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ORTHOGONAL PARTITIONS AND COVERING OF GRAPHS

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INTRODUCTION

We investigate two notions related to the notion of chromatic number. For a given graph G one may define $\bar{\omega}(G)$ as the minimum number of stable sets (i.e. subsets of the vertex set with no two vertices joined by an edge) such that every pair of non-adjacent vertices is contained in at least one of them, and dim G as the minimal d such that G is contained in a product of d complete graphs as an induced subgraph.

Determination of both $\bar{\omega}$ and dim is an NP-complete problem. (See [15], [10].) In [8] and [12], dim of some special graphs was investigated. The main purpose of this paper is to prove some analogous statements concerning the notion of $\bar{\omega}$ and extend some results of [8] and [12].

The first two sections are of technical character. In § 1 we present some remarks concerning the notion of orthogonal array (which is in our sense more general than that defined in [5]).

In § 2 we introduce the notion of orthogonal q-partitions and investigate the problem of maximal number of pairwise orthogonal q-partitions on a given set. For $q = 2$ we are able to give an exact formula for this number. For $q \geqq 3$ we give only some bounds.

In § 4 we show that $\bar{\omega}$ of a product of two graphs with large $\bar{\omega}$ is large, too. To prove an analogous statement concerning chromatic number is an open (and probably difficult) problem.

In last paragraphs using the results of the first two sections we give a formula for $\bar{\omega}(nK_2)$ and some bounds for $\bar{\omega}(nK_q)$, dim (nK_q) and dim $\binom{n}{q}$.

We state now three well known theorems which will be used throughout our paper.

Theorem 0.1. (see [5]) *Let q be a power of a prime. Then there exist $q - 1$ pairwise orthogonal latin sguares of order q.*

Theorem 0.2. (see [2]) *Let $A_1, A_2, \ldots, A_n$ be a system of subsets of* $1, 2, \ldots, t$ *such that*:

(i) $A_i \nsubseteq A_j$;

(ii) $A_i \cap A_j \neq \emptyset$;

(iii) $|A_i| \leqq \frac{1}{2}t$.

Then

$$n \leqq \binom{t-1}{\left[\frac{t}{2}\right]-1}.$$

Theorem 0.3. *For every integer n there exists a prime p such that*

$$n \leqq p \leqq 2n .$$

1. ORTHOGONAL ARRAYS

Let q be a positive integer. We say that two vectors $\mathbf{u} = (u_1, u_2, \ldots, u_{t_1})$ and $\mathbf{v} = (v_1, v_2, \ldots, v_{t_2})$ are orthogonal if for every $k, k' = 1, 2, \ldots, q$ there exists $j = 1, 2, \ldots, t$ so that $u_j = k$ and $v_j = k'$.

For two vectors $\mathbf{u} = (u_1, u_2, \ldots, u_t)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_t)$ we put $(\mathbf{u}, \mathbf{v}) = (u_1, u_2, \ldots, u_{t_1}, v_1, v_2, \ldots, v_{t_2})$.

Let π be a permutation of the set $\{1, 2, \ldots, q\}$. We identify π with the vector $(\pi(1), \pi(2), \ldots, \pi(q))$. If $\mathbf{u} = (u_j)$, $u_j \in \{1, 2, \ldots, q\}$ is another vector then we put

$$\pi(\mathbf{u}) = (\pi(u_1), \pi(u_2), \ldots, \pi(u_t)) .$$

Obviously the following holds

$$\pi(\mathbf{u}, \mathbf{v}) = (\pi(\mathbf{u}), \pi(\mathbf{v})) .$$

We say that a vector $\mathbf{w}$ has the permutation property if

$$\mathbf{w} = (\pi_1, \pi_2, \ldots, \pi_r)$$

where π_s, $s = 1, 2, \ldots, r$, are permutations of $\{1, 2, \ldots, q\}$ for some q and r.

The orthogonal array $\mathbf{OA}(q, n, t)$ is an $n \times t$ matrix (a_{ij}) where $a_{ij} \in \{1, 2, \ldots, q\}$ for every i, j, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, t$, such that every two row-vectors are orthogonal. The orthogonal array $(\mathbf{OA})$ has the permutatiin property if each row of the corresponding matrix has this property.

