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Czechoslovak Mathematical Journal, Vol. 30 (1980), No. 3, 387–417

Persistent URL: http://dml.cz/dmlcz/101690

Czechoslovak Mathematical Journal, 30 (105) 1980, Praha

FREDHOLM ALTERNATIVES AND SURJECTIVITY RESULTS FOR MULTIVALUED A-PROPER AND CONDENSING MAPPINGS WITH APPLICATIONS TO NONLINEAR INTEGRAL AND DIFFERENTIAL EQUATIONS*)

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(Received September 21, 1978)

INTRODUCTION

In [18, 20], the author established several extensions of the first Fredholm theorem to nonlinear multivalued mappings of A-proper and condensing type. It is the purpose of this paper to establish several Fredholm alternatives for multivalued A-proper and condensing mappings using our results from [18, 20], to study the ranges of the sums of various classes of nonlinear mappings and to apply some of our abstract results to establishing Fredholm alternatives for contingent integral equations and for generalized boundary value problems for nonlinear ordinary differential equations.

The organization of the paper is as follows. In the first part of Section I we introduce some basic definitions and examples and prove a Fredholm alternative for multivalued A-proper mappings. In particular, using our recent new example of A-proper maps, we obtain a Fredholm alternative for the sum of nonlinear a-stable and condensing mappings. The important feature of this result is that it cannot be obtained by either the theory of monotone mappings or the theory of condensing mappings. Our results extend the alternatives of KACHUROVSKY [11], HILDEBRANDT and WIENHOLTZ [10], HESS [9] and PETRYSHYN [29], [30]. In the second part of Section 1 we establish various solvability results for the equation $f \in A(x) - T(x)$ with A linear (unbounded) and T nonlinear and quasibounded.

In the first part of Section 2 we establish a Fredholm alternative for multivalued condensing mappings and some further extensions of the first Fredholm theorem to these mappings. In the second part of Section 2 we prove some surjectivity results involving 1-ϕ-contractive mappings and study in some detail the solvability of $f \in A(x) - T(x)$ with A unbounded using the assumption that the range of T is contained in the range of A. Some of these results include the corresponding ones of FUČÍK, WEBB, PETRYSHYN-FITZPARTICK and others.

*) This work was supported in part by NRC Grant A 4556.

In Section 3 we derive Fredholm alternatives and some surjectivity results for condensing perturbations of accretive, pseudo-contractive and a-stable maps, while applications of our abstract results to contingent integral equations and generalized BVP for nonlinear differential equations are given in Section 4.

1. FREDHOLM ALTERNATIVE FOR MULTIVALUED A-PROPER MAPPINGS AND RANGES OF THE SUMS OF NONLINEAR MAPPINGS

Let X and Y be normed linear spaces, $\{E_n\}$ and $\{F_n\}$ be two sequences of oriented finite-dimensional spaces and $\{V_n\}$, $\{W_n\}$ be two sequences of continuous linear mappings with V_n mapping E_n injectively into X and W_n mapping Y onto F_n.

Definition 1.1. A quadruple of sequences $\Gamma = \{E_n, V_n, F_n, W_n\}$ is said to be an *admissible scheme* for (X, Y) if $\dim E_n = \dim F_n$, $n \geqq 1$, $\operatorname{dist}(x, V_n(E_n)) \to 0$ as $n \to \infty$ for each x in X and $\|W_n\| \leqq M$ for all n.

Some example of such schemes are given below. To that end, assume that $X_n \subset X$ is oriented and finite-dimensional with $\operatorname{dist}(x, X_n) \to 0$ as $n \to \infty$ for each x in X and let V_n be a linear injection of X_n into X.

(a) Let $Y_n \subset Y$ be finite-dimensional and oriented, $\dim Y_n = \dim X_n$ for each n and let $\{Q_n\}$ be a sequence of continuous linear mappings of Y onto Y_n with $\|Q_n\| \leqq M$, $n \geqq 1$. Then $\Gamma_a = \{X_n, V_n; Y_n, Q_n\}$ is admissible for (X, Y).

(b) If $Y = X$, $Y_n = X_n$ and $W_n = P_n$ is a projection of X onto X_n such that $\|P_n\| \leqq M$ for all n, then $\Gamma_b = \{X_n, V_n; X_n, P_n\}$ is admissible for X.

(c) Let P_n be as in (b) and Q_n a continuous linear projection of Y onto $F_n \subset Y$ with $Q_n y \to y$ for each y in Y. Then $\Gamma_c = \{X_n, P_n; Y_n, Q_n\}$ is a *projectionally complete scheme* for (X, Y).

Let $D \subset X$, $D_n = V_n^{-1}(D)$, $T: D \to 2^Y$ and $T_n \equiv W_n T V_n \mid D_n$. Consider the equation

$$f \in T(x) \quad (f \in Y) \tag{1}$$

and associate with it the following approximate equations

$$W_n(f) \in W_n T V_n(u) \quad (u \in D_n,\ n \in N)\,. \tag{2}$$

Definition 1.2. Equation (1) is said to be *feebly approximation solvable* with respect to $\Gamma = \{E_n, V_n; F_n, W_n\}$ if for all large n, there exists a solution u_n of Eq. (2) such that $f \in T(x)$ with $x \in \operatorname{cl}\{V_n(u_n)\}$.

Denote by $BK(X)$ and $CK(X)$ the families of all nonempty bounded, closed and convex, and compact and convex subsets of X respectively.

Definition 1.3. A multivalued mapping $T: D \subset X \to 2^Y$ is said to be *approximation proper* (*A-proper*) *w.r.t.* Γ if: (i) $T_n : D_n \to CK(F_n)$ is upper semicontinuous, $n \geqq 1$; and (ii) whenever $\{V_{n_k}(u_{n_k}) \mid u_{n_k} \in D_{n_k}\}$ is bounded and $\|W_{n_k}(y_{n_k}) - W_{n_k}(f)\| \to 0$ as $k \to \infty$ for some $y_{n_k} \in TV_{n_k}(u_{n_k})$ and $f \in Y$, then some subsequence $V_{n_{k(i)}}(u_{n_{k(i)}}) \to x \in D$ and $f \in T(x)$.

Singlevalued A-proper mappings have been extensively studied by many authors (cf. [31]) and by the author in the multivalued case (cf. e.g., [17–24]).

For a bounded set $Q \subset X$, the *set-measure of noncompactness* of Q is defined ([14]) by $\gamma(Q) = \inf\{d > 0 \mid Q$ has a finite covering by sets of diameter less than $d\}$, and the *ball-measure of noncompactness* of Q is defined (([35]) by $\chi(Q) = \inf\{r > 0 \mid Q$ can be covered by a finite number of balls of radii r with centers in $X\}$. Let ϕ be either of these measures. A mapping $T: D \subset X \to CK(X)$ is called *k-ϕ-contractive* if, for each bounded $Q \subset D$, $\phi(T(Q)) \leqq k\, \phi(Q)$; it is *$\phi$-condensing* if, for each $Q \subset D$ with $\phi(Q) \neq 0$, $\phi(T(Q)) < \phi(Q)$.

We have shown in [17] that if $S: X \to CK(X)$ is a generalized contraction and $C: D \subset X \to CK(X)$ is compact, then $I - S - C$ is A-proper w.r.t. Γ_b. Moreover, a duality, strongly monotone and of (KS) type mappings are A proper (cf. [17, 23]).

Condition (C): *Whenever $\{u_n \in E_n\}$ is such that $V_n(u_n) \to x$ in X and $W_n T V_n(u_n) \to f$ in Y, then $x \in D(T)$ and $Tx = f$.*

It is easy to see that condition (C) holds for $T + F$ if: (i) T is continuous and F is demicontinuous, or (ii) Y is reflexive, T is demiclosed and K-quasibounded with $W_n K = Q_n^* K = K$ and F is demicontinuous, or (iii) T is (demi) closed, $Q_n Tx = Tx$ for $x \in X_n$ and F is continuous. Modifying somewhat the proof of our Proposition 2.1 in [19], we obtain the following extension of it.

Example A. *Let X and Y be π_1-Banach spaces with a scheme Γ_c and $T: X \to Y$ be continuous and a-stable, i.e.,*

(3) $$\|Q_n Tx - Q_n Ty\| \geqq c\|x - y\| \quad \text{for all} \quad x, y \in X_n\,, \quad n \geqq 1$$

and some constant $c > 0$. Then, if $F: X \to CK(Y)$ is a k-ball contraction with $k < c$ and $T + F$ satisfies condition (C), $T + F$ is A-proper w.r.t. Γ_c. If $c = 1$, F could be allowed to be ball-condensing.

As T one can take a c-strongly monotone mapping as was first done by Toland [36], or a c-accretive mapping (i.e., $(Tx - Ty, x - y)_+ \geqq c\|x - y\|^2$ for all x, y in X, where $(x, y)_+ = \sup\{(w, x) \mid w \in J(y)\}$ with J the normalized duality mapping) as was done by the author [19].

We shall need the following result of the author

Theorem A ([20]). *Let X and Y be normed linear spaces with a scheme Γ and $T: X \to 2^Y$ be A-proper w.r.t. Γ. Let $A: X \to 2^Y$ be odd on $X \setminus B(0, r)$ for some $r > 0$, $W_n A V_n: E_n \to CK(F_n)$ and for all large n*

(4) $$\|W_n y\| \geqq c(\|x\|) \quad \text{for} \quad x \in V_n(E_n)\,, \quad y \in A(x)\,,$$

where $c: R^+ \to R^+$ is a continuous function. Suppose that for each f in Y there is an $r_f \geqq r$ such that

$$\alpha(Tx - f, Ax) < c(\|x\|)/M\,, \quad \text{for} \quad x \in \partial B(0, r_f)\,,$$

where $\alpha(Tx - f, Ax) = \sup_{y \in Tx} d(y - f, Ax)$. Then the equation $f \in T(x)$ is feebly approximation solvable w.r.t. Γ for each f in Y.

Our first result is the following Fredholm alternative involving multivalued A-proper mappings that are in some sense close to linear A-proper mappings (i.e., condition (5) below holds). Examples of such mappings can be found in Sections 3 and 4.

Theorem 1.1. (Fredholm alternative) *Let X and Y be Banach spaces with a scheme Γ_c, $A : X \to Y$ linear continuous and A-proper w.r.t. Γ_c and $T: X \to 2^Y$ A-proper w.r.t. Γ_c and such that*

$$\limsup_{\|x\| \to \infty} \frac{\alpha(Tx, Ax)}{\|x\|} \leqq k \tag{5}$$

with k sufficiently small. Then, either $N(A) = \{0\}$, in which case the equation $f \in T(x)$ is feebly approximation solvable for each f in Y, or $N(A) \neq \{0\}$. In the latter case, assuming additionally that $R(T) \subset N(A^*)^{\perp}$ $(= R(A))$ with $\dim N(A) = \operatorname{codim} R(A)$, *the equation $f \in T(x)$ is feebly approximation solvable if and only if $f \in N(A^*)^{\perp}$.*

Proof. Suppose first that $N(A) = \{0\}$. Then, in view of Lemma 2.1 in [20], for each large n

$$\|Q_n A x\| \geqq k_0 \|x\| \quad \text{for all} \quad x \in X_n\,,$$

and some $k_0 > 0$. Since k is sufficiently small, the first assertion follows from Theorem A.

Next, suppose that $N(A) \neq \{0\}$ and $R(T) \subset R(A)$. Since $\dim N(A) = \operatorname{codim} R(A) < \infty$ there exist closed subspaces $X_1 \subset X$ and $Y_1 \subset Y$ such that $X = N(A) \oplus X_1$ and $Y = Y_1 \oplus R(A)$ with $\dim Y_1 = \dim N(A)$. Let L be a linear isomorphism of $N(A)$ onto Y_1 and P be a continuous linear projection of X onto $N(A)$. Then $C = LP : X \to Y_1$ is compact and therefore, $A_1 = A + C$ and $T_1 = T + C$ are A-proper w.r.t. Γ_c with

$$\limsup_{\|x\| \to \infty} \frac{\alpha(T_1 x, A_1 x)}{\|x\|} \leqq \limsup_{\|x\| \to \infty} \frac{\alpha(Tx, Ax)}{\|x\|} \leqq k\,.$$

Moreover, if $A_1(x) = 0$ then $A(x) = -C(x) \in Y_1 \cap R(A) = \{0\}$, i.e., $LP(x) = 0$ and so $P(x) = 0$. Thus $x \in X_1$ with $A(x) = 0$ which implies that $x = 0$ by the injectivity of A on X_1. Again, by Lemma 2.1 in [20], $\|Q_n A_1 x\| \geqq k_1 \|x\|$ for all $x \in X_n$, n large, and some $k_1 > 0$, and therefore the equation $f \in T(x) + C(x)$ if f.a. solvable for each f in Y by Theorem A.

