \in \mathrm{N}$,

(c)′ $\sum_{k=1}^{\infty} \frac{\lambda_k - \lambda_{k-1}}{\lambda_k} (|b_{nk}| + |c_{nk}|) < \infty$ *for each* $n \in \mathrm{N}$,

(d)′ $D = \limsup_{n\to\infty} \sum_{k=1}^{\infty} \frac{\lambda_k - \lambda_{k-1}}{\lambda_k} |b_{nk} - c_{nk}| < \infty\,.$

If (b)′, (c)′ *and* (d)′ *hold we may take* $M_1 = D$ *and this constant D is then the best possible.*

Theorem 2. $B_n(a)$, $C_n(a)$ *exist for each* $n \in \mathrm{N}$ *and there is a constant* M_2 *such that*

$$\limsup_{n\to\infty} |B_n(a) - C_n(a)| \leqq M_2 A_2$$

whenever $A_2 < \infty$ *if and only if*

(a) $\lim_{k\to\infty} (|b_{nk}| + |c_{nk}|) = 0$ *for each* $n \in \mathrm{N}$,

(b) $\lim_{n\to\infty} (b_{nk} - c_{nk}) = 0$ *for each* $k \in \mathrm{N}$,

(c) $\sum_{k=1}^{\infty} \lambda_k \left(\left| \Delta\left(\frac{b_{nk}}{\lambda_k}\right) \right| + \left| \Delta\left(\frac{c_{nk}}{\lambda_k}\right) \right| \right) < \infty$ *for each* $n \in \mathrm{N}$,

(d) $D = \limsup_{n\to\infty} \sum_{k=1}^{\infty} \lambda_k \left| \Delta\left(\frac{b_{nk} - c_{nk}}{\lambda_k}\right) \right| < \infty\,.$

If (a)–(d) *hold we may take* $M_2 = D$ *and this constant is then the best possible.*

The words "this constant D is the best possible" have the exact meaning "there exists a series $\sum_{k=1}^{\infty} a_k$ such that $A_1 = 1$ (or $A_2 = 1$) and $\limsup_{n\to\infty} |B_n(a) - C_n(a)| = D$" in these theorems.

2.2. For the application of Theorems 1 and 2 to the (D, λ) summability we need to choose matrices B, C in this way:

$$(1) \qquad b_{nk} = 1 \quad \text{for} \quad k \leqq n\,, \quad b_{nk} = 0 \quad \text{for} \quad k > n\,,$$

$$c_{nk} = \exp\left(-\lambda_k t_n\right) \quad \text{for each } n, \quad k \in \mathrm{N}\,.$$

In the third section we establish (imposing only a small restriction on the sequence $\{\lambda_n/\lambda_{n+1}\}_{n=1}^{\infty}$) that the existence of Tauberian constants under the conditions $A_1 < \infty$, $A_2 < \infty$ and $A_3^{(p)} < \infty$, $p \geqq 1$, is equivalent to a quite simple condition.

Firstly, we shall verify the conditions of Theorems 1 and 2 whose validity does not depend upon the choice of sequences $\{\lambda_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$.

Lemma 1. *Matrices B, C defined by* (1) *satisfy the conditions* (b)′,(c)′ *of Theorem* 1 *and* (a), (b), (c) *of Theorem 2 for arbitrary sequences $\{\lambda_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$.*

Proof. We have for each $k \in \mathrm{N}$

$$\lim_{n\to\infty} (b_{nk} - c_{nk}) = \lim_{n\to\infty} (1 - \exp(-\lambda_k t_n)) = 0$$

so that the conditions (b) and (b)′ are fulfilled. As to the condition (c)′, we evaluate for $n \in \mathrm{N}$

$$\sum_{k=1}^{\infty} \frac{\lambda_k - \lambda_{k-1}}{\lambda_k} \left(|b_{nk}| + |c_{nk}|\right) = \sum_{k=1}^{n} \frac{\lambda_k - \lambda_{k-1}}{\lambda_k} + \sum_{k=1}^{\infty} \frac{\lambda_k - \lambda_{k-1}}{\lambda_k} \exp\left(-\lambda_k t_n\right).$$

The sum $\sum_{k=1}^{n} (\lambda_k - \lambda_{k-1})/\lambda_k$ is clearly finite for each $n \in \mathrm{N}$. Moreover,

$$\sum_{k=1}^{\infty} \frac{\lambda_k - \lambda_{k-1}}{\lambda_k} \exp\left(-\lambda_k t_n\right) = \sum_{k=2}^{\infty} (\lambda_k t_n - \lambda_{k-1} t_n) \,.\, \frac{\exp\left(-\lambda_k t_n\right)}{\lambda_k t_n} + \exp\left(-\lambda_1 t_n\right)$$

is finite, too, because the sum on the righthand side is the lower integral sum belonging to the division $\mathscr{D} = \{\lambda_1 t_n < \lambda_2 t_n < \ldots\}$ of the interval $\langle \lambda_1 t_n, +\infty)$ and to the function $\mathrm{e}^{-x} x^{-1}$. The integral $\int_{\varepsilon}^{\infty} \mathrm{e}^{-x} x^{-1}\, \mathrm{d}x$ converges for arbitrary $\varepsilon > 0$.

