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SEVERAL EXTREMAL COREFLECTIVE CLASSES IN UNIFORM SPACES

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We study possible coreflective values on some very simple metrizable zero-dimensional topologically discrete uniform spaces. The results are applied to the study of a nice class of $H(\omega) - a$ spaces and to some problems concerning the existence of the largest coreflective subclass contained in a given class. Such classes are useful in the theory of uniform measures.

1. Introduction. We shall work in the category $\mathbf{U}$ of separated uniform spaces with uniformly continuous mappings. Under the symbol $\mathbf{U}(X, Y)$ we shall understand the set of all uniformly continuous mappings with a domain X and ranging in Y. If $\mathscr{C}$ is a full coreflective subcategory of $\mathbf{U}$ with F the corresponding coreflector, we shall often say shortly that $(\mathscr{C}, F)$ is a coreflection. Throughout the paper the symbols d, δ, t_f, a will denote the uniformly discrete, proximally discrete, topologically fine and Alexandrov coreflections, respectively. The first three are well known (cf. e.g. [8]), the last is described by the property that each bounded coz-function is uniformly continuous (see [3]).

Assume that X is a uniform space, F any coreflector in $\mathbf{U}$; under the symbol $X - F$ we shall understand the class of all spaces Y such that each $f \in \mathbf{U}(Y, X)$ remains uniformly continuous into FX. It can be easily proved ([13]) that $X - F$ again forms a coreflective class in $\mathbf{U}$ and if X is an injective uniform space, $X - F$ is even hereditary. (Recall that X is called *injective* if uniformly continuous mappings ranging in X extend to uniformly continuous ones from arbitrary subspaces.) Finally, we note that if $\mathscr{C}$ is a coreflective subclass of $\mathbf{U}$, the class sub $(\mathscr{C})$ of all subspaces of spaces from $\mathscr{C}$ is again coreflective ([13]).

2. Coreflections on D_2. The main tool in the paper is the examination of the following spaces:

2.1. Definition. Let $\{k_n\}$ be a sequence of natural numbers. We define the space $D(\{k_n\})$ as follows: The underlying set is $\{\langle n, i\rangle;\ n \in N,\ 1 \leqq i \leqq k_n\}$ and the family $\{\mathscr{U}_n;\ n \in N\}$, where

$$\mathscr{U}_n = \{\{\langle k, i\rangle\};\ k \leqq n\} \cup \{\{\langle k, i\rangle;\ i \leqq k_n\};\ k > n\}\ ,$$

forms a basis of covers for the uniformity of $D(\{k_n\})$. If $k_n = 2$ for all n we shall denote the space by D_2 and if $k_n = n$ for all n we shall denote the corresponding space by D_N.

It is easily seen that the spaces $D(\{k_n\})$ are complete, metrizable, zerodimensional, topologically discrete uniform spaces.

2.2. Proposition. *Let $(\mathscr{C}, F)$ be a coreflection in* $\mathbf{U}$, *$D(\{k_n\}) \notin \mathscr{C}$, then $FD(\{k_n\})$ has a discrete proximity. (All pairs of disjoint subsets are proximally far.)*

Proof. One can easily observe that it suffices to prove the statement for the space D_N. We shall identify every natural number with the set of all natural numbers that do not exceed it. Therefore $\{n\} \times n$ will mean the n^{th} column in D_N.

Suppose $D_N = A \cup B$ with $A \cap B = \emptyset$. Let us denote by N_1 the set of all $n \in N$ such that the column $\{n\} \times n$ intersects both A and B. If N_1 is finite, then A, B are far in D_N, hence also in FD_N. If N_1 is infinite, we choose points $a_n \in A \cap \{n\} \times n$, $b_n \in B \cap \{n\} \times n$ for all $n \in N_1$.