Next, suppose that $f \in R(A)$. Then $C(x) \in f - T(x) \subset R(A)$ since $R(T) \subset R(A)$ and so $C(x) = 0$ and $f \in T(x)$. Conversely, if $f \in T(x)$ is solvable, then $f \in R(A)$ since $R(T) \subset R(A)$. ∎

Corollary 1.1. (Fredholm alternative for the sum of a-stable and condensing maps.) *Let X be a π_1-Banach space and Y a π_1-Hilbert space with projectionally* *complete schemes* Γ_c *and* $\Gamma_c^* = \{R(Q_n^*), Q_n^*; R(P_n^*), P_n^*\}$ *for the pairs* (X, Y) *and* (Y^*, X^*) *respectively. Suppose that* $A, T: X \to Y$ *are continuous and a-stable with* A *linear, i.e. for all large n,*

$$(6) \qquad \|Q_n Ax\| \geqq c_1 \|x\| \quad \text{for all} \quad x \in X_n \quad \text{and some} \quad c_1 > 0,$$

(in which case, $\|P_n^* A^* f\| \geqq c_2 \|f\|$ *for all* $f \in R(Q_n^*)$ *and some* $c_2 > 0$ ([27])) *and*

$$(7) \qquad \|Q_n Tx - Q_n Ty\| \geqq c_3 \|x - y\| \quad \text{for} \quad x, y \in X_n \quad \text{and some} \quad c_3 > 0.$$

Suppose also that $F_1, F_2 : X \to Y$ *are demicontinuous* k_i*-ball contractive,* $i = 1, 2$, *respectively with* $k_1 < \min\{c_1, c_2\}$ *and* F_1 *linear nad either* $k_2 < c_3$ *or* F_2 *is ball-condensing if* $c_3 = 1$. *Then, if in addition*

$$(8) \qquad \limsup_{\|x\| \to \infty} \frac{\|Tx + F_2 x - Ax - F_1 x\|}{\|x\|} \leqq k$$

for some sufficiently small k, either $N(A + F_1) = \{0\}$, *in which case the equation* $Tx + F_2 x = f$ *is feebly approximation solvable w.r.t.* Γ_c *for each f in Y, or* $N(A + F_1) \neq \{0\}$. *In the latter case, assuming additionally that*

$$R(T + F_2) \subset N(A^* + F_1^*)^\perp \, (= R(A + F_1)),$$

the equation $Tx + F_2 x = f$ *is feebly approximation solvable w.r.t.* Γ_c *if and only if* $f \in N(A^* + F_1^*)^\perp$.

Proof. In view of Theorem 1.1 it is sufficient to show that $A + F_1$ and $T + F_2$ are A-proper w.r.t. Γ_c and that $\dim N(A + F_1) = \operatorname{codim} R(A + F_1)$. But, by Example A and inequalities (6) and (7) we have that $A + F_1$ and $T + F_2$ are A-proper w.r.t. Γ_c. Moreover, since A^* is also a-stable and F_1^* is k_1-ball condensing ([39]), again by Example A, $A^* + F_1^*$ is A-proper w.r.t. Γ_c^*. Hence, $\dim N(A + F_1) = \operatorname{codim} R(A + F_1)$ ([27]). ∎

Remark 1.1. In Corollary 1.1 we may allow F_2 to be multivalued and Y to be a π_1-Banach space provided we know that $\dim N(A + F_1) = \operatorname{codim} R(A + F_1)$ which is so if, e.g., F_1^* is also k_1-ball contractive. In particular, we have

Corollary 1.2. *Suppose that X and Y are* π_1*-Banach spaces,* $A, T: X \to Y$ *are* c_i*-accretive,* $i = 1, 3$, *respectively with A linear and* $F_1, F_2 : X \to Y$ *demicontinuous* k_i*-ball contractive,* $i = 1, 2$, *respectively with* $k_1 < \min\{c_1, c_2\}$, F_1 *linear and* F_1^* k_1*-ball contractive and either* $k_2 < c_3$ *or* F_2 *is ball-condensing if* $c_3 = 1$. *Then, if* (8) *holds, the conclusions of Corollary* 1.1 *are valid.*

Remark 1.2. When $A = T$ is linear and positive definite, $F_1 = F_2$ linear and compact and $X = Y = H$ is a π_1-Hilbert space, Corollary 1.2 reduces to the alternative proven by Hildebrandt and Wienholtz [10]. When $T = A + N$ with A and T

singlevalued and A-proper with $\|Nx\|/\|x\| \to 0$ as $\|x\| \to \infty$, Theorem 1.1 was proved by Petryshyn [30], which on the other hand extends the alternatives of Kachurovsky [11] and Hess [9] for compact and of type (S) mappings respectively.

We continue our exposition in this section by studying equations of the form $f \in A(x) - T(x)$ with A linear and $A - T$ not necessarily A-proper. Instead, we shall assume that A has a (partial) inverse A^{-1} such that $I - A^{-1}T$ is A-proper and $R(T) \subset R(A)$. The case of infinite dimensional null space of A is also studied. We need the following result which is of interest in its own right.

Theorem 1.2. *Let $\Gamma = \{E_n, V_n; E_n, P_n\}$ be an admissible scheme for (X, X) with $\|P_n\| \leqq M$ and $T: X \to 2^X$ satisfy*

$$|T| = \limsup_{\|x\| \to \infty} \frac{|Tx|}{\|x\|} < \frac{1}{M}, \tag{9}$$

where $|Tx| = \sup\{\|y\| \mid y \in T(x)\}$. Then, if $I - T$ is A-proper w.r.t. Γ the equation $f \in x - T(x)$ is feebly approximation solvable w.r.t. Γ for each f in X.

Proof. For each f in X we have

$$\limsup_{\|x\| \to \infty} \frac{\alpha(x - Tx - f, x)}{\|x\|} = \limsup_{\|x\| \to \infty} \frac{|Tx - f|}{\|x\|} < \frac{1}{M}.$$

Hence, for $\varepsilon > 0$ small there exists $r > 0$ such that $|Tx - f| \leqq (1/M - \varepsilon)\|x\|$ for all $\|x\| \geqq r$. Since $\|P_n I x\| = \|x\|$ for each $x \in E_n$, $n \geqq 1$, the conclusion of the theorem follows from Theorem A. ∎

The existence assertion of Theorem 1.2 with T singlevalued, $|T| < 1$, has been proven by GRANAS [8], VIGNOLI [38] and PETRYSHYN [28] in the compact, ϕ-condensing and $1 - \phi$-contractive case respectively and constructively by Petryshyn (see [31]) in the case when $pI - T$ is A-proper w.r.t. a projectionally complete scheme for each $p \geqq 1$ (see also MILOJEVIĆ [17] for the multivalued case).

Condition (9) can be weaken provided we require that $pI - T$ is A-proper for each $p \geqq 1$ (cf. Proposition 2.5 in [23]). More generally, we have

Theorem 1.3. *Let $A: X \to Y$ be a linear bijection and $T: X \to 2^Y$ such that $pI - A^{-1}T: X \to 2^Y$ is A-proper w.r.t. Γ_b for each $p > 1$ and either T is odd or for some $R > 0$*

$$\lambda A(x) \notin T(x) \quad for \quad \|x\| \geqq R \quad and \quad \lambda > 1. \tag{10}$$

Suppose that for each f in Y there exists an $r_f \geqq R$ such that either one of the following conditions holds:

(i) *$I - A^{-1}T$ is A-proper w.r.t. Γ_b and*

$$tA^{-1}(f) \notin (I - A^{-1}T)(\partial B(0, r_f)) \quad for \quad t \in [0, 1];$$

(ii) *For some* $\gamma > 0$,

(11) $$\|x - A^{-1}(y) - tA^{-1}(f)\| \geqq \gamma \quad for\ all \quad y \in T(x), \quad x \in \partial B(0, r_f)$$

$t \in [0, 1]$ *and* $I - A^{-1}T$ *satisfies condition* $(++)$, *i.e., whenever* $\{x_n\} \subset X$ *is bounded and* $y_n \to g$ *in* X *for some* $y_n \in (I - A^{-1}T)(x_n)$, *then* $g \in x - A^{-1}T(x)$ *for some* $x \in X$.

Then the equation $f \in A(x) - T(x)$ *is solvable for each* f *in* Y.

Proof. Suppose first that assumption (ii) holds. Then condition (11) implies that there exists an $\alpha_0 > 0$ such that for each $\alpha \in (0, \alpha_0)$

$$t A^{-1}(f) \notin ((1 + \alpha) I - A^{-1}T)(\partial B(0, r_f)) \quad \text{for} \quad t \in [0, 1].$$

Let $\alpha \in (0, \alpha_0)$ be fixed. Then the A-properness of $(1 + \alpha) I - A^{-1}T$ implies that there exists $n_0 \geqq 1$ such that for $n \geqq n_0$,

$$tP_n A^{-1}(f) \notin ((1 + \alpha) I - P_nA^{-1}TV_n)(\partial B_n), \quad t \in [0, 1],$$

where $B_n = V_n^{-1}(B(0, r_f))$. Consequently, for $n \geqq n_0$

$$\deg((1 + \alpha) I - P_nA^{-1}TV_n - P_nA^{-1}(f), B_n, 0) = \\ = \deg((1 + \alpha) I - P_nA^{-1}TV_n, B_n, 0) \neq 0$$

if T is odd.

Next, if (10) holds, then

(12) $$\lambda(1 + \alpha) x \notin A^{-1} T(x) \quad \text{for} \quad \|x\| \geqq R, \quad \lambda > 1,$$

the homotopy $F_n(t, u) = (1 + \alpha) u - tP_nA^{-1}TV_n(u)$ does not vanish on $[0, 1] \times \times \partial B_n$, and therefore [16], $\deg((1 + \alpha) I - P_nA^{-1}TV_n, B_n, 0) \neq 0$. Hence, in either case the equation $P_nA^{-1}(f) \in (1 + \alpha) u - P_nA^{-1}TV_n(u)$ is solvable in B_n for all large n which by the A-properness of $(1 + \alpha) I - A^{-1}T$ implies the solvability of $A^{-1}(f) \in (1 + \alpha) x - A^{-1} T(x)$ in $B(0, r_f)$. Let $\alpha_k \to 0$ and $x_k \in B(0, r_f)$ be such that for some $y_k \in T(x_k)$, $x_k - A^{-1}(y_k) = A^{-1}(f) - \alpha_k x_k \to A^{-1}(f)$ as $k \to \infty$. Then, by condition $(++)$ the equation $f \in A(x) - T(x)$ is solvable.

Next, if (i) holds, then repeating the above arguments with $\alpha = 0$ we get the solvability of $P_nA^{-1}(f) \in u - P_nA^{-1}TV_n(u)$ for all large n which by the A-properness of $I - A^{-1}T$ implies the solvability of $f \in A(x) - T(x)$. An alternative proof of part (i) can be obtained from Proposition 2.5 in [23]. ∎

Remark 1.2. It is easy to see that condition (i) holds if $A - T$ satisfies condition $(+)$: if $\{x_k\} \subset X$ is such that $y_k \to y$ for some $y_k \in (A - T)(x_k)$, then $\{x_k\}$ is bounded. On the other hand, this condition is implied by many well known conditions (e.g., condition (ii) of Lemma 1.1 below, K-coercivity of $A - T$, etc.). We also add that condition $(+)$ for $A - T$ does not imply (11) if A is not continuous.

Regarding condition (10) we have the following elementary lemma.

Lemma 1.1. *Condition* (10) *is implied by either one of the following conditions*:

(i) *There exists a constant* $c > 0$ *such that whenever* $0 \in A(x) - tT(x)$ *for some* x *and* $t \in [0, 1]$, *then* $\|x\| \leqq c$;

(ii) $$\limsup_{\|x\| \to \infty} \frac{|Tx|}{\|x\|} < 1/\|A^{-1}\| .$$

We complete this section by applying Theorems 1.2 and 1.3 to the sum of unbounded linear and quasibounded nonlinear mappings.