Let $t = q^2$. Then the existence of $\mathbf{OA}(q, n, q^2)$ is equivalent with the existence of $n - 2$ pairwise orthogonal latin squares of order q. (See e.g. [1, 5].) In [3], the maximal number of pairwise orthogonal latin squares is estimated. For a given q there exist at most $q - 1$ pairwise orthogonal latin squares, and if q is a power of a prime then such a system exists.

Theorem 1.1. *Let q be a power of a prime. Then*

(i) *There exists* $\mathbf{OA}(q, q+1, q^2)$.

(ii) *There exists* $\mathbf{OA}(q, q, q^2)$ *which has the permutation property.*

Proof. By Theorem 0.1 and the relation between $\mathbf{OA}$'s we have (i). It remains to prove (ii).

Take $\mathbf{OA}(q, q+1, q^2)$. In each row there are exactly q entries equal to k for each $k = 1, 2, \ldots, q$. We can suppose that the last row is of the form $(1, 1, \ldots, 1, 2, 2, \ldots \ldots, 2, \ldots, q, q, \ldots, q)$. (This follows from the obvious fact that if we rearrange columns of $\mathbf{OA}$ we obtain again $\mathbf{OA}$.) The first q rows obviously form an $\mathbf{OA}$ with the permutation property. So for $i = 1, 2, \ldots, q$ the i-th row is of the form $(\pi_{i1}, \pi_{i2}, \ldots, \pi_{iq})$.

The following theorem as well as its proof is a simple generalization of [9] (see [5], Theorem 13.2.1).

Theorem 1.2. *The existence of* $\mathbf{OA}(q, n_1, t_1)$ *and* $\mathbf{OA}(q, n_2, t_2)$ *implies the existence of* $\mathbf{OA}(q_1 q_2, n, t_1 t_2)$.

Theorem 1.3. *The existence of* $\mathbf{OA}(q, n_1, t_1)$ *and* $\mathbf{OA}(q, n_2, t_2)$ *implies the existence of* $\mathbf{OA}(q, n_1 n_2, t_1 + t_2)$.

Moreover, if the first two $\mathbf{OA}$*'s have the permutation property then the last one has the property as well.*

Proof. Denote the row-vectors of $\mathbf{OA}(q, n_1, t_1)$ and $\mathbf{OA}(q, n_2, t_2)$ by $\mathbf{u}_1, \mathbf{u}_2, \ldots \ldots, \mathbf{u}_{t_1}$ and $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v} \ldots, {}_{t_2}$, respectively. Then $\mathbf{w}_{i,i'} = (\mathbf{u}_i, \mathbf{v}_{i'})$, $i = 1, 2, \ldots, t_1$, $i' = = 1, 2, \ldots, t_2$ are pairwise orthogonal vectors and hence $\mathbf{OA}(q, n_1 n_2, t_1 + t_2)$ exists. Moreover, if $\mathbf{u}_i, \mathbf{v}_{i'}$ have the permutation property, then $\mathbf{w}_{i,i'}$ has the permutation property, too.

The following theorem gives a better estimate for the case that we start the construction with q $\mathbf{OA}$'s.

Theorem 1.4. *Let q be a power of a prime.* Let $\mathbf{OA}(q, n_j, t_j)$, $j = 1, 2, \ldots, q$, *exist. Then there exists* $\mathbf{OA}(q, q n_1 n_2 \ldots n_q, t_1 + \ldots + t_q)$. *Moreover, if the first q* $\mathbf{OA}$'s *have the permutation property then the last one has the property as well.*