Let n be an arbitrary natural number. For $k > n$ we have $|b_{nk}| + |c_{nk}| = \exp\left(-\lambda_k t_n\right)$ and $\lim_{k\to\infty} \left(|b_{nk}| + |c_{nk}|\right) = 0$, hence the condition (a) holds. It is easy to verify the condition (c) because

$$\sum_{k=1}^{\infty} \lambda_k \left(\left| \Delta\left(\frac{b_{nk}}{\lambda_k}\right) \right| + \left| \Delta\left(\frac{c_{nk}}{\lambda_k}\right) \right| \right) =$$

$$= \sum_{k=1}^{n-1} \lambda_k \left(\frac{1}{\lambda_k} - \frac{1}{\lambda_{k+1}}\right) + 1 + \sum_{k=1}^{\infty} \lambda_k \left(\frac{\exp\left(-\lambda_k t_n\right)}{\lambda_k} - \frac{\exp\left(-\lambda_{k+1} t_n\right)}{\lambda_{k+1}}\right) =$$

$$= \sum_{k=1}^{n} \frac{\lambda_k - \lambda_{k-1}}{\lambda_k} + \sum_{k=1}^{\infty} \frac{\lambda_k - \lambda_{k-1}}{\lambda_k} \exp\left(-\lambda_k t_n\right)$$

which is finite for each $n \in \mathrm{N}$ as we have proved already. This completes the proof.

Now, we shall consider the conditions (d) and (d)'. We estimate the sums in these conditions from above by integrals similarly as in the verification of the condition (c)' in the proof of Lemma 1. To obtain a necessary and sufficient condition for the existence of a finite Tauberian constant we need to restrict ourselves to those sequences of exponents $\{\lambda_n\}_{n=1}^{\infty}$ which satisfy at least one of the conditions of the next convention. For the sake of brevity we shall denote $\Lambda_n = \lambda_n/\lambda_{n+1}$.

Convention. In the next application of Theorems 1 and 2 to the (D, λ) summability we shall suppose that the sequence of exponents $\{\lambda_n\}_{n=1}^{\infty}$ fulfils at least one of these conditions:

I. $\liminf_{n\to\infty} \Lambda_n > 0$ (cf. [7], p. 750, (11), (11)'),

II. $\limsup_{n\to\infty} \Lambda_n < 1$,

III. if $\liminf_{n\to\infty} \Lambda_n = 0$ and $\limsup_{n\to\infty} \Lambda_n = 1$, then there exist $q < 1$ and $K \in \mathrm{N}$ such that in every group $\{\Lambda_n, \Lambda_{n+1}, \ldots, \Lambda_{n+K-1}\}$ we can find at least one element $\Lambda_k \leqq q$.

3. THEOREMS OF HADWIGER-WINTNER TYPE FOR THE (D, λ) SUMMABILITY

3.1. In this part of the present paper we shall formulate condition $(*)$ (cf. Theorem 3) and we shall show successively that this condition is necessary and sufficient for the existence of finite Tauberian constants $M_1, M_2, M_3^{(p)}$, $p \geqq 1$ (cf. Theorems 3, 4, 5).

Theorem 3. *A constant M_1 such that the inequality*

$$\limsup_{n\to\infty} |s_n - f(t_n)| \leqq M_1 A_1$$

holds for every series of complex numbers $\sum_{k=1}^{\infty} a_k$ for which $A_1 < \infty$, exists if and only if

$(*)$ *there is $k \geqq 0$ such that $\liminf_{n\to\infty} \lambda_{n+k} t_n > 0$ and*

there is $k' \leqq 0$ such that $\limsup_{n\to\infty} \lambda_{n+k'} t_n < \infty$

at the same time. If the condition $(*)$ holds, we may take

$$M_1 = \limsup_{n\to\infty} \left(\sum_{k=1}^{n} \frac{\lambda_k - \lambda_{k-1}}{\lambda_k} (1 - \exp(-\lambda_k t_n)) + \sum_{k=n+1}^{\infty} \frac{\lambda_k - \lambda_{k-1}}{\lambda_k} \exp(-\lambda_k t_n) \right)$$

and this constant is then the best possible in the sense that there exists a series $\sum_{k=1}^{\infty} a_k$ such that $A_1 = 1$ and $\limsup_{n\to\infty} |s_n - f(t_n)| = M_1$.

If matrices B, C are defined by (1) then the proof of Theorem 3 from Theorem 1 (with regard to Lemma 1) requires only to verify that the condition (d)′ is satisfied if and only if the condition (*) is. Consequently, we want to prove that the condition (*) is equivalent to

$$(2)\quad \limsup_{n\to\infty}\left(\sum_{k=1}^{n}\frac{\lambda_k-\lambda_{k-1}}{\lambda_k}(1-\exp(-\lambda_k t_n))+\sum_{k=n+1}^{\infty}\frac{\lambda_k-\lambda_{k-1}}{\lambda_k}\exp(-\lambda_k t_n)\right)<\infty\,.$$

We divide the whole proof of Theorem 3 in two parts.

Proof of Theorem 3 (sufficiency part). Let the condition (*) hold. Then there exist $n_0\in\mathrm{N}$ and constants L, M such that $n_0+k'\geqq 1$ and for each $n\geqq n_0$, $0<L<$ $<\lambda_{n+k}t_n$, $\lambda_{n+k'}t_n<M<\infty$. Let us take such $n\geqq n_0$. Then

$$A_1(n)=\sum_{i=1}^{n}\frac{\lambda_i-\lambda_{i-1}}{\lambda_i}(1-\exp(-\lambda_i t_n))+\sum_{i=n+1}^{\infty}\frac{\lambda_i-\lambda_{i-1}}{\lambda_i}\exp(-\lambda_i t_n)=$$

$$=\sum_{i=1}^{n+k'}\frac{\lambda_i-\lambda_{i-1}}{\lambda_i}(1-\exp(-\lambda_i t_n))+\sum_{i=n+k'+1}^{n}\frac{\lambda_i-\lambda_{i-1}}{\lambda_i}(1-\exp(-\lambda_i t_n))+$$

$$+\sum_{i=n+1}^{n+k}\frac{\lambda_i-\lambda_{i-1}}{\lambda_i}\exp(-\lambda_i t_n)+\sum_{i=n+k+1}^{\infty}\frac{\lambda_i-\lambda_{i-1}}{\lambda_i}\exp(-\lambda_i t_n)\leqq$$

$$\leqq\int_0^{\lambda_{n+k'}t_n}(1-\mathrm{e}^{-x})\,x^{-1}\,\mathrm{d}x-k'+k+\int_{\lambda_{n+k}t_n}^{\infty}\mathrm{e}^{-x}x^{-1}\,\mathrm{d}x\leqq$$

$$\leqq\int_0^{M}(1-\mathrm{e}^{-x})\,x^{-1}\,\mathrm{d}x+(k-k')+\int_L^{\infty}\mathrm{e}^{-x}x^{-1}\,\mathrm{d}x\,.$$

(Of course, we set $\sum_{i=k}^{k'}a_i=0$ if $k'<k$.)