Let X and Y be Banach spaces, $A : D(A) \subset X \to Y$ a closed linear mapping, not necessarily densily defined, with the closed range and whose null space, $N(A)$, admits an orthogonal complement in X, i.e., there exists a closed subspace U of X such that $X = N(A) \oplus U$. Let $X_0 = \overline{D(A)}$, $A_0 : D(A) \subset X_0 \to Y$ defined by $A_0(x) = A(x)$ and $A_0^* : Y^* \to X_0^*$ its adjoint. Then by Kato's result [13], $R(A) = N(A_0^*)^\perp = \{y \in Y \mid (y, y^*) = 0 \text{ for all } y^* \in N(A_0^*)\}$ and the restriction A_1 of A to $D(A) \cap U$ is injective. Hence, its inverse $A_1^{-1} : R(A) \to D(A) \cap U$ is continuous.

Now, as an easy consequence of Theorem 1.2 we have

Theorem 1.4. *Let* $A : D(A) \subset X \to Y$ *be a closed linear mapping with the closed range and* $X = N(A) \oplus U$ *for some closed subspace* U *of* X. *Let* $T : X \to 2^Y$ *be such that* $R(T) \subset R(A)$, $I - A_1^{-1}T : U \to 2^U$ *is A-proper w.r.t.* Γ_b *for* (U, U) *and*

(13) $$|T| = \limsup_{\substack{\|x\| \to \infty \\ x \in U}} \frac{|Tx|}{\|x\|} < \frac{1}{M\|A_1^{-1}\|}.$$

Then the equation $f \in A(x) - T(x)$ *is solvable if and only if* $f \in N(A_0^*)^\perp$.

Proof. Let $f \in N(A_0^*)^\perp = R(A)$. Since $|A_1^{-1}T| < 1/M$, the equation $A_1^{-1}(f) \in x - A_1^{-1} T(x)$ is solvable in $U \cap D(A)$ by Theorem 1.2, and therefore, so is $f \in A(x) - T(x)$.

Conservely, if $f \in A(x) - T(x)$ is solvable, then $f \in R(A) = N(A_0^*)^\perp$ by $R(T) \subset R(A)$. ∎

Theorem 1.5. *Suppose that* A *and* T *satisfy all the assumptions of Theorem* 1.4 *except condition* (13). *Suppose that* $A - T$ *satisfies condition* $(+)$ *on* $D(A) \cap U$, $pI - A_1^{-1}T : U \to 2^U$ *is A-proper w.r.t.* Γ_b *for each* $p \geqq 1$ *and either* T *is odd or for some* $R > 0$

(14) $$\lambda A(x) \notin T(x) \quad \text{for all} \quad \|x\| \geqq R \quad \text{in} \quad D(A) \cap U, \quad \lambda > 1 .$$

Then the equation $f \in A(x) - T(x)$ *is solvable if and only if* $f \in N(A_0^*)^\perp$.

Proof. It suffices to show that conditions (i) and (12) with $\alpha = 0$ of Theorem 1.3 hold on U (cf. its proof). Since $A - T$ satisfies $(+)$, condition (i) holds on U. Moreover, since for $x \in D(A) \cap U$ we have that $\lambda A(x) \in T(x)$ if and only if $\lambda x \in A_1^{-1} T(x)$, condition (12) with $\alpha = 0$ holds on U. ∎

Remark 1.3. Using Theorem A we see that condition (13) in Theorem 1.4 can be weaken to

$$\limsup_{\substack{\|x\| \to \infty \\ x \in U}} \frac{|A_1^{-1}Tx - Fx|}{\|x\|} \leqq \frac{k}{M}$$

with k sufficiently small and $F : U \to U$ linear continuous and such that $I - F$ is injective and A-proper w.r.t. Γ_b. However, if $N(I - F) \neq \{0\}$, then by Theorem 1.1, $f \in A(x) - T(x)$ is solvable iff $f \in A(R(I - F) \cap D(A))$ provided $R(I - A_1^{-1}T) \subset N(I - F^*)^\perp (= R(I - F))$ and $\dim N(I - F) = \operatorname{codim} R(I - F)$.

Remark 1.4. When $I - A_1^{-1}T : U \to 2^U$ is not A-proper, Theorem 1.5 still holds provided that we assume conditions $(++)$ and (11) of Theorem 1.3 instead of condition $(+)$. As observed before, (14) is implied e.g., by condition (i) or (ii) of Lemma 1.1. Theorems 1.4 and 1.5 are also valid when $A_1^{-1}T$ is a $k - \phi$-contractive mapping as shown in Section 2.

The following observations regarding Theorems 1.4 and 1.5 are perhaps in order. If X is a Hilbert space and $A : D(A) \subset X \to X$ a closed linear density defined mapping with $R(A) = N(A)^\perp$, then $R(A)$ is closed if $A_1^{-1} : R(A) \to R(A)$ is compact, and $X = N(A) \oplus R(A)$. On the other hand, if X is also separable and $A : D(A) \subset X \to X$ a closed linear density defined mapping with $N(A) = N(A^*)$ and $\{x \mid x \in D(A),\ \|x\| \leqq 1,\ \|Ax\| \leqq 1\}$ is compact, then we know that $\dim N(A) < \infty$, $R(A) = N(A)^\perp$ is closed and the partial inverse A_1^{-1} is compact. Moreover, if $T : X \to BK(X)$ is u.s.c. and bounded with $T(R(A)) \subset R(A)$, then $I - A_1^{-1}T : R(A) \to CK(R(A))$ is A-proper w.r.t. a natural scheme for X. More generally, if $A_1^{-1}T$ is ball-condensing, then $I - A_1^{-1}T$ is A-proper.

2. FREDHOLM ALTERNATIVE FOR MULTIVALUED CONDENSING MAPPINGS AND RANGES OF THE SUM OF NONLINEAR MAPPINGS

In the first part of this section we prove a Fredholm alternative for multivalued condensing mappings and a further extension of the first Fredholm theorem to $1 - \phi$-contractions. In the second part we prove some surjectivity results for these mappings and continue the study of $f \in A(x) - T(x)$.

Theorem 2.1. (Fredholm alternative) *Let X be a Banach space, $T : X \to CK(X)$ be upper semicontinuous and ϕ-condensing and $A : X \to X$ be a linear $k - \phi$-contraction with $k < 1$ and such that*

$$\limsup_{\|x\| \to \infty} \frac{\alpha(Tx, Ax)}{\|x\|} \leqq k_0 \tag{15}$$

with k_0 sufficiently small. Then, either $N(I - A) = \{0\}$, in which case the equation

$$f \in x - T(x) \tag{16}$$

is solvable for each f in X, or $N(I - A) \neq \{0\}$. In the latter case, assuming additionally that $R(I - T) \subset N(I - A)^*)^{\perp} (= R(I - A))$, Equation (16) is solvable if and only if $f \in R(I - A)$.

Proof. Suppose first that $N(I - A) = \{0\}$. Then, in view of Lemma 1.1 in [18], the solvability of Equation (16) follows from Theorem 1.1 in [18]. Next, suppose that $X_1 = N(I - A) \neq \{0\}$. By our assumptions on A and the results in [26], the range $R((I - A)^*)$ is closed, $\dim N(I - A) < \infty$ and $\dim N(I - A) = \dim N((I - A)^*)$. Since $\dim N(I - A) = \operatorname{codim} R(I - A)$, there exists a closed linear subspace X_2 of X and a finite dimensional subspace Y_2 of X such that $\dim N(I - A) = \dim Y_2$, $X = X_1 \oplus X_2 = Y_1 \oplus Y_2$, where $Y_1 = R(I - A)$, with $Y_1 = (I - A)(X_2)$ and $I - A|_{X_2}$ having a bounded inverse.

Now suppose that $f \in R(I - A)$ and let L be a linear isomorphism of $N(I - A)$ onto Y_2 and P be a continuous linear projection of X onto $N(I - A)$. Define a linear map $A_1 : X \to X$ by $A_1 = A + C$, where $C = LP : X \to Y_2$. Clearly, A_1 is a $k - \phi$ contraction. Moreover, $I - A_1$ is one-to-one. Indeed, suppose that $x - A_1x = 0$. Then $x - Ax = -Cx \in R(I - A) \cap Y_2 = \{0\}$ and consequently, $Px = 0$ since $Cx = LPx = 0$ and L is one-to-one. Hence, $x \in X_2$ with $x - Ax = 0$ and so $x = 0$, proving that $I - A_1$ is one-to-one. Next, by Lemma 1.1 in [18], there exists a constant $k_1 > 0$ such that

$$\|x - A_1x\| \geqq k_1\|x\| \quad \text{for all} \quad x \in X .$$

Since k_0 is sufficiently small and

$$\limsup_{\|x\| \to \infty} \frac{\alpha(T(x) + C(x) + f, A(x) + C(x))}{\|x\|} \leqq \limsup_{\|x\| \to \infty} \frac{\alpha(T(x) + f, A(x))}{\|x\|} \leqq k_0 ,$$

we have that the equation $f \in x - T(x) - C(x)$ is solvable by Theorem 1.1 in [18]. Since $R(I - T) \subset R(I - A)$, it follows that $C(x) \in x - T(x) - f \subset Y_1$ and so $C(x) \in Y_1 \cap Y_2 = \{0\}$. Hence, $f \in x - T(x)$.

Conversely, suppose that $f \in x - T(x)$. Then, since $R(I - T) \subset R(I - A)$, $f \in R(I - A)$. ∎

Remark 2.1. If $T : X \to X$ is asymptotically linear, i.e. there exists a continuous linear mapping $A : X \to X$ such that for all x in X

$$T(x) = A(x) + N(x) \quad \text{with} \quad \|N(x)\|/\|x\| \to 0 \quad \text{as} \quad \|x\| \to \infty ,$$

the Fredholm alternative for T compact was obtained by Kachurovsky [14] and for T k-ball-contractive, $k < 1$, by Petryshyn [29]. Our alternative extends these to multivalued ϕ-condensing nonasymptotically linear mappings.

Let $c_1 : R^+ \to R^+$ be continuous with $c_1(r) < c(r)$ for all r and c as defined before. Then we have

Theorem 2.2. *Let X be a Banach space, $T: X \to CK(X)$ u.s.c. and $1 - \phi$ contractive and $A: X \to CK(X)$ odd. Suppose that to each f in X there corresponds a constant $r_f > 0$ such that*

$$\|x - y\| \geqq c(\|x\|) \quad \textit{for all} \quad y \in A(x) \quad \textit{with} \quad \|x\| = r_f \tag{17}$$

and

$$\alpha(T(x) + f, A(x)) < c_1(\|x\|) \quad \textit{for all} \quad \|x\| = r_f\,. \tag{18}$$

Then, if $I - T$ satisfies condition $(++)$, the equation $f \in x - T(x)$ has at least one solution for each f in X.