a) First we show that the sets $A_1 = \{a_n;\ n \in N_1\}$, $B_1 = \{b_n;\ n \in N_1\}$ are far in FD_N. As D_N is metrizable, there is no strictly finer uniformity with the same proximity (metrizable spaces are even proximally fine, see [1]). Therefore FD_N has a strictly finer proximity than D_N, hence we can find $N_2 \subset N$ infinite and $c_m, d_m \in \in \{m\} \times m$ for all $m \in N_2$ such that the sets $C = \{c_m;\ m \in N_2\}$, $D = \{d_m;\ m \in N_2\}$ are far in FD_N. Now we define for $n \in N_1$:

$$\varphi(n) = \min\{m \in N_2;\ m \geqq n\}\ ,$$

$$h(a_n) = c_{\varphi(n)}\ , \quad h(b_n) = d_{\varphi(n)}\ ;$$

$h(\langle n, i\rangle) \in \{\varphi(n)\} \times \varphi(n)$ is defined arbitrarily for $\langle n, i\rangle \neq a_n, b_n$, only we need it to be one-to-one on each $\{n\} \times n$, $n \in N_1$. Finally, for $n \in N_1$ we put $h(\langle n, i\rangle) = \langle n, i\rangle$. Obviously the mapping h is uniformly continuous from D_N into D_N. hence $h : FD_N \to FD_N$ is uniformly continuous and maps A_1 into C, B_1 into D, these being far in FD_N, hence A_1, B_1 are far in FD_N as well.

b) From the result in a) we easily check that A, B are far in FD_N. We define $g : D_N \to D_N$ as follows:

$$g(\langle n, i\rangle) = \begin{cases} \langle n, i\rangle & \text{for} \quad n \in N_1\ , \\ a_n & \text{for} \quad n \in N_1 \quad \text{and} \quad \langle n, i\rangle \in A\ , \\ b_n & \text{for} \quad n \in N_1 \quad \text{and} \quad \langle n, i\rangle \in B\ ; \end{cases}$$

$g \in \mathbf{U}(D_N, D_N)$, hence $g \in \mathbf{U}(FD_N, FD_N)$, $g(A)$ and $g(B)$ are far in FD_N, hence also A, B are far in FD_N.

2.3. Corollary. *Let $\{k_n\}$ be any sequence of natural numbers, F a coreflector in* $\mathbf{U}$. *Then either $FD(\{k_n\}) = D(\{k_n\})$ or $FD(\{k_n\})$ is finer than $\delta D(\{k_n\})$. (Observe that the latter is uniformly discrete if $\{k_n\}$ is bounded.)*

2.4. Corollary. *The following properties of a uniform space X are equivalent:*

(1) *X is $D(\{k_n\}) - \delta$ for all sequences $\{k_n\}$,*

(2) *X is $D(\{k_n\}) - d$ for all bounded sequences $\{k_n\}$,*

(3) *X is $D_N - \delta$,*

(4) *X is $D_2 - d$.*

Moreover, the coreflective class described here is the largest coreflective class not containing D_2 (or others).

2.5. Proposition. *Every metrizable space is in the coreflective hull of the space D_2 in* $\mathbf{U}$.

The proof is immediate from the following observation (see [1]): If M, S are metrizable uniform spaces and $f : M \to S$ is not uniformly continuous, then there is $g \in \mathbf{U}(D_2, M)$ such that fg is not uniformly continuous.

2.6. Corollary. *If $(\mathscr{C}, F)$ is a coreflection in* $\mathbf{U}$, *then either $\mathscr{C}$ is contained in $D_2 - d$ or $\mathscr{C}$ contains all metrizable spaces.*

3. Spaces $H(\omega) - a$. Corollaries 2.4 and 2.6 give a condition of extremality of the coreflective class $D_2 - d$. Now we turn to the study of spaces which are hereditarily in $D_2 - d$. We will show that this class is again coreflective, has some nice extremal properties and has an interesting description in terms of summability of functions, uniform continuity of algebraic products of functions and others.

Recall that the *hedgehog* $H(A)$ over a set A is the cone over a uniformly discrete space A, that is, the set of all $\langle a, x\rangle$, $a \in A$, $0 \leqq x \leqq 1$, where $\langle a, 0\rangle = \langle b, 0\rangle$ for all $a, b \in A$, metrized by the metric $d(\langle a, x\rangle, \langle a, y\rangle) = |x - y|$ and $d(\langle a, x\rangle, \langle b, y\rangle) = x + y$ for $a \neq b$. $H(A)$ is an inejctive uniform space (see [11]). Uniform spaces projectively generated by mappings into hedgehogs are the so called *distally coarse* spaces, that is, those X for which any distally continuous map (i.e. such maps that the preimages of uniformly discrete families are gain uniformly discrete) ranging in X is uniformly continuous. For details and proofs we refer to [4].