Proof. Denote by $\mathbf{v}^j_{i_j}$ the i_j-th row-vector of $\mathbf{OA}(q, n_j, t_j)$. Take $\mathbf{OA}(q, q, q^2)$ with the permutation property (the existence follows from Theorem 1.1) and denote its i-th row by $\boldsymbol{\pi}_i = (\pi_{i1}, \pi_{i2}, \ldots, \pi_{iq})$. The vectors

$$\mathbf{v}_{i_0, i_1, i_2, \ldots, i_q} = (\pi_{i_0 1}(\mathbf{u}^1_{i_1}), \pi_{i_0 2}(\mathbf{u}^2_{i_2}), \ldots, \pi_{i_0 q}(\mathbf{u}^q_{i_q}))$$

for $i_0 \in \{1, 2, \ldots, q\}$ and $i_j \in \{1, 2, \ldots, n_j\}$, for each $j \in \{1, 2, \ldots, q\}$, form $\mathbf{OA}(q, q n_1 n_2 \ldots n_q, t_1 + t_2 + \ldots + t_q)$. If the vectors $\mathbf{u}^j_{i_j}$, $j \in \{1, 2, \ldots, q\}$, have the permutation property then the vectors $\mathbf{v}_{i_0, i_1, \ldots, i_q}$ have the property as well.

Definition. Let $\mathcal{A} = (A_1, A_2, \ldots, A_q)$, $\mathcal{B} = (B_1, B_2, \ldots, B_q)$ be two partitions into q parts (q-partitions) of a set with t elements. We say that $\mathcal{A}$ and $\mathcal{B}$ are *orthogonal* if $A_i \cap B_{i'} \neq 0$ for every $i, i' \in \{1, 2, \ldots, q\}$.

We denote by $f(t, q)$ the maximal size of a system of pairwise orthogonal q-partitions of the t-element set.

Theorem 2.1.

$$f(t, 2) = \binom{t-1}{\left[\frac{t}{2}\right] - 1}.$$

Proof. Let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ be a system of pairwise orthogonal 2-partitions. For every $i = 1, \ldots, n$ choose the partition class A_i with cardinality $\leqq t/2$. The sets $A_1, A_2, \ldots, A_n$ have the following properties:

(i) they form an antichain in the partial order of subsets of $\{1, \ldots, t\}$;

(ii) $A_i \cap A_j \neq 0$ for every $i \neq j$;

(iii) $|A_i| \leqq \frac{t}{2}$ for every i.

Hence, by Erdös Ko Rado theorem [2], we get

$$f(t, 2) \leqq \binom{t-1}{\left[\frac{t}{2}\right] - 1}.$$

To prove $\geqq$ consider the 2-partitions of $\{1, \ldots, t\}$ defined as follows: To every subset with cardinality $\left[\frac{t}{2}\right]$ containig 1, assign a 2-partition consisting of this set and its complement.

For $q > 2$ we are far from being able to find an exact formula for $f(t, q)$. We give here some estimates.

Theorem 2.2.

$$f\left(\binom{q}{2} t, q\right) \geqq f(t, 2).$$

Proof. Suppose that on the set $X = \{1, 2, \ldots, t\}$ there exists a system of n pairwise orthogonal 2-partitions. Consider $\binom{q}{2}$ copies X^{kj} $(1 \leqq k < j \leqq q)$ of the set X and on each set X^{kj} a system of pairwise orthogonal 2-partitions $\mathcal{A}_i^{kj} = (A_i^{kj}, B_i^{kj})$, $i = 1, \ldots, n$. For every i define a q-partition $\mathcal{C}_i = (C_{i1}, C_{i2}, \ldots, C_{iq})$ of the set $Y =$

$= \bigcup_{k<j} X^{kj}$ by formula

$$C_{ik} = \bigcup_{j=1}^{k=1} A_i^{jk} \cup \bigcup_{j=k+1}^{q} B_i^{kj}, \quad k = 1, \ldots, q .$$

It is easy to verify that $\mathscr{C}_i$, $i = 1, \ldots, n$, is a system of pairwise orthogonal q-partitions.

Theorem 2.3. $\mathbf{OA}(q, n, t)$ *exists if and only if* $f(t, q) \geqq n$.

Proof. The proof follows from the fact that each q-partition $\mathscr{A}$ of the set $\{1, 2, \ldots, t\}$ is in a $1-1$ correspondence with the vector $\boldsymbol{u} = (u_1, u_2, \ldots, u_t)$ defined as follows: $u_j = k$ iff $j \in A_k$ where A_k is the k-th class of $\mathscr{A}$.