Proof. Let f be a fixed element in X and define $T_\beta(x) = \beta\, T(x)$ for $x \in X$, where $\beta \in (\beta_0, 1)$ with $\beta_0 > 0$ such that $1 - \beta_0 + c_1(r_f) < c(r_f)$. Since $T(\partial B(0, r_f))$ is bounded, we can select $\beta_0 < \beta_1 < 1$ such that

$$(1 - \beta)\,\|y\| \leqq 1 - \beta_0 \quad \text{for all} \quad y \in T(\partial B(0, r_f)) \quad \text{and} \quad \beta \in (\beta_1, 1)\,.$$

Then, for each $\beta \in (\beta_1, 1)$ and $x \in X$ with $\|x\| = r_f$, we have that

$$\begin{aligned} \alpha(T_\beta(x) + f, A(x)) &= \sup_{y \in T(x)} \inf_{z \in A(x)} \|\beta y + f - z\| \leqq \\ &\leqq \sup_{y \in T(x)} \inf_{z \in A(x)} [(1 - \beta)\,\|y\| + \|y + f - z\|] \leqq \\ &\leqq 1 - \beta_0 + \alpha(T(x) + f, A(x)) \leqq 1 - \beta_0 + c_1(r_f)\,. \end{aligned}$$

Set $c_2(r) = 1 - \beta_0 + c_1(r)$. Thus, for each $\beta \in (\beta_1, 1)$,

$$\alpha(T_\beta(x) + f,\ A(x)) \leqq c_2(\|x\|) \quad \text{for} \quad \|x\| = r_f$$

and

$$\|x - y\| \geqq c\|x\| \quad \text{for all} \quad y \in A(x) \quad \text{with} \quad \|x\| = r_f\,.$$

Since $c_2(r_f) < c(r_f)$, by Theorem 1.1 in [18] (see its proof), we have that the equation $f \in x - T_\beta(x)$ is solvable for each $\beta \in (\beta_1, 1)$. Let $\beta_n \in (\beta_1, 1)$ with $\beta_n \to 1$ and $x_n \in B(0, r_f)$ be such that $f \in x_n - T_{\beta_n}(x_n)$. Then $f = x_n - \beta_n y_n$ for some $y_n \in T(x_n)$, and $x_n - y_n = (\beta_n - 1)\, y_n + f \to f$ as $n \to \infty$ since $\{y_n\}$ is bounded. By condition $(++)$, there exists $x \in B(0, r_f)$ such that $f \in x - T(x)$. ∎

Theorem 2.2 and Remark 2.2 below imply

Corollary 2.1. *Let $A: X \to CK(X)$ be u.s.c. positively homogeneous and ϕ-condensing, $T: X \to CK(X)$ u.s.c. and $1 - \phi$ contractive, A and T satisfy condition (15) and $x = 0$ if $x \in A(x)$. Then, if $I - T$ satisfies condition $(++)$, $I - T$ is surjective.*

Remark 2.2. In [18] we have shown that condition (16) holds if A is u.s.c., positively homogeneous ϕ-condensing and such that $x \in A(x)$ implies $x = 0$. If A is not odd, then requiring that T be odd we can still obtain the solvability of $f \in x - T(x)$ provided also that $c(r) \to \infty$ as $r \to \infty$, as a consequence of Theorem 2.4 below.

We continue our exposition by establishing the solvability of $f \in x - T(x)$ under several new growth conditions on $I - T$. The following result was proven constructively in [21] for ball-condensing mappings as a special case of our Theorem 1. [21] for A-proper mappings.

In case of 1-ball-condensing mappings it can also be obtained from the corresponding theorem for uniform limits of A-proper mappings (see the Note at the end of the paper). The growth condition (19) below was first used by WILLE [41] and later by BROWDER [2], Milojević-Petryshyn [24] and Milojević [21] (see also Notices Amer. Math. Soc., January 1977, 77T-B27.)

Theorem 2.3. *Let X be a Banach space and $T : X \to CK(X)$ u.s.c. and $1 - \phi$-contractive such that $I - T$ satisfies condition $(++)$ and*

$$\|u\| + \frac{(u, Jx)}{\|x\|} \to \infty \quad \text{as} \quad \|x\| \to \infty \quad \text{for each} \quad u \in x - T(x) \tag{19}$$

where $J : X \to X^*$ is a section of the normalized duality mapping. Then the equation $f \in x - T(x)$ is solvable for each f in X.

Proof. Let f in X be fixed. By condition (19), there exists an $r_f > 0$ and $\gamma > 0$ such that

$$\|x - u - tf\| \geqq \gamma \quad \text{for} \quad u \in T(x) \quad \text{with} \quad \|x\| \geqq r_f\,, \quad t \in [0, 1]\,, \tag{20}$$

$$\|x - u\| + \frac{(x - u, Jx)}{\|x\|} > 0 \quad \text{for} \quad u \in T(x)\,, \quad \|x\| \geqq r_f\,. \tag{21}$$

By the boundedness of T, there exists $\beta_0 > 0$ such that for each $\beta \in (\beta_0, 1)$

$$\|x - \beta u - tf\| \geqq \gamma/2\,, \quad u \in T(x) \quad \text{with} \quad \|x\| = r_f\,, \quad t \in [0, 1]\,.$$

Fix $\beta \in (\beta_0, 1)$ and for each $Q \subset \bar{B}(0, r_f)$ the homotopy $H : [0, 1] \times \bar{B}(0, r_f) \to \to CK(X)$ given by $H_\beta(t, x) = \beta Tx + tf$ is such that

$$\phi(H_\beta([0, 1] \times Q)) \leqq \phi(\beta\, T(Q) + \{tf \mid t \in [0, 1]\}) \leqq \phi(\beta\, T(Q)) \leqq \beta\, \phi(Q)\,.$$

Hence, by the homotopy theorem ([33]) $\deg(I - \beta T - f, B(0, r_f), 0) = \deg(I - \beta T, B(0, r_f), 0)$. Next, define the homotopy $F_\beta : [0, 1] \times \bar{B}(0, r_f) \to CK(X)$ by $F_\beta(t, x) = = \beta t\, T(x)$. We claim that

$$x \notin F_\beta(t, x) \quad \text{for each} \quad t \in [0, 1]\,, \quad \|x\| = r_f\,.$$

If not, then there would exist $t_0 \in [0, 1]$ and $\|x_0\| = r_f$ such that for some $u_0 \in \in T(x_0)$, $x_0 - \beta t_0 u_0 = 0$. But, then $t_0 \neq 0, 1$, and

$$\|x_0 - u_0\| + \frac{(x_0 - u_0, J(x_0))}{\|x_0\|} = \left\|x_0 - \frac{1}{\beta t_0}\, x_0\right\| + \|x_0\| - \frac{1}{\beta t_0}\, \frac{(x_0, J(x_0))}{\|x_0\|} =$$

$$= \left(-1 + \frac{1}{\beta t_0}\right)\|x_0\| + \|x_0\| - \frac{1}{\beta t_0}\, \|x_0\| = 0\,,$$

a contradiction. Hence, $x \notin F_\beta(t, x)$ on $[0, 1] \times \partial\beta(0, r_f)$, and so

$$\deg(I - \beta T, B(0, r_f), 0) = \deg(I - B(0, r_f), 0) \neq 0 .$$

Thus, $\deg(I - \beta T - f, B(0, r_f), 0) \neq 0$ implying that for each $\beta_k \to 1$ with $\beta_k \in (\beta_0, 1)$, there exists $x_k \in B(0, r_f)$ such that $f \in x_k - \beta_k T(x_k)$. Let $u_k \in T(x_k)$ be such that $f = x_k - \beta_k u_k$. Then $x_k - u_k = f + (\beta_k - 1) u_k \to f$ as $k \to \infty$ and by condition $(+ +)$, there exists x in X such that $f \in x - T(x)$. ∎

Remark 2.3. From the proof of Theorem 2.3 we see that condition (19) can be replaced by conditions (20) and (21).

Under some different growth conditions, we have (see [34] for the single-valued case and the references there in).

Theorem 2.4. *Let $T: X \to CK(X)$ be u.s.c. and $1 - \phi$-contractive. Suppose that for some $R > 0$, either T is odd on $X \setminus B(0, R)$ or*

$$\lambda x \notin T(x) \quad \text{for all} \quad \|x\| \geqq R , \quad \lambda > 1 , \tag{22}$$

and that to each f in X there corresponds $r_f \geqq R$ and $\gamma > 0$ such that

$$\|x - y - tf\| \geqq \gamma \quad \text{for all} \quad y \in T(x) \quad \text{with} \quad \|x\| = r_f , \quad t \in [0, 1] . \tag{23}$$

Then, if $I - T$ satisfies condition $(+ +)$, the equation $f \in x - T(x)$ is solvable for each f in X.

Proof. Let f in X be fixed and observe that by the boundedness of T there exists $\beta_0 \in (0, 1)$ such that for each $\beta \in (\beta_0, 1)$

$$\|x - \beta y - tf\| \geqq \gamma/2 \quad \text{for} \quad y \in T(x) , \quad \|x\| = r_f , \quad t \in [0, 1] .$$

Since for each $Q \subset \bar{B}(0, r_f)$, the homotopy $H : [0, 1] \times \bar{B}(0, r_f) \to CK(X)$ given by $H_\beta(t, x) = \beta\, T(x) + tf$ is such that:

$$\phi(H_\beta([0, 1] \times Q)) \leqq \phi(\beta\, T(Q) + \{tf \mid t \in [0, 1]\}) \leqq \phi(\beta\, T(Q)) \leqq \beta\, \phi(Q) ,$$

The homotopy theorem ([33]) implies

$$\deg(I - \beta T - f, B(0, r_f), 0) = \deg(I - T, B(0, r_f), 0) .$$

If T is odd, the last degree is nonzero, while if T satisfies (22) it is again nonzero which can be easily seen by using the homotopy $F_\beta(t, x) = \beta t\, T(x)$. Hence, for each $\beta_k \in (\beta_0, 1)$ with $\beta_k \to 1$ there exists $x_k \in B(0, r_f)$ such that $f \in x_k - \beta_k T(x_k)$. This and condition $(+ +)$ imply that $f \in x - T(x)$ for some x in X. ∎

Remark 2.4. When T is ϕ-condensing, condition (23) of Theorem 2.4 can be weaken to

$$tf \notin (I - T)(\partial B(0, r_f)) \quad \text{for each} \quad t \in [0, 1] .$$

The usefulness of this observation can be seen from the following easy application of Theorem 2.4.

Theorem 2.5. *Let $A : D(A) \subset X \to Y$ be a closed linear mapping with the closed range and $X = N(A) \oplus U$ for some closed subspace U of X. Let $T : X \to CK(Y)$ be such that $R(T) \subset R(A)$ and either T is odd or for some $R > 0$*

$$\lambda A(x) \notin T(x) \quad for \quad \|x\| \geqq R \quad in \quad D(A) \cap U\,, \quad \lambda > 1\,. \tag{24}$$

Suppose that either one of the following conditions holds:

(i) *$A - T$ satisfies condition $(+)$ on $D(A) \cap U$ and $I - A_1^{-1}T : U \to CK(U)$ is u.s.c. and ϕ-condensing, where A_1^{-1} is the inverse of the restriction A_1 of A to $D(A) \cap U$;*

(ii) *$I - A_1^{-1}T : U \to CK(U)$ is u.s.c., $1 - \phi$-contractive and satisfies condition $(++)$ on U and for each $f \in R(A)$ there exists an $r_f \geqq R$ such that for some $\gamma > 0$.*

$$\|x - A_1^{-1}(y) - t\, A_1^{-1}(f)\| \geqq \gamma \quad for \quad y \in T(x)\,, \qquad \|x\| = r_f \quad in \quad D(A) \cap U\,, \quad t \in [0, 1]\,. \tag{25}$$

Then the equation $f \in A(x) - T(x)$ is solvable if and only if $f \in N(A_0^*)^\perp$ $(=R(A))$, where A_0^* is as defined in Section 1.

Proof. (i) Let $f \in N(A_0^*)^\perp$. By condition $(+)$ there exists an $r_f \geqq R$ such that $tA_1^{-1}(f) \notin (I - A_1^{-1})(\partial B(0, r_f))$ for all $t \in [0, 1]$. Moreover, condition (24) implies that

$$\lambda x \notin A_1^{-1}\, T(x) \quad \text{for} \quad \|x\| \geqq R \quad \text{in} \quad U\,, \quad \lambda > 1\,.$$

Hence, by Theorem 2.4 and Remark 2.4 the equation $A_1^{-1}(f) \in x - A_1^{-1}\, T(x)$ is solvable in U, and therefore, so is $f \in A(x) - T(x)$.

Conversely, if $f \in A(x) - T(x)$ is solvable, then $f \in N(A_0^*)^\perp = R(A)$ since $R(T) \subset$ $\subset R(A)$.

(ii) Since $I - A_1^{-1}T$ satisfies all the assumptions of Theorem 2.4, the conclusion follows as in part (i). ∎

Remark 2.5. Since A is not continuous, condition $(+)$ for $A - T$ does not imply (25). However, conditions (24) and (25) of Theorem 2.5 are implied by

$$\limsup_{\substack{\|x\| \to \infty \\ x \in U}} \frac{|Tx|}{\|x\|} < 1/\|A_1^{-1}\|$$

and consequently, Theorem 2.5 (ii) with T singlevalued extends the corresponding results of Kachurovsky [12] and Petryshyn [32] for $A_1^{-1}T$ compact and $1 - \phi$-contractive, respectively. Moreover, in view of Theorem 2.1, the last condition can be weaken in a manner as indicated in Remark 1.3 (with F ϕ-condensing). For bijective A and singlevalued T such that $A^{-1}T$ or TA^{-1} is $k - \phi$-contractive, $k \leqq 1$, Theorem 2.5 has been previously proven by Fučík [4] $(k = 0)$, Webb [40] and Petryshyn-

Fitzpatrick [34] under various stronger additional assumptions which include the oddness of T or the boundedness of the set of solutions of $A(x) - t\,T(x) = 0$ for all $t \in [0, 1]$, etc. For nonlinear and noninjective A with TA^{-1} 1-ball-contractive, see Milojević-Petryshyn [23, 24]. For other assumptions on A and T see, e.g., Fučík [5], Fučík-Kučera-Nečas [7] and the references therin.