The following theorem seems to be of the main importance. The symbols I, R, ω will stand respectively for a compact interval, the real line and a countable uniformly discrete space.

3.1. Theorem. *The following properties of a uniform space X are equivalent:*

(1) *X is hereditarily $D(\{k_n\}) - d$ for each bounded sequence $\{k_n\}$,*

(2) *X is hereditarily $D_N - \delta$,*

(3) *X is hereditarily $D_2 - d$,*

(4) *each countable uniformly discrete union of boundedly finite uniformly discrete families is a uniformly discrete family,*

(5) *X is hereditarily $(I \times \omega) - a$,*

(6) *X is hereditarily $R - a$,*

(7) *X is $H(\omega) - a$,*

(8) *if $f_n \in \mathbf{U}(X, I)$ is a countable family such that $\{\operatorname{coz} f_n\}_n$ is a uniformly discrete family, then the function $\sum f_n$ is uniformly continuous,*

(9) *X is hereditarily $R^2 - a$,*

(10) *for any subspace Y of X and any two functions $f, g \in \mathbf{U}(Y, R)$, their (algebraic) product $f \,.\, g$ is uniformly continuous whenever $f \,.\, g$ is bounded,*

(11) *for any subspace Y of X, $f, g \in \mathbf{U}(Y, R)$ with g bounded and $f \,.\, g$ bounded, the function $f \,.\, g$ is uniformly continuous.*

Proof. The equivalence of (1)–(3) follows immediately from 2.4.

(1) ⇒ (4): Take $\{A_n\}_n$ a uniformly discrete countable family and let $A_n = \bigcup \{B_{n_i};\ i = 1, 2, \ldots, k_n\}$ with $\{k_n\}$ bounded and $\{B_{n_i}\}_i$ uniformly discrete. The mapping $g(x) = \langle n, i \rangle$ for $x \in B_{n_i}$ is uniformly continuous from $\bigcup A_n$ (as a subspace of X) into $D(\{k_n\})$. (1) implies that g remains uniformly continuous into $dD(\{k_n\})$, hence the family $\{B_{n_i}\}_{i,n}$ is uniformly discrete.

(4) ⇒ (5): $I \times \omega$ is metrizable, so each open subset is a cozero set, hence $a(I \times \omega) = \inf(I \times \omega,\ pt_f(I \times \omega))$, where p stands for the precompact reflector and the infimum is taken in the usual order "finer than". Therefore the uniformly discrete families in $a(I \times \omega)$ are just the $I \times \omega$-uniformly discrete unions of boundedly finite uniformly discrete families in $I \times \omega$. Moreover, $a(I \times \omega)$, being the infimum of two distally coarse spaces, is distally coarse (in view of the reflectivity of the class of all distally coarse spaces). Therefore if X fulfils (4) then each $f \in \mathbf{U}(X, I \times \omega)$ is distally continuous into $a(I \times \omega)$, hence $f \in \mathbf{U}(X, a(I \times \omega))$. The rest follows from the evident heredity of the property (4).

(5) ⇒ (6): Take a subspace Y of X, $f \in \mathbf{U}(Y, R)$. The cover $\{V, W\}$, where

$$V = \bigcup \{[\![n - \tfrac{1}{2},\ n + 1]\!];\ n \text{ is an odd integer}\}\,,$$

$$W = \bigcup \{[\![n - \tfrac{1}{2},\ n + 1]\!];\ n \text{ is an even integer}\}\,,$$

is a uniform cover of R. For any cover $\mathscr{U}$ uniform on aR the covers $f^{-1}(\mathscr{U}) \mid f^{-1}(V)$, $f^{-1}(\mathscr{U}) \mid f^{-1}(W)$ are uniform, $\{f^{-1}(V), f^{-1}(W)\}$ is uniform as well, hence the cover $f^{-1}(\mathscr{U})$ is uniform on Y.

(6) ⇒ (3): $R - a$ is a coreflective property, $D_2 \notin R - a$, hence by 2.4 $R - a$ is contained in $D_2 - d$.