Theorems 1.1 – 1.4 and Theorem 2.3 imply immediately

Theorem 2.4. *Let $t, t_1, t_2, n, n_1, n_2, q, q_1, q_2$ be positive integers. Then*

(i) $f(t_1, q_1) \geqq n, f(t_2, q_2) \geqq n$ *implies* $f(t_1 t_2, q_1 q_2) \geqq n$;

(ii) $f(t_1, q) \geqq n_1, f(t_2, q) \geqq n_2$ *implies* $f(t_1 + t_2, q) \geqq n_1 n_2$.

Theorem 2.5. *Let q be a power of a prime. Then*

(i) $f(q^2, q) = q + 1$;

(ii) $f(t_j, q) \geqq n_j$ *for* $j = 1, \ldots, q$ *implies* $f(t_1 + \ldots + t_q, q) \geqq q n_1 n_2 \ldots n_q$.

An easy computation gives the following

Corollary 2.6. *For q a power of a prime and t of the form*

$$t = \alpha_j q^j + \alpha_{j-1} q^{j-1} + \ldots + \alpha_1 q + \alpha_0$$

it is

$$f(t, q) \geqq \prod_{i=2}^{j} \left[(q + 1)^{q^{i-2}} q^{(q^{i-1}-1)/(q-1)}\right]^{\alpha_i}$$

and hence e.g. $f(t, q) \geqq (q + 1)^{[t/q^2]}$.

Remark 2.7. For $q = 3,4$ Theorem 2.2 gives a better result than Corollary 2.6. Nevertheles, we have not been able to generalize the construction given in 2.2 to improve the estimates given in 2.6 for $q > 3$.

Theorem 2.8. *Let t, p, q be positive integers, $p \leqq q$. Then*

$$f(t, q) \leqq f(t, p)/f(q, p) .$$

Particularly, $f(t, q) \leqq f(t, q - 1)$.

Proof. Let $\mathfrak{S}$ and $\mathfrak{T}$ be systems of orthogonal q-partitions and p-partitions on the t-set and q-set, respectively. To every pair $\mathscr{A}, \mathscr{B}$, $\mathscr{A} = (A_1, \ldots, A_q)$, $\mathscr{B} = (B_1, \ldots, B_p)$,

$\mathscr{A} \in \mathfrak{S}$, $\mathscr{B} \in \mathfrak{T}$, assign $\mathscr{C}(\mathscr{A}, \mathscr{B}) = (C_1, \ldots, C_p)$ defined as follows:

$$C_i = \bigcup\{A_j / j \in B_i\}, \quad i = 1, \ldots, p.$$

$\{\mathscr{C}(\mathscr{A}, \mathscr{B}) / \mathscr{A} \in \mathfrak{S}, \mathscr{B} \in \mathfrak{T}\}$ obviously forms a system of pairwise orthogonal p-partitions.

Corollary 2.9.

$$f(t, q) \geqq (2q)^{t/4q^2}$$

for every integers $t, q, t \geqq 8q^3$.

Proof. Immediately follows from 2.6 and 0.3.

Proposition 2.10. *Let $t > q$ be positive integers. Then*

$$f(t, q) \leqq \binom{t-1}{\left\lceil \frac{t}{q} - 1 \right\rceil}.$$

Proof. Let $\mathfrak{S}$ be a maximal system of pairwise orthogonal q-partitions of a set $\{1, \ldots, t\}$. From every $\mathscr{A}_j \in \mathfrak{S}$, $j = 1, \ldots, f(t, q)$ choose one of its partition classes $A^j \in \mathscr{A}_j$ with $|A^j| \leqq t/q$. The system A^j has the following properties:

$$|A^j| \leqq t/q, \quad A^i \not\subset A^j \quad \text{and} \quad A^i \cap A^j \neq 0 \quad \text{for every} \quad i \neq j.$$

Thus the Erdös Ko Rado theorem yields

$$|\mathfrak{S}| \leqq \binom{t-1}{\left\lceil \frac{t}{q} - 1 \right\rceil}.$$

3. $\bar{\omega}(nK_q)$

We shall now give estimates of the number $\bar{\omega}$, defined in the introduction, for the graphs nK_q. (The graph nK_q consists of n disjoint copies of the complete graph K_q.) The notion of $\bar{\omega}$ was in fact studied in a "complementary form" in several papers, e.g. [4], [15]. The number $\omega(G)$ is defined in [6] as the minimal size of a set S such that G is isomorphic to an intersection graph of a system of subsets of the set S.