3. CONDENSING PERTURBATIONS OF ACCRETIVE AND PSEUDO-CONTRACTIVE MAPPINGS

The results of the previous sections will now be used to establish Fredholm alternatives and other surjectivity type of results for condensing like perturbations of accretive and pseudo-contractive like maps. The fixed point type and some surjectivity result for such maps have been proven in [19, 22] under different boundary conditions; for bifurcation theory see [36, 37].

Consider the equation

$$(26) \qquad f \in Ax - Bx + Nx \quad (x \in D(A),\ f \in H),$$

where H is a real π_1-Hilbert space and the mappings A, B and N satisfy:

(A1) *A is a densily defined, positive definite, self-adjoint linear mapping whose essential spectrum $\sigma_e(A)$ is bounded below, i.e. there is a number $\gamma > 0$ such that for each $\varepsilon > 0$, $\sigma(A) \cap (-\infty, \gamma - \varepsilon)$ consists of a nonempty set of isolated eigenvalues, each of finite multiplicity, with*

$$\lambda_0 < \lambda_1 < \ldots < \lambda .$$

(A2) *$B : H \to CK(H)$ is a demicontinuous k_1-ball-contractive mapping with $k_1\gamma^{-1} < 1$.*

(A3) *Let H_0 be the completion of $D(A)$ in the metric $[x, y] = (Ax, y)$ and $\|x\|_0 = (Ax, x)^{1/2}$ for all $x, y \in D(A)$ and $N : H_0 \to 2^H$ be of the form $N = N_1 + N_2$, where $N_1 : H_0 \to H$ is continuous and monotone and $N_2 : H \to CK(H)$ is demiclosed and compact.*

By assumption (A1) A has a bounded inverse $A^{-1} : H \to H$ which is self-adjoint, positive and γ^{-1}-ball-contractive. Furthermore, the positive self-adjoint square root $A^{1/2}$ of A is a linear homeomorphism of $H_0 = D(A^{1/2})$ with $\|x\|_0 = \|A^{1/2}x\|$ onto H and the positive square root $A^{-1/2} \equiv (A^{-1})^{1/2} \equiv (A^{1/2})^{-1} : H \to H$ is continuous and $\gamma^{-1/2}$-ball-contractive (cf. [36]). Hence, the mapping $L \equiv A^{-1/2}BA^{-1/2} : H \to CK(H)$ is k-ball-contractive with $k = k_1\gamma^{-1}$ and by (A3), $M \equiv A^{-1/2}NA^{-1/2} : H \to 2^H$ is the sum of the monotone $M_1 = A^{-1/2}N_1A^{-1/2}$ and the compact mapping $M_2 = A^{-1/2}N_2A^{-1/2}$.

An element $x \in D(A)$ is a solution of $f \in Ax - Bx + Nx$ if and if only if $y = A^{1/2}x$ and y is a solution of

$$(27) \qquad A^{-1/2}(f) \in y - Ly + My \quad (y \in H,\ A^{-1/2}(f) \in H).$$

We also observe that if $\Gamma_0 = \{A^{1/2}X_n, P_n\}$, $X_n \subset D(A)$, is a projectionally complete scheme for H, then for each approximate solution x_n of (26) obtained by the Galerkin method, the vector $y_n = A^{1/2}x_n$ satisfies $P_nA^{-1/2}(f) \in y_n - P_nLy_n + P_nMy_n$ and conversely and therefore Equation (26) is approximation solvable w.r.t. $\Gamma = \{X_n, P_n\}$ for H_0 if and only if Equation (27) is approximation solvable w.r.t. Γ_0 for H. We shall discuss the (approximation) solvability of Eq. (27) w.r.t. any projectionally complete scheme $\Gamma = \{X_n, P_n\}$ with $X_n \subset X_{n+1}$ which then implies the solvability of Eq. (26).

Theorem 3.1. *Under the assumptions* (A1)–(A3), *Equation* (26) *is solvable in* $D(A)$ *for each* f *in* H *if either one of the following conditions holds*:

(i) N_1 *is odd on* $H \setminus B(0, r)$ *for some* $r > 0$ *and*

$$\limsup_{\|x\| \to \infty} \frac{|Lx - M_2x|}{\|x\|} < 1 ;$$

(ii) B *is odd and* 1-*homogeneous* (*i.e.* $B(tx) = tB(x)$ *for all* $t > 0$ *and* $x \in H$), $Ax \notin Bx$ *for* $0 \neq x \in D(A)$ *and*

$$\limsup_{\|x\| \to \infty} \frac{|Mx|}{\|x\|} < c$$

with c *sufficiently small.*

(iii) $$\limsup_{\|x\| \to \infty} \frac{|Lx - Mx|}{\|x\|} < 1 .$$

Proof. (i) We may assume that $M_1(0) = 0$. Since $A_1 = I + M_1$ is strongly monotone, it is A proper, odd and satisfies $\|P_nA_1x\| \geqq \|x\|$ for $x \in X_n$, $n \geqq 1$. Since $T = I - L + M$ is also A-proper by Example A, A_1 and T satisfy all the hypotheses of Theorem A.

(ii) Since L is k-ball-contractive, $k < 1$, $A_1 = I - L$ is A proper, odd, 1-homogeneous and $0 \in A_1(x) = x - L(x)$ implies $x = 0$ since $Ax \notin Bx$ for $0 \neq x \in D(A)$. By Lemma 2.1 in [20], there exists a constant $c_1 > 0$ such that for large n

$$\|P_n(y)\| \geqq c_1\|x\| \quad \text{for} \quad y \in A_1(x), \quad x \in X_n .$$

Hence, A_1 and $T = I - L + M$ satisfy all the hypotheses of Theorem A.

(iii) This part is a direct consequence of Theorem 1.2. ∎

Let us observe that by the properties of $A^{1/2}$, the growth condition (i) in Theorem 3.1 holds if

$$\limsup_{\|y\|_0 \to \infty} \frac{\|By - N_2y\|}{\|y\|_0} < 1/\|A^{-1/2}\| .$$

Similar observations hold for (ii) and (iii). As observed before, Eq. (26) is feebly approximation solvable under the conditions of Theorem 3.1 provided $\Gamma =$

$= \{A^{1/2}X_n, P_n\}$. However, just the solvability of Eq. (26) holds say, in part (i) without the oddness of N_1, as established in Theorem 3.3 (cf. also Theorem 3.4).

Theorem 3.2. *Let N be single valued in Theorem 3.1 and such that for some c-strongly monotone continuous linear mapping $G : H \to H$ and a demicontinuous linear k_0-ball-contractive mapping $F : H \to H$ with $k_0 < \min\{c, c_1\}$, and*

$$\|P_n G^* x\| \geqq c_1 \|x\| \quad \text{for} \quad x \in X_n,$$

we have

$$\limsup_{\|x\| \to \infty} \frac{|x - Lx + Mx - Gx - Fx|}{\|x\|} \leqq m$$

for some sufficiently small m. Then, either $N(G + F) = \{0\}$, in which case Equation (26) is solvable in $D(A)$ for each f in H, or $N(G + F) \neq \{0\}$. In the latter case, assuming additionally that

$$R(I - L + M) \subset N(G^* + F^*)^{\perp} \; (= R(G + F))$$

or equivalently,

$$R(A - B + N) \subset R(A^{1/2}G + A^{1/2}F),$$

Equation (26) is solvable if and only if $A^{-1/2}(f) \in N(F^* + G^*)^{\perp}$.

Proof. The mapping $F_0 = M_2 - L$ is k-ball-condensing and $T = I + M_1$ is strongly monotone, and therefore, F, G, T and F_0 satisfy all the assumptions of Corollary 1.1 (cf. Remark 1.1). ∎

Corollary 3.1. *Let B and N be singlevalued, B be linear with $\|B\| \max\{1, \gamma^{-1}\} < 1$ and*

$$\limsup_{\|x\| \to \infty} \frac{\|Mx\|}{\|x\|} \leqq m$$

with m sufficiently small. Then, either $N(A - B) = \{0\}$, in which case Equation (26) is solvable in $D(A)$ for each f in H or $N(A - B) \neq \{0\}$. In the latter case, assuming additionally that either $R(I - L + M) \subset N(I - L^*)^{\perp} = [A^{1/2}N(A - B^*)]^{\perp}$ or $R(N) \subset R(A - B)$, Equation (26) is solvable if and only if $f \in [N(A - B^*)]^{\perp}$, where $N(A - B^*)$ denotes the null space of $A - B^*$.

Proof. We shall show that the mappings $G = I$, $F = -L$, A, L and M satisfy all the hypotheses of Theorem 3.2. Since $A^{1/2}$ is a linear homeomorphism, $N(I - L) = \{0\}$ if and only if $N(A - B) = \{0\}$ and $u \in N(I - L^*)^{\perp}$ if and only if $(u, A^{1/2}y) = 0$ for all $y \in N(A - B^*)$ by the self-adjointness of $A^{-1/2}$. Next, if $R(N) \subset R(A - B)$, then $R(I - L + M) \subset N(I - L^*)^{\perp} = R(I - L)$. Indeed, for each $x \in H_0$, there exists $y \in H_0$ such that $N(x) = Ay - By$. Set $u = A^{1/2}x$ and $v = A^{1/2}y$. Then $u, v \in H$ and $NA^{-1/2}u = A^{1/2}v - BA^{-1/2}v$ or, $A^{-1/2}NA^{-1/2}u = v - A^{-1/2}BA^{-1/2}v$, i.e. $Mu = (I - L)v$. Hence, for each $u \in H$ there exists $v \in H$ such that $Mu = (I - L)v$, i.e. $R(M) \subset R(I - L)$ and therefore $R(I - L + M) \subset R(I - L)$. The conclusions now follow from Theorem 3.2. ∎

In a similar fashion, one proves (taking $G = I + M$, $F = 0$).

Corollary 3.2. *Let N be singlevalued and N linear, continuous and $(Nx, x) \geqq 0$ for $x \in H_0$ and*

$$\limsup_{\|x\| \to \infty} \frac{|Lx|}{\|x\|} \leqq m$$

with m sufficiently small. Then, either $N(A + N) = \{0\}$, in which case Equation (26) is solvable in $D(A)$ for each f in H or $N(A + N) \neq \{0\}$. In the latter case, assuming additionally that either $R(I - L + M) \subset N(I + M^*)^\perp = [A^{1/2}N(A + N^*)]^\perp$ $(= R(I + M))$ or $R(B) \subset R(A + N)$, Equation (26) is solvable if and only if $f \in [N(A + N^*)]^\perp$.

For treating the case of $k - \phi$-contractions, where ϕ is either the ball or set-measure of noncompactness, we introduce

(A2′) *$B : H \to CK(H)$ is an upper semicontinuous $k_1 - \phi$-contraction with $k_1\gamma^{-1} < 1$.*

Theorem 3.3. *Let H be a Hilbert space and let the assumptions (A1), (A2′) and (A3) hold and*

$$\limsup_{\|x\| \to \infty} \frac{|Lx - M_2x|}{\|x\|} < 1 .$$

Then Equation (26) is solvable for each f in H.