(5) ⇒ (7): Suppose $f : X \to H(\omega)$ is uniformly continuous. Take any finite open cover $\mathscr{U}$ of $H(\omega)$. $\mathscr{U}$ can be refined by a uniform cover $\mathscr{V}$ of the form $\{\langle n, x \rangle;$

$x < \varepsilon\} \cup \mathscr{V}_1$, where $\mathscr{V}_1$ is a finite open cover of the space $B = \{\langle n, x\rangle; \frac{1}{2}\varepsilon \leqq x \leqq 1\}$ as a subspace of $H(\omega)$. If we denote the subspace $f^{-1}[B]$ of X by Y, then the mapping $f_1 = f \mid Y$ is uniformly continuous into B, the latter being uniformly homeomorphic to $I \times \omega$, hence $f_1 \in U(Y, aB)$, and hence the cover

$$f_1^{-1}[\mathscr{V}_1] \cup \{f^{-1}[\{\langle n, x\rangle; x < \varepsilon\}]\}$$

is a uniform cover of X refining $f^{-1}(\mathscr{U})$.

(7) ⇒ (3): $D_2 \notin H(\omega) - a$ and the same argument as in (6) ⇒ (3) applies.

(4) ⇒ (8): Take a countable family $f_n \in \mathbf{U}(X, I)$ with the family $\{\operatorname{coz} f_n\}_n$ uniformly discrete. We shall prove that the function $f = \sum f_n$ is proximally continuous. Take any two far subsets A, B of I. We may and shall suppose that $0 \notin A \cup B$. For each n the sets $f_n^{-1}[A], f_n^{-1}[B]$ are far in X and, moreover, they are both contained in $\operatorname{coz} f_n$. Applying (4) we conclude that the family

$$\{f_n^{-1}[A];\ n \in \omega\} \cup \{f_n^{-1}[B];\ n \in \omega\}$$

is uniformly discrete in X and hence $f^{-1}[A] = \bigcup f_n^{-1}[A]$ and $f^{-1}[B] = \bigcup f_n^{-1}[B]$ are far in X. Therefore f is proximally continuous into I, hence it is uniformly continuous.

(8) ⇒ (9): First we observe that the property (8) is closed under uniform sums, quotients and subspaces (it is a direct verification). So (8) describes a hereditary coreflection in $\mathbf{U}$. For a while we shall denote the corresponding coreflector by F. The only thing we must prove is that FR^2 is finer than aR^2.

a) First we prove that $F(I^2 \times \omega)$ is finer than $a(I^2 \times \omega)$. Take a finite open cover $\mathscr{P} = \{P_i\}_{i=1}^k$ of $I^2 \times \omega$. Taking the standard k-element open cover $\mathscr{U}$ of I, then for each n we can easily find functions $p_n', q_n' : I^2 \times \{n\} \to I$ uniformly continuous and such that the cover $p_n'^{-1}(\mathscr{U}) \wedge q_n'^{-1}(\mathscr{U})$ refines $\mathscr{P} \mid I^2 \times \{n\}$ for each n. Now we define

$$p_n(x) = \begin{cases} p_n'(x) & \text{for } x \in I^2 \times \{n\}\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

$$q_n(x) = \begin{cases} q_n'(x) & \text{for } x \in I^2 \times \{n\}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

The sequences $\{p_n\}$, $\{q_n\}$ both fulfil the hypothesis of (8), hence $p = \sum p_n$, $q = \sum q_n$ are elements of $\mathbf{U}(F(I^2 \times \omega), I)$ and of course $p^{-1}(\mathscr{U}) \wedge q^{-1}(\mathscr{U})$ refines $\mathscr{P}$, which gives that $F(I^2 \times \omega)$ is finer than $a(I^2 \times \omega)$.

b) In the second step we prove that $F(I \times R)$ is finer than $a(I \times R)$. We take

$$V = \{I \times [\![n - \tfrac{1}{2},\ n + 1]\!];\ n \text{ an odd integer}\}\,,$$
$$W = \{I \times [\![n - \tfrac{1}{2},\ n + 1]\!];\ n \text{ an even integer}\}\,.$$

$\{V, W\}$ is a uniform cover of $I \times R$ and both V, W are isomorphic to $I^2 \times \omega$. If $\mathscr{P}$ is any finite open cover of $I \times R$, then in virtue of a) and the heredity of F, both

$\mathscr{P} \mid V$, $\mathscr{P} \mid W$ are uniform covers for the uniformity inherited from $F(I \times R)$, hence $\mathscr{P}$ is uniform on $F(I \times R)$.

c) Now we show that $F(I \times R \times \omega)$ is finer that $a(I \times R \times \omega)$. We again take a finite open cover $\mathscr{P} = \{P_i\}_{i=1}^k$ of $I \times R \times \omega$ and go on in a similar way as in the proof of a) showing that $\mathscr{P}$ is uniform on $F(I \times R \times \omega)$.

d) Similar arguments as in b) show again in virtue of c) that FR^2 is finer than aR^2 and the proof is complete.