Proposition 3.1. $\bar{\omega}(G) = \omega(\bar{G})$ *where $\bar{G}$ denotes the complement of G. Hence $\bar{\omega}(G)$ equals to the minimal size of a set S such that to every vertex of G one can assign a subset of S so that two vertices are adjacent iff the corresponding subsets are disjoint. (This assignment is called a disjoint representation of G.)*

The proof immediately follows from Proposition 1, Chapter 17, § 4 in [0].

Theorem 3.2.

(i) $\bar{\omega}(nK_q) \leqq t$ *iff* $\mathbf{OA}(q, n, t)$ *exists;*

(ii) $\bar{\omega}(nK_q) = \mathrm{Min}\{t \mid$ *there exists at least n orthogonal q-partitions of the set* $\{1, 2, \ldots, t\}\}$.

Proof. We shall write $nK_q = K_q^1 + \ldots + K_q^n$ where $V(K_q^i) = \{x_1^i, \ldots, x_q^i\}$. Let $A_1, \ldots, A_t$ be a system of stable sets so that every pair of nonadjacent vertices of nK_q is contained in at least one of A_i's. Without loss of generality we may suppose A_j to be maximal, i.e. $|A_j| = n$ for each j. To every $j \in \{1, \ldots, t\}$ assign a column vector $(\boldsymbol{u}_{ij})_{1 \leqq i \leqq n}$ so that

$$(*) \qquad u_{ij} = k \quad \text{iff} \quad x_k^i \in A_j .$$

The above defined system of column vectors forms an $\mathbf{OA}(q, n, t)$. On the other hand, if an $\mathbf{OA}(q, n, t)$ exists then the formula $(*)$ defines a system of A_j's with the required properties.

(ii) follows from (i) and Theorem 2.3.

The proof of the following theorem is easy and therefore it is omitted.

Theorem 3.3. *Let* $A_1, \ldots, A_{q.r}$ *be a system of stable sets of* nK_q *such that every pair of nonadjacent vertices is contained in at least one of them. Then the sets* $A_{(s-1)q+1}, A_{(s-1)q+2}, \ldots, A_{s.q}$ *form a partition of* $V(nK_q)$ *for every* $s = 1, \ldots, r$ *iff the corresponding* $\mathbf{OA}$ *(constructed in the proof of the above theorem) has the permutation property.*

Theorem 3.2 and the results of § 2 imply

Corollary 3.4.

$$\text{(i) } \bar{\omega}(nK_2) = \mathrm{Min}\left\{ t \Big/ \binom{t-1}{\left[\frac{t}{2}\right] - 1} \geqq n \right\},$$

$$\text{(ii) } \frac{\log n}{\frac{1}{q}\log q + \left(1 - \frac{1}{q}\right)\log\frac{q}{q-1}} \leqq \bar{\omega}(nK_q) \leqq \frac{q^2 \log n}{\log q}(1 + o(1)) \textit{ for } q \geqq 3.$$

4. PRODUCT

By a categorical product of two graphs G, H we mean the following graph $G \times H$:

$V(G \times H) = V(G) \times V(H)$;

$E(G \times H) = \{\{\langle x_1, y_1\rangle, \langle x_2, y_2\rangle\} / \{x_1, x_2\} \in E(G), \{y_1, y_2\} \in E(H)\}$.