Proof. By the previous discussion, we need show the solvability of $h \in x - Lx + Mx$ for each $h \in H$. Since M_1 is monotone,

$$\|x + M_1x - y - M_1y\| \geqq \|x - y\| , \quad x, y \in H ,$$

and consequently, the inverse $(I + M_1)^{-1}$ exists [25] with

$$\|(I + M_1)^{-1} u - (I + M_1)^{-1} v\| \leqq \|u - v\| \quad \text{for} \quad u, v \in H$$

and

$$\limsup_{\|x\| \to \infty} \frac{|(I + M_1)^{-1} (Lx - M_2x)|}{\|x\|} \leqq \limsup_{\|x\| \to \infty} \frac{|Lx + M_2x|}{\|x\|} < 1 .$$

Since $(I + M_1)^{-1} (L - M_2)$ is $k_1\gamma^{-1} - \phi$-contractive, by Theorem 2.4 the equation $g \in x - (I + M_1)^{-1} (L - M_2) x$ is solvable for each $g \in H$, and therefore, so is the equation $h \in x - Lx + Mx$ for each $h \in H$. ∎

In a similar fashion, using Theorems 2.2 and 2.4, we can prove

Theorem 3.4. *Let H be a Hilbert space and the assumptions* (A1) *and* (A2′) *hold. Suppose that M is $k_2 - \phi$-contractive and that either one of the following conditions holds*:

(i) *N is odd and 1-homogeneous, $Ax \notin -Nx$ for $0 \neq x \in D(A)$, $k_2\gamma^{-1} < 1$ and*

$$\limsup_{\|x\| \to \infty} \frac{|Lx|}{\|x\|} < c$$

with c sufficiently small;

(ii) $$(k_1 + k_2)\gamma^{-1} < 1 \quad \textit{and} \quad \limsup_{\|x\| \to \infty} \frac{|Lx - Mx|}{\|x\|} < 1 .$$

Then Equation (26) *is solvable in $D(A)$ for each f in H.*

Proof. (i) By the properties of $-M$ and Lemma 1.1 in [18], there exists a constant $c_1 > 0$ such that $\|x + y\| \geqq c_1\|x\|$ for all $y \in M(x)$ and $x \in H$. Hence, since $-M$ and $L - M$ satisfy all the assumptions of Theorem 2.2, the theorem is valid in this case. We add that the roles of M and L can be interchanged in this part.

(ii) This part follows immediately from Theorem 2.4. ∎

Theorem 3.5. *Suppose that* (A1) *and* (A2′) *hold and that N is linear and continuous with $(k_1 + \|M\|)\gamma^{-1} < 1$. Suppose that*

$$\limsup_{\|x\| \to \infty} \frac{|Lx|}{\|x\|} \leqq k_0$$

with k_0 sufficiently small. Then, either $N(A + N) = \{0\}$ in which case Equation (26) *is solvable in $D(A)$ for each f in H, or $N(A + N) \neq \{0\}$. In the latter case, assuming additionally that either*

$$R(I - L + M) \subset N(I + M^*)^{\perp} = [A^{1/2}N(A + N^*)]^{\perp}$$

or $R(B) \subset R(A + N)$, Equation (26) *is solvable if and only if $f \in [N(A + N^*)]^{\perp}$.*

Proof. The theorem will follow from Theorem 2.1 if we could show that $-M$ and $T = L - M$ satisfy its assumptions. Clearly,

$$\limsup_{\|x\| \to \infty} \frac{\alpha(Tx, -Mx)}{\|x\|} \leqq \limsup_{\|x\|} \frac{|Lx|}{\|x\|} \leqq k_0$$

and $N(I + M) = \{0\}$ if and only if $N(A + N) = \{0\}$. Since $A^{1/2}$ is a linear homeomorphism, and $A^{-1/2}$ is self-adjoint, it follows that $N(I + M^*)^{\perp} = [A^{1/2}N(A + N^*)]^{\perp}$ and that $R(B) \subset R(A + N)$ implies $R(I - L + M) \subset R(I + M) = N(I + M^*)^{\perp}$. ∎

Finally, as an application of Theorem 2.3, we have

Theorem 3.6. *Suppose that* (A1) *and* (A2′) *hold and that* N *is linear and continuous with* $(k_1 + \|M\|)\gamma^{-1} < 1$. *Then, if* $T = I - L + M$ *and*

$$\|u\| + \frac{(u, x)}{\|x\|} \to \infty \quad as \quad \|x\| \to \infty \quad for\ each \quad u \in T(x)$$

Equation (26) *is solvable for each* f *in* H.

Let us now look at some approximation solvability results involving condensing like perturbations of a-stable, c-accretive and strongly pseudo-contractive mappings. We shall just illustrate some applications of Theorems 1.2 and 1.3.

Theorem 3.7. *Let* X *be a* π_1*-Banach space with a projectionally complete scheme* $\Gamma = \{X_n, P_n\}$, $T: X \to X$ *continuous, and* a*-stable and* $F: X \to CK(X)$ *demicontinuous and* k*-ball-contractive with* $k < c$ *or ball-condensing if* $c = 1$. *Suppose that*

$$\limsup_{\|x\| \to \infty} \frac{|x - Tx - Fx|}{\|x\|} < 1 .$$

Then the equation $f \in T(x) + F(x)$ *is feebly approximation-solvable for each* f *in* X.

Proof. By Example A, $I - (I - T - F) = T + F$ is A-proper and the conclusion follows by Theorem 1.2. ∎

Theorem 3.8. *Let* X *and* Γ *be as in Theorem* 3.7, $T: X \to X$ *continuous and* c*-accretive and* $F: X \to CK(X)$ *demicontinuous and* k*-ball-contractive. Suppose that either* F *and* T *are odd or*

(28) *for some* $R > 0$, $\lambda x \notin (I - T - F)(x)$ *for all* $\|x\| \geqq R$ *and* $\lambda > 1$.

(a) *If* $k < c$ *or* F *is ball-condensing if* $c = 1$ *and to each* f *in* X *there corresponds* $r_f \geqq R$ *such that*

(29) $tf \notin (T + F)(\partial B(0, r_f))$, $t \in [0, 1]$, *the equation* $f \in T(x) + F(x)$

is feebly approximation solvable for each f *in* X.

(b) *If* $k = c$, *with* $c \geqq 0$, $T + F$ *satisfies condition* $(++)$ *and for some* $\gamma > 0$.

(30) $\|Tx + u - tf\| \geqq \gamma$ *for all* $u \in F(x)$, $\|x\| = r_f$, $t \in [0, 1]$,

then the equation $f \in T(x) + F(x)$ *is solvable for each* f *in* X.

Proof. (a) By Example A, for each $\alpha \geqq 0$,

$$(1 + \alpha)I - (I - T - F) = \alpha I + T + F$$

is A-proper since $k < c + \alpha$ and conditions (10) and (12) of Theorem 1.3 (with $A = I$) hold. Hence, the conclusion follows by Theorem 1.3 part (i).

(b) In a similar fashion as in part (a), the conclusion now follows from Theorem 1.3 part (ii). ∎

For each pair $x, y \in X$, define $(x, y)_- = \inf\{(x, w) \mid w \in J(y)\}$, where J is the normalized duality mapping.

Definition 3.1. A mapping $T: X \to X$ is said to be *k-strongly pseudo-contractive* if for some $k < 1$ ([1]).

$$(Tx - Ty, x - y)_- \leqq k\|x - y\|^2 \quad (x, y \in X).$$

In view of Theorem 3.8 and the fact that $T = I - A$ is $(1 - k)$-accretive whenever A is k-strongly pseudo-contractive, we have

Theorem 3.9. *Let X and Γ be as in Theorem 3.7, $T: X \to X$ continuous and k-strongly pseudo-contractive and $F: X \to CK(X)$ demicontinuous and k_1-ball-contractive. Suppose that either F and T are odd or for some $R > 0$*

$$\lambda x \notin Tx + Fx \quad \text{for all} \quad \|x\| \geqq R \quad \text{and} \quad \lambda > 1.$$

(a) *If $k_1 < 1 - k$ or F is ball-condensing if $k = 0$ and to each f in X there corresponds $r_f \geqq R$ such that*

$$tf \notin (I - T - F)(\partial B(0, r_f)), \quad t \in [0, 1],$$

then the equation $f \in x - T(x) - F(x)$ is feebly approximation-solvable for each f in X.

(b) *If $k_1 = 1 - k$ with $k \leqq 1$, $I - T - F$ satisfies condition $(++)$ and for some $\gamma > 0$*

$$\|x - Tx - u - tf\| \geqq \gamma \quad \text{for all} \quad u \in F(x), \quad \|x\| = r_f,$$

$t \in [0, 1]$, then the equation $f \in x - T(x) - F(x)$ is solvable for each f in X.

4. APPROXIMATION SOLVABILITY OF INTEGRAL AND DIFFERENTIAL EQUATIONS

In this section we apply our Fredholm alternatives and the results of Section 3 to establishing some approximation solvability results for contingent integral equations and for generalized BVP for nonlinear odinary differential equations on bounded and unbounded domains.

1. Contingent integral equations. Let Q be a bounded domain in R^n, $F: Q \times R^n \to BK(R^n)$ and $K(t, s)$ be a matrix of dimension n, i.e., $K: Q \times Q \to R^{n^2}$. For

$x : Q \to R^n$ we define for $t \in Q$

$$\int_Q K(t, s)\, F(s, x(s))\, ds = \{y(t) = \int_Q K(t, s) f(s)\, ds \mid y : Q \to R^n$$

is measurable and $f(s) \in F(s, x(s))$ a. e., f measurable$\}$.

For $h \in L_2(Q) \equiv L_2(Q, R^n)$, consider the contingent integral equation

$$x(t) \in \int_Q K(t, s)\, F(s, x(s))\, ds + h(t) \quad (t \in Q)\,. \tag{31}$$

Assumption (A) (1) *$F(t, x)$ satisfies Caratheodory conditions:*

(i) *$F(t, x)$ is measurable in t for each fixed $x \in R^n$* (i.e., $\{t \in Q \mid F(t, x) \cap A \neq \emptyset\}$ *is measurable for each closed $A \subset R^n$*), *and* (ii) *$F(t, x)$ is closed* (i.e., *if $x_i \to x_0$, $y_i \to y_0$, $y_i \in F(t, x_i)$, then $y_0 \in F(t, x_0)$, $t \in Q$ fixed.*)

(2) *For $\lambda \in R$,*

$$\sup_{y \in F(s,x)} |\lambda x - y| \leqq \beta|x| + \sum_{k=1}^{n} s_k(s)\, |x|^{1-p_k} + m(s)\,,$$

where $m \in L_2(Q, R)$, $s_k(s) \in L_{2/p_k}(Q, R)$, $0 < p_k < 1$ and β is a sufficiently small constant.

For each $x \in L_2(Q)$, define $\mathscr{F}(x) = \{f \in L_2(Q) \mid f(s) \in F(s, x(s))$ a.e.$\}$. By Assumption (A), $\mathscr{F}(x) \neq \emptyset$ (cf. [3,15]), bounded, closed and convex subset of $L_2(Q)$ and and $\mathscr{F} : L_2(Q) \to BK(L_2(Q))$ is closed. Define $A : L_2(Q) \to L_2(Q)$ by $A\,x(t) = \lambda \int_Q K(t, s)\, x(s)\, ds$ and assume that $K(t, s) \in L_2(Q \times Q, R^{n^2})$. Since A is compact, $r = \dim N(I - A) < \infty$ and let $\{z_1, \ldots, z_r\} \subset L_2(Q)$ be a basis of the null space $N(I - A^*)$, i.e., they are linearly independent and

$$z_i(t) - \lambda \int_Q K(s, t)\, z_i(s)\, ds = 0 \quad (t \in Q,\ i = 1, \ldots, r)\,. \tag{32}$$

Assumption (B). *Assume that for each $f \in \mathscr{F}(x)$ with $x \in L_2(Q)$,*

$$\int_Q \left(\int_Q K(t, s)\, [f(s) - \lambda\, x(s)]\, ds \right) z_i(t)\, dt = 0\,, \quad 1 \leqq i \leqq r\,.$$

Set $T = (1/\lambda)\, A\mathscr{F}$. Since $(T - A)\, x(t) = \{\int K(t, s)\, [f(s) - \lambda\, x(s)]\, ds \mid f \in \mathscr{F}(x)\}$, $R(T - A) \subset N(I - A^*)^{\perp} = R(I - A)$ by Assumption (B). Moreover, $T : L_2(Q) \to CK(L_2(Q))$ is compact and closed and Equation (31) is equivalent to the operator equation

$$h \in x - T(x) \quad (x \in L_2(Q))\,. \tag{33}$$

Theorem 4.1. *Suppose that Assumption* (A) *holds.*

(a) *If the equation*

$$x(t) - \lambda \int_Q K(s, t)\, x(s)\, ds = 0$$

has a unique zero solution, then Equation (31) *is feebly approximation solvable w.r.t. any projectionally complete scheme* Γ_0 *for* (L_2, L_2) *for each* $h \in L_2(Q)$.