(9) $\Rightarrow$ (10): If $f, g \in \mathbf{U}(Y, R)$, then the cartesian product $f \times g$ defined by $(f \times g)(x) = \langle fx, gx \rangle$ is an element of $\mathbf{U}(Y, R^2)$. Using (9) we have that $f \times g \in \mathbf{U}(Y, aR^2)$. Take $p : R^2 \to R$ defined by $p(x, y) = x \,.\, y$. Then p is continuous, hence uniformly continuous on $t_f R^2$. By assumption there is a positive real K such that $|(f \,.\, g)(x)| \leqq K$ for all $x \in X$, hence for $S = p^{-1}([\![-K, K]\!])$, $p_1 = p \mid S$ the following assertions hold:

a) aS is a subspace of aR^2, because S is closed in R^2,

b) $f \times g$ ranges in S, hence $f \times g \in \mathbf{U}(Y, aS)$,

c) p_1 ranges in a compact interval, hence $p_1 \in \mathbf{U}(aS, R)$.

Therefore $f \,.\, g = p_1 \circ (f \times g)$ is uniformly continuous. (Here "$\circ$" stands for the composition of mappings and "." for the algebraic product.)

(10) $\Rightarrow$ (11) is evident.

(11) $\Rightarrow$ (3): One can immediately see that D_2 does not fulfil the condition (11), so it suffices to prove that (11) is a coreflective property (this is routine by verifying the closedness under sums and quotients) and to use 2.4.

3.2. Corollary. *The class $H(\omega) - a$ described in Theorem* 3.1 *is the largest hereditary coreflective subclass in* $\mathbf{U}$ *not containing D_2 and the largest coreflective subclass in* $\mathbf{U}$ *not containing $H(\omega)$. The proof follows from* 2.4, 2.5 *and the injectivity of $H(\omega)$.*

Taking into account 2.5 we obtain that if a hereditary coreflective subclass $\mathscr{C}$ of $\mathbf{U}$ is not contained in $H(\omega) - a$, then $\mathscr{C}$ contains sub (co ($\mathscr{M}$)), where co ($\mathscr{M}$) stands for the coreflective hull of metric spaces in $\mathbf{U}$. Hušek and Rice have proved recently that assuming the nonexistence of uniformly sequential cardinals ($\nexists u$) the class co ($\mathscr{M}$) is productive (see [7]), hence sub (co ($\mathscr{M}$)) = $\mathbf{U}$. So under this set theoretical assumption we have the following surprising result:

3.3. Corollary. ([$\nexists u$]). *There exists a largest nontrivial hereditary coreflective subclass of* $\mathbf{U}$, *namely the class $H(\omega) - a$.*

4. Coreflections on D_N. Given a coreflection ($\mathscr{C}$, F) in $\mathbf{U}$, then $D_N \in \mathscr{C}$ if and only if $D_2 \in \mathscr{C}$. One can observe this easy fact taking into account that D_2 is a retract of D_N and using 2.5 to prove the converse. Moreover, we know that the first non-identical coreflective value on D_N is δD_N (see 2.2). We will show that the space δD_N

behaves very well again with respect to coreflections, which will be useful for studying spaces in the class $D_N - d$.

A uniform space X will be called *distally minimal* if each strictly finer uniformity has strictly more uniformly discrete families.