Theorem 4.1.

$$\bar{\omega}(G) + \bar{\omega}(H) \geqq \bar{\omega}(G \times H) > \tfrac{1}{2}[\log_2 \bar{\omega}(G) + \log_2 \bar{\omega}(H)] .$$

Proof. Let $C_1, C_2, \ldots, C_l$ and $D_1, D_2, \ldots, D_k$ be systems of stable sets covering all pairs of nonadjacent vertices of G and H, respectively. Then $C_i \times V(H)$, $V(G) \times \times D_j$, $i = 1, \ldots, l$, $j = 1, \ldots, k$, is a system covering all pairs of nonadjacent vertices of $G \times H$. This proves the first inequality.

For the proof of the second inequality we shall need the following three propositions.

Let G be a graph. Define an equivalence $\sim$ on the set of its vertices as follows: $x \sim y$ iff x and y have the same neighbourhoods. From each equivalence class choose one point and consider the subgraph induced on this set. Denote this graph by $\mathrm{m}(G)$.

Proposition 4.2. $|V(\mathrm{m}(G))| \geqq \sqrt{(2\bar{\omega}(G))}$ *for every graph* G.

Proof. It follows from the well known theorem of Erdős, Goodman and Posa [4] which states that $\omega(G) \leqq |V(G)|^2/4$, and from the fact that $\bar{\omega}(\mathrm{m}(G)) = \bar{\omega}(G)$.

Proposition 4.3.

$$\mathrm{m}(G) \times \mathrm{m}(H) = \mathrm{m}(G \times H) .$$

Proof. It suffices to realize that a neighbourhood of a vertex $\langle x, y \rangle$ of the product is the product of neighbourhoods of vertices x and y.

Proposition 4.4. $\bar{\omega}(F) \geqq \log_2 |V(F)|$ *for* $F = \mathrm{m}(F)$.

Proof. Consider the disjoint representation $\mathcal{S}$ of the graph F. From $F = \mathrm{m}(F)$ it follows that $\mathcal{S}$ is a system of distinct sets and hence $|V(F)| \leqq 2^{\bar{\omega}(F)}$.

Now we can prove the second inequality of 4.1:

$$\begin{aligned} \bar{\omega}(G \times H) &= \bar{\omega}(\mathrm{m}(G \times H)) \geqq \log_2 (|V(\mathrm{m}(G \times H))|) = \\ &= \log_2 (V(\mathrm{m}(G)) \,.\, V(\mathrm{m}(H))) = \log_2 |V(\mathrm{m}(G))| + \log_2 |V(\mathrm{m}(H))| > \\ &> \tfrac{1}{2}[\log_2 \bar{\omega}(G) + \log_2 \bar{\omega}(H))] . \end{aligned}$$

Remark 4.5. We have just proved:

If we denote $f(r) = \min \{\bar{\omega}(G \times H)/\bar{\omega}(G) = \bar{\omega}(H) = r\}$ then $f(r) \to \infty$ with $r \to \infty$.

An analogous statement for the chromatic number is not known (see [14]).

5. DIMENSION

The notion of dimension of graphs, defined in the introduction, was introduced in [11] and [8]. For a survey of recent results concerning this notion see [10].

The following proposition is shown in [11].

Proposition 5.1. *The dimension of G equals the minimal number of equivalences $E_1, \ldots, E_n$ such that*

1) $E(G) = \bigcup_{i=1}^{n} E_i$;

2) $\bigcap_{i=1}^{n} E_i = 0$.

In [8], dimension of the graphs nK_2 was studied and it was shown that $\dim nK_2 = \log_2^+ n + 1$. We give here some estimates of $\dim nK_q$ for $q \geqq 3$.

Proposition 5.2. *If there exists* $\mathbf{OA}(q, n, qr)$ *with the permutation property then* $\dim nK_q \leqq r$.

Proof follows immediately from Theorem 3.3 and Proposition 5.1.

Theorem 5.3.

(i) $\dim(qK_q) = q$ *for q a power of a prime*;

(ii) $\dim(n_1 n_2 K_q) \leqq \dim(n_1 K_q) + \dim(n_2 K_q)$;

(iii) $\dim(nK_{q_1 q_2}) \leqq \dim nK_{q_1} \cdot \dim nK_{q_2}$.