(b) *If* $N(I - A) \neq \{0\}$, *Assumption* (B) *holds and* $\{z_1, \ldots, z_r\}$ *satisfy condition* (31), *then* Equation (31) *is feebly approximation solvable w.r.t.* Γ_0 *for a given* $h \in L_2(Q)$ *if and only if*

$$\text{(C)} \qquad \int_Q h(s)\, z_i(s)\, ds = 0 \quad (i = 1, 2, \ldots, r)\,.$$

Proof. Since $L_2(Q)$ is a separable Hilbert space, there exists a projectionally complete scheme $\Gamma_0 = \{E_n, V_n; E_n, P_n\}$ for (L_2, L_2), where V_n is the identity injection of E_n into L_2. Moreover, since A is a compact mapping, $I - A : L_2(Q) \to L_2(Q)$ is A-proper with respect to Γ_0. Since $T: L_2(Q) \to CK(L_2)$ is compact and closed, $I - T$ is A-proper w.r.t. Γ_0 (see [17]) and $P_n(I - T) : B(0, r) \cap E_n \to CK(E_n)$ is upper semi-continuous for each $r > 0$. Next, observing that Equation (31) is equivalent to Equation (33), the conclusions of our theorem will follow from Theorem 1.1 provided we show that $I - A$ and $I - T$ satisfy all the assumption of that theorem. In view of the above discussion, we need only show that condition (5) of Theorem 1.1 holds and, in case (b), that $\dim N(I - A) = \operatorname{codim} R(I - A)$ and $R(I - T) \subset N(I - A^*)^{\perp}$. By Assumption (A), we get

$$\limsup_{\|x\| \to \infty} \frac{\alpha((I - T)\,x,\, (I - A)\,x)}{\|x\|} \leqq \limsup_{\|x\| \to \infty} \frac{\alpha(T(x), A(x))}{\|x\|} \leqq$$

$$\leqq \limsup_{\|x\| \to \infty} \frac{1}{\|x\|} \sup_{f \in \mathscr{F}(x)} \left\{ \left(\int_Q \left(\int_Q K(t, s)\, (f(s) - \lambda\, x(s))\, ds \right)^2 dt \right) \right\}^{1/2} \leqq$$

$$\leqq \limsup_{\|x\| \to \infty} \frac{1}{\|x\|}\, k\{\beta\|x\| + M \sum_{k=1}^{n} \|x\|^{1 - p_k} + N\} = k\beta\,,$$

where

$$k^2 = \int_Q \int_Q K^2(t, s)\, ds\, dt\,, \quad M = \max_{1 \leqq i \leqq n} \left\{ \int_Q |s_i(s)|^{2/p_i}\, ds \right\}^{p_i/2}$$

and $N = \|m\|$. Thus, since β is sufficiently small we have that $k\beta < k_1$ and condition (5) holds. In case (b), since A is a linear compact mapping, we have that $\dim N(I - A) = \operatorname{codim} R(I - A)$. To show that $R(I - T) \subset N(I - A^*)^{\perp}$, we first observe that $I - T = I - A + (A - T)$ and $R(I - A) \subset N(I - A^*)^{\perp}$ and then that $R(A - T) \subset N(I - A^*)^{\perp}$ by Assumption (B). ∎

In case when all mappings are singlevalued, the existence assertions of Theorem 4.1 have been proven by Kachurovski [12] under a more restrictive Assumption (A2) which implies that $\|Tx - Ax\|/\|x\| \to 0$ as $\|x\| \to \infty$.

2. Variational BVP for nonlinear ordinary differential equations. Consider formal ordinary differential operators

$$\mathscr{A}(u) = \sum_{i,j=0}^{m} (-1)^i \frac{\mathrm{d}^i}{\mathrm{d}x^i} (A_{ij}(x)\, u^{(j)}(x))$$

and

$$\mathscr{B}(u) = \sum_{j=0}^{m-1} (-1)^j \frac{\mathrm{d}^j}{\mathrm{d}x^j} [B_j(x; u(x), \ldots, u^{(m-1)}(x))] .$$

For $m \in N$, $p \in [1, \infty]$ we consider the Sobolev spaces $W_p^m(a, b) = \{u : D^i u \in L_p(a, b)$ for all $1 \leqq i \leqq m\}$ with the norm $\|u\|_{m,p} = (\sum_{i=1}^{m} \|u^{(i)}\|_{L_p}^p)^{1/p}$. Let $\mathring{W}_p^m(a, b)$ be the closed subspace of $W_p^m(a, b)$ defined by $\mathring{W}_p^m(a, b) = \{u \in W_p^m(a, b) : u(a) = \ldots = u^{(m-1)}(a) = u(b) = \ldots = u^{(m-1)}(b) = 0\}$. Let V be a closed subspace of the Hilbert space $W_2^m(a, b)$ containing $\mathring{W}_2^m(a, b)$ as a closed subspace, i.e. $\mathring{W}_2^m(a, b) \subset V \subset W_2^m(a, b)$.

Consider the following formal differential equation

$$\mathscr{A}(u) - \mathscr{B}(u) = f\,, \quad f \in L_1(a, b)\,. \tag{34}$$

Our conditions on A_{ij} and B_j are given by

Assumption (D) (1) *Suppose that* $A_{ij} \in L_\infty(a, b)$ *for* $i, j = 0, \ldots, m$ *and that there exists a constant* $C_0 > 0$ *such that* $A_{mm}(x) \geqq C_0$ *for almost all* $x \in (a, b)$.

(2) *The function* $B_j(x, \xi_0, \ldots, \xi_{m-1}) : [a, b] \times R^m \to R^1$ *is continuous for each* $j = 0, \ldots, m - 1$.

Let $\phi \in W_2^m(a, b)$ and $h \in V$ be such that the scalar product in $W_2^m(a, b)$, $(h, u) = 0$ for each $u \in \mathring{W}_2^m(a, b)$.

Definition 4.1. A function $u \in V$ is said to be a *weak solution* of the boundary value problem $\{V, h, \phi\}$ for differential equation (34) if for each $v \in V$ the following indentity holds:

$$\sum_{i,j=0}^{m} \int_a^b A_{ij}(x)\,(u^{(j)}(x) + \phi^{(j)}(x))\, v^{(i)}(x)\,\mathrm{d}x +$$
$$+ \sum_{j=0}^{m-1} \int_a^b B_j(x; u(x) + \phi(x), \ldots, u^{(m-1)}(x) + \phi^{(m-1)}(x))\, v^{(j)}(x)\,\mathrm{d}x =$$
$$= \int_a^b f(x)\, v(x)\,\mathrm{d}x + (h, v)_{W_2^m}\,.$$

Now, introduce in V the inner product as follows

$$(u, v)_V = \int_a^b A_{mm}(x)\, u^{(m)}(x)\, v^{(m)}(x)\, dx + \int_a^b u(x)\, v(x)\, dx .$$

Then the norm $\|u\|_V = (u, u)_V^{1/2}$ is equivalent to the norm $\|u\|_{m,2}$ on V.

Define the operator $L : V \to V$ by

$$(Lu, v)_V = \sum_{i,j=0}^{m} \int_a^b A_{ij}(x)\, u^{(j)}(x)\, v^{(i)}(x)\, dx , \quad u, v \in V ,$$

and $B : V \to V$ by

$$(Bu, v)_V = \sum_{j=0}^{m-1} \int_a^b B_j(x; u(x) + \phi(x), \ldots, u^{(m-1)}(x) + \phi^{(m-1)}(x))\, v^{(j)}(x)\, dx +$$

$$+ (h, v)_{W_2{}^m} - \sum_{i,j=0}^{m} \int_a^b A_{ij}(x)\, \phi^{(j)}(x)\, v^{(i)}(x)\, dx , \quad \text{for all} \quad u, v \in V .$$

Then, as shown in Fučík [6], the operator $A = I - L$ is linear continuous and compact and $B : V \to V$ is continuous and compact. Moreover, the existence of a weak solution to the boundary value problem (V, h, ϕ) for equation (34) is equivalent to the solvability of the operator equation

$$(35) \qquad Lu - Bu = u - Au - Bu = w_f , \quad (u \in V)$$

where $w_f \in V$ is such that $\int_a^b f(x)\, v(x)\, dx = (\omega_f, v)_V$ for all $v \in V$. By the generalized boundary value problem (V, h, ϕ) for Equation (34) we shall mean a problem of finding its weak solution. Since A is a compact linear operator, the dimension of the null space $\dim N(I - A) = r < \infty$ and let $\{\phi_1, \ldots, \phi_r\} \subset V$ be a basis of the null space $N(I - A^*)$.

Assumption (E) *Suppose that for each* $1 \leqq i \leqq r$, $(\phi_i, Bu)_V = 0$ *for each* $u \in V$.

Since $X_1 = N(I - A)$ and $Y_1 = N(I - A^*)$ are finite-dimensional subspaces of the Hilbert space V, we know that

$$V = X_1 \oplus X_2 = Y_1 \oplus Y_2 \quad \text{with} \quad X_2 = R(I - A^*) \quad \text{and} \quad Y_2 = R(I - A) .$$

Assumption (F) *There exist constants* $c_0 > 0$, $c_1 > 0$ *and* $0 \leqq \delta \leqq 1$ *such that for each* $0 \leqq j \leqq m - 1$

$$|B_j(x; \xi_0, \ldots, \xi_{m-1})| \leqq c_0 + c_1 (\sum_{j=0}^{m-1} |\xi_i|^2)^{\delta/2}$$

with c_1 *small enough if* $\delta = 1$.

Theorem 4.2. *Suppose that Assumptions* (D) *and* (F) *hold.*

(i) *If* $N(I - A) = \{0\}$, *then the generalized boundary value problem* (V, h, ϕ) *for Equation* (34) *is feebly approximation-solvable with respect to a projectionally complete scheme* Γ_0 *for the pair* (V, V) *for each* $f \in L_1(a, b)$.

(ii) *If* $N(I - A) \neq \{0\}$ *and Assumption* (E) *holds, then the generalized boundary value problem* (V, h, ϕ) *for* Equation (34) *is feebly approximation-solvable for a given* $f \in L_1(a, b)$ *if and only if*

$$\text{(C}'\text{)} \qquad \int_a^b f\phi_i \, \mathrm{d}x = 0 \quad \textit{for all} \quad i = 1, \ldots, r\,.$$

Proof. Since V is a separable Hilbert space, there exists a projectionally complete scheme Γ_0 for the pair (V, V). Since A and B are compact, the mappings $I - A$ and $I - A - B : V \to V$ are A-proper with respect to Γ_0. Next, by Assumption (F) we get for each $u, v \in V$

$$|(Bu, v)_V| \leqq \sum_{j=0}^{m-1} \int_a^b |B_j(x; u(x) + \phi(x), \ldots, u^{(m-1)}(x) + \phi^{(m-1)}(x))| \,.$$

$$.\, |v^{(j)}(x)| \, \mathrm{d}x + |(h, v)_{W_2^m}| + \left| \sum_{i,j=0}^{m} \int_a^b A_{ij}(x)\, \phi^{(j)}(x)\, v^{(i)}(x) \, \mathrm{d}x \right| \leqq$$

$$\leqq \sum_{j=0}^{m-1} \int_a^b (c_0 + c_1 (\sum_{i=0}^{m-1} |u^{(i)}(x) + \phi^{(i)}(x)|^2)^{\delta/2}\, |v^{(j)}(x)| \, \mathrm{d}x +$$

$$+ |(h, v)_{W_2^m}| + |(L\phi, v)_V|$$

and so

$$\frac{\|Bu\|}{\|u\|_V} = \frac{1}{\|u\|_V} \sup_{v \neq 0} \frac{|(Bu, v)_V|}{\|v\|_V} \leqq \tilde{c}$$

for all large $\|u\|_V$ with $\tilde{c} \leqq k_1$ since c_1 is sufficiently small when $\delta = 1$. Thus,

$$\limsup_{\|u\|_V \to \infty} \frac{\|Bu\|}{\|u\|} \leqq \tilde{c}$$

when $\delta = 1$ and $\tilde{c} = 0$ when $\delta < 1$. This shows that the mappings $I - A$ and $I - A - B$ satisfy condition (5) of Theorem 1.1. Next, we show in case (b) that $R(I - A - B) \subset N(I - A^*)^\perp = R(I - A)$. To that end, we need only show that $R(B) \subset N(I - A^*)^\perp$, i.e. that $(\phi_i, Bu) = 0$ for each $u \in V$ and $i = 1, \ldots, r$. But, this follows from Assumption (E). Finally, since A is compact, $\dim N(I - A) = \operatorname{codim} R(I - A)$, and condition (C′) is equivalent to $w_f \in R(I - A)$, the theorem follows from Theorem 1.1. ∎

The existential assertion of Theorem 4.2. (i) was proved in Fučík [6].