4.1. Lemma. *The space δD_N is distally minimal.*

Proof. Take X strictly finer than δD_N.

a) First we observe that a cover $\mathscr{U}$ is uniform on δD_N whenever the following holds ($\{n\} \times n$ again stands for the n^{th} column in D_N): There exists $k \in N$ such that for each $n \in N$ and for each star-refinement $\mathscr{W}$ of $\mathscr{U}$ there are points $x_1, x_2, \ldots, x_k \in \in \{n\} \times n$ such that $\{n\} \times n \subset \bigcup \{\mathrm{St}(x_i, \mathscr{W});\ i = 1, 2, \ldots, k\}$. To prove this we put

$$P_1(n) = \mathrm{St}(x_1, \mathscr{W} \wedge \{n\} \times n),$$

$$P_i(n) = \mathrm{St}(x_i, \mathscr{W} \wedge \{n\} \times n) \setminus \bigcup_{j<i} P_j \quad \text{for} \quad 1 < i \leqq k,$$

$$P_i = \bigcup_{n \in N} P_i(n) \quad \text{for} \quad 1 \leqq i \leqq k.$$

Then the cover

$$\{P_i;\ i = 1, 2, \ldots, k\} \wedge \{\{n\} \times n;\ n \in N\}$$

is uniform on δD_N and refines $\mathscr{U}$.

b) Now suppose $\mathscr{U}$ is uniform on X and not uniform on δD_N. Using a) we can find for each $k \in N$ some $n_k \in N$ and $\mathscr{W}$ star-refining $\mathscr{U}$ such that we cannot find less than k points in $\{n_k\} \times n_k$, the $\mathscr{W}$-stars of which cover $\{n_k\} \times n_k$. Therefore we can find points $x_1, x_2, \ldots, x_k \in \{n_k\} \times n_k$ such that the stars

$$\{\mathrm{St}(x_i, \mathscr{W} \wedge \{V_k, W_k\})\}_{i,k}$$

form a disjoint system, where

$$V_k = \{x \in \{n_k\} \times n_k;\ \text{there exist } i \neq j \text{ such that } 1 \leqq i, j \leqq k \text{ and } x \in \mathrm{St}(x_i, \mathscr{W}) \cap \mathrm{St}(x_j, \mathscr{W})\},$$

$$W_k = \{n_k\} \times n_k \setminus V_k.$$

Now it follows immediately that the family

$$\{\{\langle n_k, x_i \rangle\};\ i = 1, 2, \ldots, k;\ k \in N\}$$

is uniformly discrete in X and, of course, not uniformly discrete in δD_N.

4.2. Proposition. *Let $(\mathscr{C}, F)$ be the coreflection in* **U**, *$\delta D_N \notin \mathscr{C}$, then $F\delta D_N$* **is** *uniformly discrete.*

Proof. The proof proceeds very similarly as that of 2.2. Using 4.1 we know that in $F\delta D_N$ we add a certain uniformly discrete family. The same procedure as in the proof of 2.2 gives that each countable partition of $F\delta D_N$ is uniform.

4.3. Corollary. *$\delta D_N - d$ is the largest coreflective subclass of **U** not containing the space δD_N.*

4.4. Corollary. *The following properties of a uniform space X are equivalent:*

(1) *X is $D(\{k_n\}) - d$ for all sequences $\{k_n\}$,*
(2) *X is $D_N - d$,*
(3) *X is simultaneously $D_2 - d$ and $\delta D_N - d$.*

*Moreover, the class $D_N - d$ is the largest coreflective subclass of **U** not containing both D_2 and δD_N. The proof is immediate by combining* 2.4 *and* 4.3.

A theorem similar to 3.1 can be proved for hereditarily $D_N - d$ spaces. These spaces are studied in [5] and [6] and they are called $H(\omega) - t_f$ spaces after the property (6).

4.5. Theorem. *The following properties of a uniform space X are equivalent:*

(1) *X is hereditarily $D(\{k_n\}) - d$ for each sequence $\{k_n\}$,*
(2) *X is hereditarily $D_N - d$,*
(3) *X is hereditarily $D_2 - d$ and hereditarily $\delta D_N - d$,*
(4) *each countable uniformly discrete union of finite uniformly discrete families is a uniformly discrete family,*
(5) *X is hereditarily $(I \times \omega) - t_f$,*
(6) *X is $H(\omega) - t_f$,*
(7) *each countable family $\{f_n\}$ of bounded uniformly continuous functions such that $\{\mathrm{coz} f_n\}_n$ is uniformly discrete has a uniformly continuous sum,*
(8) *X is hereditarily $R - t_f$,*
(9) *X is hereditarily $R^n - t_f$ for each natural n,*
(10) *for each subspace Y of X the set **U**(Y, R) is a ring under pointwise operations,*
(11) *for any subspace Y of X, $f, g \in$ **U**(Y, R) with g bounded, the function $f \,.\, g$ is uniformly continuous.*

Remarks. The proof of this theorem may be found in [5] or [6]. Only the property (3) is new and for this property the equivalence with (1) follows from 4.4.