Thus $\limsup_{n\to\infty}A_1(n)\leqq\int_0^M(1-\mathrm{e}^{-x})\,x^{-1}\,\mathrm{d}x+(k-k')+\int_L^\infty\mathrm{e}^{-x}x^{-1}\,\mathrm{d}x<\infty$. So (2) holds and the condition (d)′ is satisfied. According to Theorem 1 the constant $M_1=\limsup_{n\to\infty}A_1(n)$ is the best possible.

We prove one lemma before the necessity part of the proof.

Lemma 2. *Let* $\liminf_{n\to\infty}\lambda_{n+k}t_n=0$ *hold for each* $k\geqq 0$ ($\limsup_{n\to\infty}\lambda_{n+k'}t_n=\infty$ *for each* $k'\leqq 0$, *respectively*). *Then there exists an increasing sequence of natural numbers* $\{n_i\}_{i=1}^{\infty}$ *such that*

$$(3)\qquad \lim_{i\to\infty}\lambda_{n_i+k}t_{n_i}=0\quad\textit{for each}\quad k\geqq 0$$

$$(4)\qquad (\lim_{i\to\infty}\lambda_{n_i+k'}t_{n_i}=\infty\quad\textit{for each}\quad k'\leqq 0\,,$$

respectively).

Proof. Firstly, let $\liminf_{n\to\infty} \lambda_{n+k}t_n = 0$ hold for each $k \geqq 0$. Let us consider increasing sequences

$$\{n_k^{(0)}\}_{k=1}^{\infty} \supset \{n_k^{(1)}\}_{k=1}^{\infty} \supset \dots$$

which are defined in this way:

$$n \in \{n_k^{(0)}\} \Leftrightarrow \lambda_n t_n < \tfrac{1}{2},$$

$$\dots$$

$$n \in \{n_k^{(m)}\} \Leftrightarrow \lambda_{n+m} t_n < \frac{1}{m+2},$$

$$\dots$$

Every sequence $\{n_k^{(m)}\}$, $m = 0, 1, \dots$, is, of course, infinite. Evidently $\bigcap_{m=0}^{\infty} \{n_k^{(m)}\}$ is empty. That is why there surely exists an increasing sequence of natural numbers $\{j_i\}_{i=1}^{\infty}$, $\lim_{i\to\infty} j_i = \infty$ such that $\{n_k^{(j_i)}\} \setminus \{n_k^{(j_i+1)}\} \neq \emptyset$. Let $\{n_i\}_{i=1}^{\infty}$ be a sequence of natural numbers such that $n_i \in \{n_k^{(j_i)}\} \setminus \{n_k^{(j_i+1)}\}$.

Let us choose from it an increasing sequence and call it again $\{n_i\}_{i=1}^{\infty}$. Then $\lim_{i\to\infty} \lambda_{n_i+k} t_{n_i} = 0$ for each $k \geqq 0$. Indeed, for each $k \geqq 0$ and for each $\varepsilon > 0$ there exists $m \in \mathrm{N}$ such that $m \geqq k$ and $(m+2)^{-1} < \varepsilon$. Further, there exists $i_0 \in \mathrm{N}$ such that $n_i \in \{n_k^{(m)}\}$ for all $i \geqq i_0$. It means that

$$\lambda_{n_i+k} t_{n_i} \leqq \lambda_{n_i+m} t_{n_i} < \frac{1}{m+2} < \varepsilon$$

for all $i \geqq i_0$. Therefore the relation (3) holds.

Now let $\limsup_{n\to\infty} \lambda_{n+k'} t_n = \infty$ hold for each $k' \leqq 0$. It is again possible to choose an increasing sequence of natural numbers $\{n_i\}_{i=1}^{\infty}$ such that the condition (4) is satisfied. It is enough to define

$$n \in \{n_k^{(0)}\} \Leftrightarrow \lambda_n t_n > 1,$$

$$\dots$$

$$n \in \{n_k^{(m)}\} \Leftrightarrow \lambda_{n-m} t_n > m + 1,$$

$$\dots$$

and to choose a sequence $\{n_i\}_{i=1}^{\infty}$ exactly in the same way as in the previous part of this proof.

Remark 1. (See [6], p. 121.) Let $-1 < L < 0$ and let $\{u_k\}_{k=1}^{\infty}$ be a sequence of real numbers such that $L \leqq u_k \leqq 0$ for each $k \in \mathrm{N}$. Further, let $\{k_n\}_{n=1}^{\infty}$ and $\{k_n'\}_{n=1}^{\infty}$ be any sequences of natural numbers such that $k_n \leqq k_n'$ for each $n \in \mathrm{N}$. Let us put

$$s_n = \sum_{k=k_n}^{k'_n} u_k, \quad p_n = \prod_{k=k_n}^{k'_n} (1 + u_k).$$

Then the sequence $\{s_n\}_{n=1}^{\infty}$ is bounded if and only if there exists $K_1 > 0$ such that $p_n \geqq K_1$ for each $n \in \mathrm{N}$.