Proof. According to (ii) of 1.1, there exists $\mathbf{OA}(q, q, q^2)$ with the permutation property and hence according to 5.2, $\dim qK_q \leqq q$. The second inequality follows from the easy fact that $\dim(K_q + x) = q$. ($K_q + x$ is the disjoint union of K_q and a single vertex x.)

(ii) As $n_1 n_2 K_q$ is an induced subgraph of $n_1 K_q \times n_2 K_q$ we have $\dim(n_1 n_2 K_q) \leqq \dim(n_1 K_q \times n_2 K_q) \leqq \dim n_1 K_q + \dim n_2 K_q$.

(iii) It is easily seen that $\dim G = r$ if one can assign to each vertex x of G a vector $\mathbf{v}(x)$ with r coordinates so that $x = y$ implies $\mathbf{v}(x) = \mathbf{v}(y)$ and $(x, y) \in E(G)$ iff $v_i(x) \neq v_i(y)$ for every $i = 1, \ldots, r$. ($v_i(x)$ denotes the i-th coordinate of $\mathbf{v}(x)$.) We shall call such an assignment an encoding.

Denote by x_j^i and x_k^i the j-th and the k-th vertex of the i-th copy of K_{q_2} in the graph nK_{q_2} and nK_{q_2}, respectively. Put $nK_{q_1 q_2} = K^1_{q_1 q_2} + \ldots + K^n_{q_1 q_2}$, where $V(K^i_{q_1 q_2}) = \{x^i_{1,1}, \ldots, x^i_{j,k}, \ldots, x_{q_1, q_2}\}$. Let $\mathbf{u}$ and $\mathbf{v}$ be an encoding of nK_{q_1} and nK_{q_2}, respectively. Define a mapping $\mathbf{w}$ as follows: $\mathbf{w}_{\alpha,\beta}(x^i_{j,k}) = \langle \mathbf{u}_\alpha(x^i_j), \mathbf{v}_\beta(x^i_k) \rangle$. It can be verified easily that $\mathbf{w}$ is an encoding.

Theorem 5.4.

$$\dim nK_q \leqq \frac{q \log n}{\log q}(1 + o(1)) \quad for \quad q \geqq 3\,.$$

Proof follows from 1.4, 5.2 and 0.3.

Let $n > q$ be positive integers. Define the graph $\binom{n}{q}$ whose vertices are q-point subsets of the set $\{1, \dots, n\}$ with two vertices adjacent if and only if they are disjoint subsets.

The graphs $\binom{n}{q}$ have interesting properties. It is not difficult to see that they are universal, i.e. every finite graph is an induced subgraph of some $\binom{n}{q}$. Kneser conjectured that the chromatic number of $\binom{2n+k}{n}$ is $k+2$. Lovasz proved this in [7] using methods of algebraic topology. In [13] we studied $\bar{\omega}$ of $\binom{n}{q}$ and proved that $\bar{\omega}\binom{n}{q} = n$ for $q \leqq \frac{1}{2}n$.

In [12] the dimension of Kneser graphs was studied. It was proved that

$$(*) \qquad \log_2^+ \log_2^+ n - o(1) \leqq \dim \binom{n}{q} \leqq (q-1)\, q^2 \log_2^+ \log_2^+ n\,.$$

Using 2.9 and [12, Remark 3.7] we obtain a slight improvement of $(*)$:

$$\dim \binom{n}{q} \leqq \frac{4(q-1)\, q^2}{1 + \log_2 q} \log_2^+ \log_2^+ n\,.$$

($\log_2^+ n$ denotes the smallest integer not less then $\log_2 n$).

6. PROBLEMS

There are many open problems left. We shall mention only three of them.

1) Is nK_q an induced subgraph of a product of dim nK_q copies of K_q? (I.e., is the condition in Prop. 5.2 also necessary?)
2) Find a better estimates for $f(n, q)$!
3) Let G be an arbitrary graph. Is it true that

$$\bar{\omega}(nG) \geqq \bar{\omega}(nK_{\chi(G)})\ ?$$

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