All the properties in 4.5 are formulated in the same manner as in 3.1. Note that the condition (9) in 3.1 can also be proved for any natural n, which makes the analogy between 3.1 and 4.5 complete.

We observe again that e.g. the condition (6) proves that hereditarily $D_N - d$ spaces form a coreflective subclass of **U**. However, it is not true in general that the class her $(\mathscr{C})$ of spaces which are hereditarily in $\mathscr{C}$ forms a coreflective subclass if $\mathscr{C}$ is coreflective. For example, her $(\delta D_N - d)$ is not coreflective. Another example is the class of hereditarily Alexandrov spaces (see [3]).

The condition (3) of 4.5 together with 4.4 gives the following useful extremal property for $H(\omega) - t_f$ spaces:

4.6. Corollary. *Let $(\mathscr{C}, F)$ be a hereditary coreflection in **U** and let neither D_2 nor δD_N be contained in $\mathscr{C}$. Then $\mathscr{C}$ is contained in $H(\omega) - t_f$.*

As an application of 4.6 we present the following theorem:

4.7. Theorem. *$H(\omega) - t_f$ is the largest coreflective subclass of **U** contained in each of the following classes:*

(A) *The class* Ext *of all X such that each uniformly continuous real valued function defined on a subspace of X extends to X.*
(B) *The class of all X such that each real valued function on a uniformly discrete subspace of X extends to a uniformly continuous function on X.*
(C) *The class of all X such that for each free uniform measure μ on X the support* supp $\tilde{\mu}$ *of the corresponding Radon measure on a Samuel compactification $\tilde{X}$ of X lies in the completion $\hat{X}$ of X.*
(D) *The class of all X such that each bounded subset of X is precompact.*

Remarks. Conditions (A), (B) need no explanation. A linear form μ on $\mathbf{U}(X, R)$ is called a *free uniform measure* on X, if for each $H \subset \mathbf{U}(X, R)$ uniformly equicontinuous and point-bounded the restriction $\mu \nearrow H$ is continuous in the topology of pointwise convergence. The reader is refered to [9] for results and further references. If pX denotes the precompact reflection of X, the Samuel compactification $\tilde{X}$ of X is the completion $\widehat{pX}$ of pX. Finally, recall that a subset B of X is called bounded in X, if each $f \in \mathbf{U}(X, R)$ is bounded on B. We refer to [12] for some properties of the class (D).

Proof of 4.7. The result concerning classes (A), (B) is proved in [5] or [6]. One can also very easily see that (C) $\Rightarrow$ (D) (see [2]). Moreover, it is proved in [10] that the condition (7) from 4.5 implies the property (C). The class (D) is obviously hereditary, hence with each coreflection $(\mathscr{C}, F)$ contained in (D), (D) also contains sub $(\mathscr{C})$. Therefore, according to the above remarks and to 4.6, it suffices to prove that for any (hereditary) coreflection $(\mathscr{C}, F)$ contained in (D), neither D_2 nor δD_N are contained in $\mathscr{C}$.

If D_2 is contained in $\mathscr{C}$, then each metrizable space is in $\mathscr{C}$ (see 2.5), which is not possible, as for example $H(\omega)$ is metrizable, bounded in itself, but not precompact.

Similarly $aH(\omega) = \inf(H(\omega), pt_f H(\omega))$ is bounded in itself and not precompact, hence it is not in (D). Hence $FaH(\omega)$ must contain a 2-discrete uniform cover $\mathscr{U}$ such that the number of members of $\mathscr{U}$ in its trace on $\{\langle n, x \rangle; 0 < x \leqq 1\}$ increases to infinity with the increase of n. This means that we can embed δD_N into $aH(\omega)$ in such a way that the uniformity of $FaH(\omega)$ restricted to δD_N is strictly finer, hence $F\delta D_N \neq \delta D_N$, which concludes the proof.

The conditions (8) from 3.1 or (7) from 4.2 make it possible to define natural concepts of σ-additivity of uniform measures on the corresponding classes of spaces, but we do not intend to go into any details here.

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