Proof of Theorem 3 (necessity part). Let us assume that the condition $(*)$ is not satisfied. It means that the assumptions of Lemma 2 are satisfied. Thus either (3) or (4) holds. Firstly, let (3) hold. To each $i \in \mathrm{N}$ we find $l(i) \in \mathrm{N}$ such that

$$\lambda_{n_i+l(i)-1} t_{n_i} \leqq 1 < \lambda_{n_i+l(i)} t_{n_i} . \tag{5}$$

We have $\lim_{i\to\infty} l(i) = \infty$. Namely, if we choose $K \in \mathrm{N}$ then according to (3) there exists $i_0 \in \mathrm{N}$ such that $\lambda_{n_i+K} t_{n_i} \leqq 1$ for all $i \geqq i_0$ and hence $l(i) \geqq K + 1$ for all $i \geqq i_0$. For $i \geqq 1$ let us consider the sum

$$S_i = \sum_{j=n_i+1}^{n_i+l(i)} \frac{\lambda_j - \lambda_{j-1}}{\lambda_j} \exp\left(-\lambda_j t_{n_i}\right).$$

By (5) we obtain for $i \geqq 1$

$$S_i \geqq \mathrm{e}^{-1} \sum_{j=n_i+1}^{n_i+l(i)} (1 - \Lambda_{j-1}) + (1 - \Lambda_{n_i+l(i)-1}) \left(\exp\left(-\lambda_{n_i+l(i)} t_{n_i}\right) - \mathrm{e}^{-1}\right) \geqq \tag{6}$$

$$\geqq \mathrm{e}^{-1} \sum_{j=n_i+1}^{n_i l(i)} (1 - \Lambda_{j-1}) - 1 .$$

Now if the alternative I holds there exists $\delta > 0$ such that for each $n \geqq 2$ we have $\Lambda_{n-1} > \delta$ and so $-1 + \delta < \Lambda_{n-1} - 1 < 0$. If we use Remark 1, putting $u_n = \Lambda_{n-1} - 1$, $k_i = n_i + 1$, $k'_i = n_i + l(i)$, $n \geqq 2$, $i \geqq 1$, we obtain that the sequence

$$\left\{\sum_{j=n_i+1}^{n_i+l(i)} \left(1 - \frac{\lambda_{j-1}}{\lambda_j}\right)\right\}_{i=1}^{\infty}$$

is bounded if and only if there exists $K_1 > 0$ such that

$$\prod_{j=n_i+1}^{n_i+l(i)} \frac{\lambda_{j-1}}{\lambda_j} = \frac{\lambda_{n_i}}{\lambda_{n_i+l(i)}} \geqq K_1$$

for each $i \in \mathrm{N}$. But according to (3) and (5) we have

$$0 \leqq \limsup_{i\to\infty} \frac{\lambda_{n_i}}{\lambda_{n_i+l(i)}} = \limsup_{i\to\infty} \frac{\lambda_{n_i} t_{n_i}}{\lambda_{n_i+l(i)} t_{n_i}} \leqq \limsup_{i\to\infty} \lambda_{n_i} t_{n_i} = 0 .$$

This relation implies that such a constant $K_1 > 0$ does not exist and by Remark 1

$$\limsup_{i\to\infty} \sum_{j=n_i+1}^{n_i+l(i)} (1 - \Lambda_{j-1}) = \infty . \tag{7}$$

It follows from (6) and (7) that $\limsup_{i\to\infty} S_i = \infty$ and, because $A_1(n_i) \geqq S_i$, also $\limsup_{n\to\infty} A_1(n_i) = \limsup_{i\to\infty} A_1(n) = \infty$.

If the alternative II holds, then there exist $0 < q < 1$ and $j_0 \in \mathrm{N}$ such that $1 - A_{j-1} > 1 - q$ for all $j \geqq j_0$. Then, we have with regard to (6) $S_i \geqq e^{-1} l(i) . (1 - q) - 1$ for $i \in \mathrm{N}$ with $n_i + 1 \geqq j_0$ and so

$$\limsup_{n\to\infty} A_1(n) \geqq \limsup_{i\to\infty} A_1(n_i) \geqq \limsup_{i\to\infty} S_i = \infty$$

for $\lim_{i\to\infty} l(i) = \infty$.

If the alternative III holds and $m \in \mathrm{N}$ is arbitrary, then according to (6) $S_i \geqq e^{-1}(1 - q)\, m - 1$ for $i \in \mathrm{N}$ with $l(i) \geqq Km$. Consequently, $\limsup_{i\to\infty} A_1(n_i) = \infty$ again and $\limsup_{n\to\infty} A_1(n) = \infty$ as well.

Now, let the condition (4) be satisfied. For each $i \in \mathrm{N}$ we find $l(i) \leqq 0$, the integer such that (5) holds. We have $\lim_{i\to\infty} l(i) = -\infty$ again. Further

$$(8)\quad S'_i = \sum_{j=n_i+l(i)}^{n_i} \frac{\lambda_j - \lambda_{j-1}}{\lambda_j}\left(1 - \exp\left(-\lambda_j t_{n_i}\right)\right) \geqq \left(1 - e^{-1}\right) \sum_{j=n_i+l(i)}^{n_i} \left(1 - A_{j-1}\right).$$

We can deduce $\limsup_{i\to\infty} S'_i = \infty$ from (8) identically as we have deduced $\limsup_{i\to\infty} S_i = \infty$ from (6) in the previous part of this proof, if at least one of the conditions I, II, III holds. In view of the fact that $A_1(n_i) \geqq S'_i$, we obtain $\limsup_{n\to\infty} A_1(n) = \infty$. Theorem 3 is proved.

Corollary 1. *Let $\liminf_{n\to\infty} A_n > 0$ (i.e. the alternative I holds). Then a finite Tauberian constant for the (D, λ) summability under the condition $A_1 < \infty$ belonging to a sequence $\{t_n\}_{n=1}^{\infty}$ exists if and only if*

$$(9)\qquad 0 < \liminf_{n\to\infty} \lambda_n t_n \leqq \limsup_{n\to\infty} \lambda_n t_n < \infty .$$

(Cf. [2], p. 227, Theorem 7.1.)