3. BVP for nolinear second-order ordinary differential equations on unbounded domain. Let us now briefly indicate how the results of Section 3 can be applied to some boundary value problems on unbounded domain. The global bifurcation phenomenon for such problems has been studied by many authors (see [36] and the references therein).

Let $Q = (0, \infty)$, $L_2 = L_2(Q)$, $W_2^m = W_2^m(Q)$ and $C_0^\infty(Q)$ be the family of infinitely differentiable functions with compact support in Q and $\mathring{W}_2^m$ be the completion of $C_0^\infty(Q)$ in W_2^m.

Consider the boundary value problem

$$(36) \qquad -(p(x)\,u')' + q(x)\,u - f(x, u) + g(x, u) = h(x)\,(x \in Q,\, u(0) = 0)\,,$$

where $h \in L_2$. Suppose that p and g satisfy the usual conditions ([cf. 36]):

(a_1) *$p : [0, \infty) \to R$ is continuous, $p \in C^1(Q)$, p' is bounded and* $0 < P_1 \leqq p(x) \leqq \leqq P_2$ *for all* $x \geqq 0$.

(a_2) *$q : [0, \infty) \to R$ is continuous with* $\liminf_{x\to\infty} q(x) \geqq \gamma > 0$ *and* $0 < Q_1 \leqq \leqq g(x) \leqq Q_2$ *for all* $x \geqq 0$.

Conditions on f and g depend on the abstract results of Section 3 to be used. For later use we state some of them.

(b_1) *$f : [0, \infty) \times R \to R$ is continuous,* $|f(x, \xi_1) - f(x, \xi_2)| \leqq k|\xi_1 - \xi_2|$ *for all $\xi_1, \xi_2 \in R$ and some $k > 0$ with $f(x, 0) \in L_2$.*

Suppose that $g(x, u) = g_1(x, u) + g_2(x, u)$ with

(b_2) *$g_1 : [0, \infty) \times R \to R$ is continuous and*

$$(g_1(x, \xi_1) - g_1(x, \xi_2))\,(\xi_1 - \xi_2) \geqq 0 \quad \text{for all} \quad \xi_1, \xi_2 \in R\,.$$

(b_3) *$g_2 : [0, \infty) \times R \to R$ is continuous and the mapping $N_2 : \mathring{W}_2^1 \to L_2$ given by $N_2(u)(x) = g_2(x, u(x))$ is continuous and compact.*

(b_4) $|g_1(x, \xi)| \leqq a(x) + \alpha|\xi|_x$ *for all* $x \geqq 0$, *all* $\xi_1, \xi_2 \in R$, *and some* $a \in L_2$, $\alpha > 0$ *and* $\beta \in (0, 1]$.

Define $A_1(u)(x) = -(p(x)\,u')' + q(x)\,u(x)$ for $u \in D(A_1) = C_0^\infty(Q)$. By conditions ($a_1$) and ($a_2$) and the results in [36], A_1 has a unique self-adjoint extenion, call it A, in L_2 with $D(A) = \mathring{W}_2^1 \cap W_2^2$, A has a bounded inverse $A^{-1} : L_2 \to L_2$ which is self-adjoint, positive and γ^{-1}-set-contractive. Furthermore, $A^{-1/2} : L_2 \to L_2$ is $\gamma^{-1/2}$-set-contractive and $A^{1/2}$ is a homeomorphism of $D(A^{1/2}) = \mathring{W}_2^1$ onto L_2. As shown in [36], each $u \in \mathring{W}_2^1$ is continuous on $[0, \infty)$, $u(x) \to 0$ as $x \to \infty$, $\max_{x\geqq 0} |u(x)| \leqq \leqq \|u\|_{1,2}$ and $\|u\|_p \leqq \|u\|_{1,2}$ for all $p \geqq 2$. Define $\|u\|_0 = \|A^{1/2}u\|_2$ for $u \in \mathring{W}_2^1$ and denote by H_0 the space $\mathring{W}_2^1$ normed by the equivalent norm $\|\cdot\|_0$.

For each $u : [0, \infty) \to R$, define $B(u)(x) = f(x, u(x))$ and $N_1(u)(x) = g_1(x, u(x))$, $x \geqq 0$. In view of assumption (b_1) we have that $B : L_2 \to L_2$ satisfies $\|Bu - Bv\|_2 \leqq \leqq k\|u - v\|_2$ for all $u, v \in L_2$, and therefore, is $k - \phi$-contractive. Moreover, (b_2) and (b_4) imply that N_1 is continuous, bounded and monotone from L_2 into L_2 with $\|N_1 u\|_2 \leqq \|a\|_2 + \alpha\|u\|_2^\beta$ for all u in L_2. Since $\mathring{W}_2^1$ is continuously imbedded into L_2, the mapping $N_1 : \mathring{W}_2^1 \to L_2$ is also bounded, continuous and monotone.

Definition 4.2. A function $u \in \mathring{W}_2^1$ is called *a weak solution* of Equation (36) if

$$\int_0^\infty \{p(x)\, u'(x)\, v'(x) + q(x)\, u(x)\, v(x)\}\, dx - \int_0^\infty f(x, u(x))\, v(x)\, dx +$$

$$+ \int_0^\infty g(x, u(x))\, v(x)\, dx = \int_0^\infty h(x)\, v(x)\, dx \quad \text{for all} \quad v \in C_0^\infty(0, \infty)\,.$$

By the above discussion for any solution y in L_2 of the corresponding operator equation

$$(37) \qquad y - A^{-1/2}BA^{-1/2}y + A^{-1/2}NA^{-1/2}y = A^{-1/2}(h)\,,$$

the function $u = A^{-1/2}y$ is a weak solution of Equation (36).

Theorem 4.3. *Suppose that conditions* (a_1), (a_2), (b_1), (b_2) *and* (b_3) *hold. Then* Equation (37) *is solvable in* L_2 *if either one of the following conditions holds*:

(i) $k \max\{\|A^{-1/2}\|^2, \gamma^{-1}\} < 1$, $N_1 : L_2 \to L_2$ *is continuous and bounded and*

$$\limsup_{\|u\|_2 \to \infty} \frac{\|N_2 u\|_2}{\|u\|_2} = 0\,.$$

(ii) f *is odd and* 1-*homogeneous* (i.e., $f(x, -\xi) = -f(x, \xi)$ *and* $f(x, t\xi) = tf(x, \xi)$ *for all* $\xi \in R$, $t > 0$ *and* $x \geqq 0$), $k\gamma^{-1} < 1$, *condition* (b_4) *holds with* α *sufficiently small if* $\beta = 1$ *and* $A_1(u) - f(x, u) \neq 0$ *for* $u \in D(A_1)$, $u \neq 0$.

(iii) $N : L_2 \to L_2$ *is continuous and bounded*, $k\gamma^{-1} < 1$ *and*

$$\limsup_{\|v\|_0 \to \infty} \frac{\|Bv - Nv\|_2}{\|v\|_0} < 1/\|A^{-1/2}\|\,.$$

Proof. (i) Since $\|Lu\|_2 \leqq k\|A^{-1/2}\|^2\, \|u\|_2 + \|A^{-1/2}\|\, \|F(0)\|_2$, and

$$\limsup_{\|u\| \to \infty} \frac{\|Lu - M_2 u\|_2}{\|u\|_2} < 1\,,$$

the conclusion follows from Theorem 3.3.

(ii) Since B is odd and 1-homogeneous with $Au \neq Bu$ for $0 \neq u \in D(A)$,Theorem 3.1 (ii) is applicable.

(iii) Since, setting $x = A^{1/2}v$,

$$\limsup_{\|u\|_2 \to \infty} \frac{\|Lu - Nu\|_2}{\|u\|_2} \leqq \|A^{-1/2}\| \limsup_{\|A^{1/2}v\| \to \infty} \frac{\|Bv - Nv\|_2}{\|v\|_0} < 1\,,$$

Theorem 3.1 (iii) is applicable. ∎

In a similar fashion, using Corollary 3.1, we have

Theorem 4.4. (Fredholm alternative). *Suppose that conditions* (a_1), (a_2) *and* $(b_2) - (b_4)$ *hold with* α *sufficiently small if* $\beta = 1$. *Let* B *be linear with* $\|B\| \max . \{1, \gamma^{-1}\} < 1$. *Then, either* $N(A - B) = \{0\}$, *in which case* Equation (37) *is solvable in* $D(A)$ *for each* h *in* L_2 *or* $N(A - B) \neq \{0\}$. *In the latter case, assuming additionally that* $R(N) \subset R(A - B)$, Equation (37) *is solvable if and only if* $f \in [N(A - B^*)]^{\perp}$.

In view of Corollary 3.2, a similar alternative holds when N is linear.

Note added in proof. Theorem 2.4 is valid for A-proper like mappings. Namely, let $K : X \to Y^*$ with $\|Kx\| \to \infty$ as $\|x\| \to \infty$, $G : X \to Y$ bounded, $(Gx, Kx) = \|Gx\| \|Kx\|$ for $x \in X$ and $Gx \neq 0$ for $\|x\|$ large.

Theorem. *Let* $T : X \to Y$ *be such that* $\|Tx\| + (Tx, Kx)/\|Kx\| \to \infty$ *as* $\|x\| \to \infty$ *and either* (i) $\deg(\mu W_n G V_n, V_n^{-1}(B(0, r), 0) \neq 0$ *for all large* $r > 0$ *with* $\mu = 1$ *if* T *is* A*-proper and* $\mu \in (0, \beta)$ *for some* β *small when* $T + \mu G$ *is* A*-proper, or* (ii) *there are* $K_n : V_n(E_n) \to F_n^*$ *and a linear isomorphism* $M_n : E_n \to F_n$ *such that* $(W_n y, K_n V_n u) = (y, K V_n u)$ *for* $u \in E_n$, $y \in Y$ *and* $(M_n u, K_n V_n u) >$ *for* $0 \neq u \in E_n$. *Then,*

(a) *Equation* $Tx = f$ *is f.a. solvable for each* $f \in Y$ *if, in addition,* $H(t, x) = t\,T(x) + (1 - t)\,Gx$ *is an* A*-proper homotopy on* $[0, 1] \times X$ (cf. Notices Amer. Math. Soc. January 1977, 77T-B27, and [21]);

(b) $T(X) = Y$ *if, in addition,* $H_\mu(t, x) = t\,T(x) + \mu\,Gx$ *is an* A*-proper homotopy at* 0 *on* $[0, 1] \times X \setminus B(0, R_0)$ *for some* R_0 *large and all* $\mu \in (0, \beta)$ *and* T *either satisfies condition* $(++)$ *or is pseudo* A*-proper* (cf. Notices AMS, loc. cit. and for details, P. S. Milojević, On the solvability and continuation type results for nonlinear equations with applications. II, to appear).

If G and $T + \mu G$ are A-proper for $\mu \geqq 0$ ($\mu > 0$ resp.) and either T is bounded or $(Tx, Kx) \geqq -c\|Kx\|$ for $\|x\| \geqq R_0$ and some $c > 0$, then $H(t, x)$ ($H_\mu(t, x)$, resp.) is A-proper on $[0, 1] \times X$ (at 0 on $[0, 1] \times X \setminus B(0, R)$, $R \geqq R_0$, resp.); cf. the above cited papers for details. Let us also add that an extensive study of $Ax + Tx = f$ with a linear mapping A either densily defined and closed or Fredholm of index zero can be found in the forthcoming papers of the author: "Approximation solvability of some nonlinear operator equations with applications" (Proc. Intern. Symp. on Funct. Diff. Eq. and Bifurcation, 1979, Lecture Notes in Math. Springer-Verlag, Editor A. Ize, to appear) and "Approximation solvability results for equations involving nonlinear perturbations of Fredholm mappings with applications to differential equations", (Advances in Functional Analysis, Holomorphy and Approximation Theory, Marcel Dekker, New York, Editor G. Zapata, to appear).

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