Proof. Sufficiency of (9) follows directly from Theorem 3. On the contrary, if there exists a finite Tauberian constant for the (D, λ) summability under the condition $A_1 < \infty$ belonging to a sequence $\{t_n\}_{n=1}^{\infty}$, then according to Theorem 3 there exist $k \geqq 0$ and $k' \leqq 0$ such that

$$(10)\qquad \liminf_{n\to\infty} \lambda_{n+k} t_n > 0 ,$$

$$(11)\qquad \limsup_{n\to\infty} \lambda_{n+k'} t_n < \infty .$$

Since $\liminf_{n\to\infty} A_n > 0$, there exists $K > 0$ such that $A_n \geqq K$ for all $n \in \mathrm{N}$. Further we have

$$(12)\qquad \lambda_{n+k} t_n = A_{n+k-1}^{-1} \cdot \ldots \cdot A_n^{-1} \lambda_n t_n \leqq K^{-k} \lambda_n t_n ,$$

$$(13)\qquad \lambda_{n+k'} t_n = A_{n+k'} \cdot \ldots \cdot A_{n-1} \lambda_n t_n \geqq K^{-k'} \lambda_n t_n ,$$

where (13) holds for all $n > -k'$. The relations (10) and (12) imply $\liminf_{n\to\infty} \lambda_n t_n > 0$, (11) and (13) yield $\limsup_{n\to\infty} \lambda_n t_n < \infty$. This completes the proof.

Remark 2. The condition (*) can be formulated in the symmetrical form: there exists $k \geqq 0$ such that $\liminf_{n\to\infty} \lambda_{n+k} t_n > 0$ and $\limsup_{n\to\infty} \lambda_{n-k} t_n < \infty$.

Remark 3. According to the first part of the proof of Theorem 3 the condition (*) is sufficient for the existence of a finite Tauberian constant for the (D, λ) summability under the condition $A_1 < \infty$, belonging to a sequence $\{t_n\}_{n=1}^{\infty}$ for any sequence of exponents $\{\lambda_n\}_{n=1}^{\infty}$. We mention an example showing that the condition (*) is not necessary if the sequence $\{\lambda_n\}_{n=1}^{\infty}$ satisfies none of the conditions I, II, III.

Define $\{\lambda_n\}_{n=1}^{\infty}$, $\{t_n\}_{n=1}^{\infty}$ as follows:

$$\lambda_n : 6, 6\frac{1}{6}, 6\frac{2}{6}, \dots, 6\frac{5}{6}, 7, \dots, k!, \quad k! + \frac{1}{k!}, \dots, k! + \frac{k! - 1}{k!}, k! + 1, \dots,$$

$$t_n : \frac{1}{4!}, \frac{1}{4! + 1}, \frac{1}{4! + 2}, \dots, \frac{1}{4! + 5}, \frac{1}{4! + 3!}, \dots, \frac{1}{(k + 1)!},$$

$$\frac{1}{(k + 1)! + 1}, \dots, \frac{1}{(k + 1)! + k! - 1}, \frac{1}{(k + 1)! + k!},$$

Then $\lambda_n t_n \leqq 1$ and thus $\sum_{k=1}^{n} (1 - A_{k-1}) (1 - \exp(-\lambda_k t_n)) \leqq \int_0^1 (1 - e^{-x}) x^{-1}\, dx$. If we denote by $\{n_i\}_{i=3}^{\infty}$ the sequence of those natural numbers for which $\lambda_{n_i} = i!$, $t_{n_i} = 1/(i + 1)!$, we have $\lim_{i\to\infty} \lambda_{n_i+k} t_{n_i} = 0$ for each $k \geqq 0$. It means that the condition (*) is not satisfied. Take any $n \in \mathrm{N}$. Let $\lambda_n = k! + i/k!$, $0 \leqq i \leqq k!$. Then $t_n = 1/((k + 1)! + i)$. Only the numbers $\lambda_n t_n, \lambda_{n+1} t_n, \dots, \lambda_m t_n, \dots, \lambda_p t_n$, where $\lambda_m = (k + 1)!$, $\lambda_p = (k + 1)! + 1$, can lie in the interval $\langle 0, 1 \rangle$. Really, $\lambda_{p+1} t_n = (k + 2)!/((k + 1)! + i) \geqq k + 1$. Since

$$\frac{\lambda_{m-1}}{\lambda_m} = \frac{k! + 1}{k! + i/k!} \leqq 2 \quad \text{and} \quad \frac{\lambda_p}{\lambda_m} = \frac{(k + 1)! + 1}{(k + 1)!} \leqq 2$$

for each $k \in \mathrm{N}$, $i = 0, 1, \dots, k!$, we have

$$\sum_{j=n+1}^{\infty} \frac{\lambda_j - \lambda_{j-1}}{\lambda_j} \exp(-\lambda_j t_n) = \sum_{j=n+1}^{m-1} \frac{\lambda_j - \lambda_{j-1}}{\lambda_j} \exp(-\lambda_j t_n) +$$

$$+ \frac{\lambda_m - \lambda_{m-1}}{\lambda_m} \exp(-\lambda_m t_n) + \sum_{j=m+1}^{p} \frac{\lambda_j - \lambda_{j-1}}{\lambda_j} \exp(-\lambda_j t_n) +$$

$$+ \sum_{j=p+1}^{\infty} \frac{\lambda_j - \lambda_{j-1}}{\lambda_j} \exp(\lambda_j t_n) \leqq \int_{\lambda_n t_n}^{\lambda_{m-1} t_n} e^{-x} x^{-1}\, dx + 1 + \int_{\lambda_m t_n}^{\lambda_p t_n} e^{-x} x^{-1}\, dx + 1 +$$

$$+\int_{\lambda_{p+1}t_n}^{\infty} e^{-x}x^{-1}\,dx \leqq \int_{\lambda_n t_n}^{2\lambda_n t_n} e^{-x}x^{-1}\,dx + 1 + \int_{\lambda_m t_n}^{2\lambda_m t_n} e^{-x}x^{-1}\,dx + 1 +$$

$$+\int_1^{\infty} e^{-x}x^{-1}\,dx \leqq \int_a^{2a} x^{-1}\,dx + 2 + \int_b^{2b} x^{-1}\,dx + \int_1^{\infty} e^{-x}x^{-1}\,dx = 2 + 2\log 2 +$$

$$+\int_1^{\infty} e^{-x}x^{-1}\,dx\,.$$

Therefore $A_1(n) \leqq 2 + 2\log 2 + \int_0^1 (1 - e^{-x})\,x^{-1}\,dx + \int_1^{\infty} e^{-x}x^{-1}\,dx$. This implies $\limsup_{n\to\infty} A_1(n) < \infty$ and so there exists a finite Tauberian constant for the (D, λ) summability under the condition $A_1 < \infty$, belonging to the sequence $\{t_n\}_{n=1}^{\infty}$. The sequence of exponents $\{\lambda_n\}_{n=1}^{\infty}$ does not satisfy anyone of the conditions I, II, III.

3.2. In the rest of this paper we shall again suppose that at least one of the conditions I, II, III is satisfied.

Theorem 4. *A constant M_2 such that the inequality*

$$\limsup_{n\to\infty} |s_n - f(t_n)| \leqq M_2 A_2$$

holds for every series of complex numbers $\sum_{k=1}^{\infty} a_k$, for which $A_2 < \infty$, exists if and only if the condition $(*)$ is satisfied. If the condition $(*)$ holds, we may put

$$M_2 = \limsup_{n\to\infty}\left(\sum_{k=1}^{n} \frac{\lambda_k - \lambda_{k-1}}{\lambda_k}(1 - \exp(-\lambda_k t_n)) + \sum_{k=n+1}^{\infty} \frac{\lambda_k - \lambda_{k-1}}{\lambda_k}\exp(-\lambda_k t_n) + \frac{2\lambda_n}{\lambda_{n+1}}\exp(-\lambda_{n+1}t_n)\right)$$

and this constant is then the best possible in the sense that there exists a series $\sum_{k=1}^{\infty} a_k$ such that $A_2 = 1$ and $\limsup_{n\to\infty} |s_n - f(t_n)| = M_2$.

Proof. With regard to Lemma 1, the proof of Theorem 4 requires only to verify that the condition (d) of Theorem 2 is satisfied if and only if the condition $(*)$ holds.

By definition of the matrices B and C we have for each $n \in \mathrm{N}$

$$A_2(n) = \sum_{k=1}^{\infty} \lambda_k \left| \Delta\left(\frac{b_{nk} - c_{nk}}{\lambda_k}\right)\right| = \sum_{k=1}^{n-1} \lambda_k \left| \frac{1 - \exp(-\lambda_k t_n)}{\lambda_k} - \frac{1 - \exp(-\lambda_{k+1}t_n)}{\lambda_{k+1}}\right| +$$

$$+ \lambda_n \left| \frac{1 - \exp(-\lambda_n t_n)}{\lambda_n} - \frac{-\exp(-\lambda_{n+1}t_n)}{\lambda_{n+1}}\right| + \sum_{k=n+1}^{\infty} \lambda_k \left| \frac{-\exp(-\lambda_k t_n)}{\lambda_k} -\right.$$

$$\left. - \frac{-\exp(-\lambda_{k+1}t_n)}{\lambda_{k+1}}\right| = \sum_{k=1}^{n} \frac{\lambda_k - \lambda_{k-1}}{\lambda_k}(1 - \exp(-\lambda_k t_n)) + \frac{2\lambda_n}{\lambda_{n+1}}\exp(-\lambda_{n+1}t_n) +$$

$$+\sum_{k=n+1}^{\infty} \frac{\lambda_k - \lambda_{k-1}}{\lambda_k} \exp(-\lambda_k t_n) = A_1(n) + 2\Lambda_n \exp(-\lambda_{n+1} t_n).$$

Since we know by Theorem 3 that

(14) $$\limsup_{n\to\infty} A_1(n) < \infty \Leftrightarrow (*) \quad \text{holds}$$

and moreover $0 < 2\Lambda_n \exp(-\lambda_{n+1} t_n) < 2$, the equality $A_2(n) = A_1(n) + 2\Lambda_n \exp . (-\lambda_{n+1} t_n)$, $n \in \mathrm{N}$, yields that $\limsup_{n\to\infty} A_2(n) < \infty$ if and only if $\limsup_{n\to\infty} A_1(n) < \infty$. The assertion of this theorem follows now from (14).

3.3. We turn to the condition $A_3^{(p)} < \infty$, $p \geqq 1$. We shall need the next lemma.

Lemma 3. (R. P. Agnew, [1]). *Let* (a_{nk}), $n \geqq 1$, $k \geqq 1$, *be an in finite matrix of complex numbers. Let* $\lim_{n\to\infty} a_{nk} = 0$ *for each* $k \in \mathrm{N}$ *and* $\limsup_{n\to\infty} \sum_{k=1}^{\infty} |a_{nk}| = C < \infty$. *Then*

$$\limsup_{n\to\infty} \left|\sum_{k=1}^{\infty} a_{nk} s_k\right| \leqq C \limsup_{k\to\infty} |s_k|$$

for every sequence of complex numbers $\{s_k\}_{k=1}^{\infty}$ *such that* $\limsup_{k\to\infty} |s_k| < \infty$. *Moreover, there exists a sequence* $\{s_k\}_{k=1}^{\infty}$ *with* $\limsup_{k\to\infty} |s_k| = 1$ *and* $\limsup_{n\to\infty} \left|\sum_{k=1}^{\infty} a_{nk} s_k\right| = C$.

We present this lemma without a proof. It is used in proofs of Theorems 1 and 2 and we shall use it in the proof of the next theorem.

Remark 4. It is possible to improve slightly the assertions of Theorems 3 and 4 (compare the proof of our Lemma 3 in [1]) in this sense: If $M_1 = \limsup_{n\to\infty} A_1(n) = \infty$ or $M_2 = \limsup_{n\to\infty} A_2(n) = \infty$ then there exists a series $\sum_{k=1}^{\infty} a_k$ such that $A_1 = 1$ or $A_2 = 1$, respectively, and $\limsup_{n\to\infty} |s_n - f(t_n)| = \infty$. We shall use this fact essentially in the proof of the next theorem.

Theorem 5. *Let* $p \geqq 1$. *A constant* M *such that the inequality*

$$\limsup_{n\to\infty} |s_n - f(t_n)| \leqq M A_3^{(p)}$$

holds for every series of complex numbers $\sum_{k=1}^{\infty} a_k$ *with* $A_3^{(p)} < \infty$ *exists if and only if the condition* $(*)$ *holds.*

Proof. Let the condition $(*)$ hold and let $A_3^{(p)} < \infty$. We denote

$$q_n^{(p)} = \left(\frac{1}{\lambda_n} \sum_{k=1}^{n} \frac{\lambda_k^p |a_k|^p}{(\lambda_k - \lambda_{k-1})^{p-1}}\right)^{1/p}, \quad n \in \mathrm{N}, \quad q_0^{(1)} = 0.$$

Then for any $n \in \mathrm{N}$,

$$\left|s_n - f(t_n)\right| \leqq \sum_{k=1}^{n} \left|a_k\right| (1 - \exp(-\lambda_k t_n)) + \sum_{k=n+1}^{\infty} \left|a_k\right| \exp(-\lambda_k t_n) =$$

$$= \sum_{k=1}^{n} \lambda_k \left|a_k\right| \frac{1 - \exp(-\lambda_k t_n)}{\lambda_k} + \sum_{k=n+1}^{\infty} \lambda_k \left|a_k\right| \frac{\exp(-\lambda_k t_n)}{\lambda_k} =$$

$$= \sum_{k=1}^{n} (\lambda_k q_k^{(1)} - \lambda_{k-1} q_{k-1}^{(1)}) \frac{1 - \exp(-\lambda_k t_n)}{\lambda_k} + \sum_{k=n+1}^{\infty} (\lambda_k q_k^{(1)} - \lambda_{k-1} q_{k-1}^{(1)}) \frac{\exp(-\lambda_k t_n)}{\lambda_k} =$$

$$= \sum_{k=1}^{n-1} q_k^{(1)} \lambda_k \left(\frac{1 - \exp(-\lambda_k t_n)}{\lambda_k} - \frac{1 - \exp(-\lambda_{k+1} t_n)}{\lambda_{k+1}} \right) +$$

$$+ q_n^{(1)} \lambda_n \left(\frac{1 - \exp(-\lambda_n t_n)}{\lambda_n} - \frac{\exp(-\lambda_{n+1} t_n)}{\lambda_{n+1}} \right) +$$

$$+ \sum_{k=n+1}^{\infty} q_k^{(1)} \lambda_k \left(\frac{\exp(-\lambda_k t_n)}{\lambda_k} - \frac{\exp(-\lambda_{k+1} t_n)}{\lambda_{k+1}} \right) \leqq$$

$$\leqq \sum_{k=1}^{n-1} q_k^{(1)} \lambda_k \Delta \left(\frac{1 - \exp(-\lambda_k t_n)}{\lambda_k} \right) + q_n^{(1)} \lambda_n \left(\frac{1 - \exp(-\lambda_n t_n)}{\lambda_n} + \frac{\exp(-\lambda_{n+1} t_n)}{\lambda_{n+1}} \right) +$$

$$+ \sum_{k=n+1}^{\infty} q_k^{(1)} \lambda_k \Delta \left(\frac{\exp(-\lambda_k t_n)}{\lambda_k} \right) = v_n \,.$$

Since the condition $(*)$ holds, we have $M_2 = \limsup_{n \to \infty} A_2(n) < \infty$ according to Theorem 4, where

$$A_2(n) = \sum_{k=1}^{n-1} \lambda_k \left| \Delta \left(\frac{1 - \exp(-\lambda_k t_n)}{\lambda_k} \right) \right| + \lambda_n \left| \frac{1 - \exp(-\lambda_n t_n)}{\lambda_n} - \frac{- \exp(-\lambda_{n+1} t_n)}{\lambda_{n+1}} \right| +$$

$$+ \sum_{k=n+1}^{\infty} \lambda_k \left| \Delta \left(\frac{\exp(-\lambda_k t_n)}{\lambda_k} \right) \right| .$$

It is easy to deduce from Hölder's inequality that $A_3^{(p)} \leqq A_3^{(p')}$ for $1 \leqq p \leqq p'$. As $A_3^{(p)} < \infty$, we have $A_3^{(1)} = \limsup_{n \to \infty} q_n^{(1)} < \infty$ as well, and so we can use Lemma 3 on v_n. We obtain

$$\limsup_{n \to \infty} \left|s_n - f(t_n)\right| \leqq M_2 A_3^{(1)} \leqq M_2 A_3^{(p)} .$$

Consequently, there exists a finite Tauberian constant M for the (D, λ) summability under the condition $A_3^{(p)} < \infty$, belonging to a sequence $\{t_n\}_{n=1}^{\infty}$. Indeed, we may take $M = M_2$. (This constant, of course, need not be the best possible.)

Suppose the condition $(*)$ does not hold. Then we find a series of complex numbers $\sum_{k=1}^{\infty} a_k$ such that $A_3^{(p)} = 1$ and $\limsup_{n \to \infty} \left|s_n - f(t_n)\right| = \infty$. Firstly, by Theorem 3

$$(15) \qquad \limsup_{n\to\infty} \sum_{k=1}^{\infty} \frac{\lambda_k - \lambda_{k-1}}{\lambda_k} |b_{nk} - c_{nk}| = \infty$$

where matrices B, C are defined by (1). Define a series $\sum_{k=1}^{\infty} a_k$ such that

$$(16) \qquad |a_k| = \frac{\lambda_k - \lambda_{k-1}}{\lambda_k}, \quad k \in \mathrm{N}, \quad \text{and}$$

$$\limsup_{n\to\infty} |B_n(a) - C_n(a)| = \limsup_{n\to\infty} |\sum_{k=1}^{\infty} a_k(b_{nk} - c_{nk})| = \infty .$$

This is possible by (15) (see the proof of Lemma 3). Then $A_1 = 1$ for this series. Let us consider $A_3^{(p)}$. Evidently by (16)

$$q_n^{(p)} = \left(\frac{1}{\lambda_n} \sum_{k=1}^{n} \frac{\lambda_k^p |a_k|^p}{(\lambda_k - \lambda_{k-1})^{p-1}}\right)^{1/p} = \left(\frac{1}{\lambda_n} \sum_{k=1}^{n} (\lambda_k - \lambda_{k-1})\right)^{1/p} = 1$$

and thus $A_3^{(p)} = 1$. It means we have found such a series as required. It follows from the existence of such a series that there exists no finite Tauberian constant for the (D, λ) summability under the condition $A_3^{(p)} < \infty$, belonging to a sequence $\{t_n\}_{n=1}^{\infty}$. We have proved necessity of the condition $(*)$. This completes the proof.

3.4. Let $\{\lambda_n\}_{n=1}^{\infty}$, $\{t_n\}_{n=1}^{\infty}$ be sequences of real positive numbers such that the condition $(*)$ holds. We define $M_3^{(p)} = \inf \{M, \limsup_{n\to\infty} |s_n - f(t_n)| \leqq M A_3^{(p)}$ for every series $\sum_{k=1}^{\infty} a_k$ such that $A_3^{(p)} < \infty\}$ for $p \geqq 1$. It is clear that $M_3^{(p)}$ is the least Tauberian constant for the (D, λ) summability under the condition $A_3^{(p)} < \infty$, belonging to a sequence $\{t_n\}_{n=1}^{\infty}$. We have inequalities corresponding to M_1, M_2 and $M_3^{(p)}$:

Proposition 1. *We have, for* $1 \leqq p < p'$,

$$M_1 \leqq M_3^{(p')} \leqq M_3^{(p)} \leqq M_2 .$$

Proof. We have

$$(17) \qquad A_2 \leqq A_3^{(p)} \leqq A_3^{(p')} \leqq A_1$$

for any series of complex numbers $\sum_{k=1}^{\infty} a_k$ and for $1 \leqq p \leqq p'$ (cf. [7], p. 749, (9)). By Theorem 3 there exists a series $\sum_{k=1}^{\infty} a_k$ such that $A_1 = 1$ and $\limsup_{n\to\infty} |s_n - f(t_n)| = M_1$. Then (17) implies $A_3^{(p')} \leqq 1$, $p' > 1$, for this series. According to Theorem 5, $\limsup_{n\to\infty} |s_n - f(t_n)| \leqq M_3^{(p')} A_3^{(p')}$ holds. Then clearly $M_1 \leqq M_3^{(p')} A_3^{(p')} \leqq M_3^{(p')}$. For the proof of the second inequality take an arbitrary $\varepsilon > 0$. Since $M_3^{(p')} =$

$= \sup \{\limsup_{n\to\infty} |s_n - f(t_n)|, A_3^{(p')} = 1\}$, there exists a series $\sum_{k=1}^{\infty} a_k$ such that $A_3^{(p')} = 1$. and $M_3^{(p')} - \varepsilon \leqq \limsup_{n\to\infty} |s_n - f(t_n)| = M_3^{(p)} A_3^{(p)} \leqq M_3^{(p)}$ for $1 \leqq p < p'$ by (17). It follows that $M_3^{(p')} \leqq M_3^{(p)}$, $1 \leqq p < p'$. The last inequality is clear by the proof of Theorem 5. This completes the proof.

Remark 5. Theorems 3 and 4 guarantee existence of a series $\sum_{k=1}^{\infty} a_k$ with $A_1 = 1$ or $A_2 = 1$ and $\limsup_{n\to\infty} |s_n - f(t_n)| = M_1$ or $\limsup_{n\to\infty} |s_n - f(t_n)| = M_2$, respectively. It is not clear if the constant $M_3^{(p)}$, $p \geqq 1$, defined before Proposition 1 is the best possible in this sense. Only some particular results are known, e.g.: if $p \geqq 1$, the condition (*) holds and $\lambda_{n+1} t_n \geqq \log 2$ for each $n \in \mathrm{N}$, then

$$\limsup_{n\to\infty} |s_n - f(t_n)| \leqq M_3^{(p)} A_3^{(p)}$$

for every series $\sum_{k=1}^{\infty} a_k$ such that $A_3^{(p)} < \infty$. Moreover, $M_3^{(p)} = M_1$ and there exists a series $\sum_{k=1}^{\infty} a_k$ such that $A_3^{(p)} = 1$ and $\limsup_{n\to\infty} |s_n - f(t_n)| = M_1$.

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Author's address: 306 14 Plzeň, Nejedlého sady 14 (Vysoká škola strojní a elektrotechnická).