A. Anjaneyulu

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ON PRIMARY SEMIGROUPS

A. ANJANEYULU, Guntur

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A study on primary semigroups was initiated by M. SATYANARAYANA [5] and his results were extended to semiprimary semigroups by HARBANS LAL [1]. Their study is limited to commutative semigroups only. In this note we extend the notions of primary and semiprimary semigroups to noncommutative semigroups. We introduce left primary, right primary and primary ideals and also a class of semigroups, namely pseudosymmetric semigroups, which includes the classes of commutative, quasi commutative, normal and duo semigroups. In this paper we deal with semigroups (not necessarily commutative) in which every ideal is primary and with semigroups in which every ideal is semiprimary which is a generalization of primary and semiprimary commutative semigroups.

Definition 1. An ideal P of a semigroup S is said to be *pseudosymmetric* provided $xy \in P$ for some $x, y \in S$ implies $xsy \in P$ for all $s \in S$.

Definition 2. A semigroup S is said to be *pseudosymmetric* provided its every ideal is pseudosymmetric.

Remark 1. Every commutative semigroup is a pseudosymmetric semigroup.

Proposition 1. *Every duo semigroup is a pseudosymmetric semigroup.*

Proof. It is known (Theorem 2 and Remark 2 [3]) that a semigroup S is duo if and only if for any $a \in S$, we have $aS \cup a = Sa \cup a$. Let A be any ideal of S and let $xy \in A$. Now $xs \in xS \cup x = Sx \cup x$ for any $s \in S$. So $xs = tx$ where t is an element of S or an empty symbol. Thus $xsy = txy \in A$. Therefore A is a pseudosymmetric ideal. Hence S is a pseudosymmetric semigroup.

Remark 2. Every normal, quasi commutative, left zero and right zero semigroup is pseudosymmetric.

Proposition 2. *Every idempotent semigroup is a pseudosymmetric semigroup.*

Proof. Let A be an ideal in an idempotent semigroup S and let $xy \in A$ for some

$x, y \in S$. Since S is an idempotent semigroup, we have $yx = yxyx \in A$ and also $xsy = xsyxsy \in A$ for all $s \in S$. So A is a pseudosymmetric ideal. Therefore S is a pseudosymmetric semigroup.

Lemma 1. *Every prime ideal in a pseudosymmetric semigroup S is completely prime.*

Proof. Let P be any prime ideal in a pseudosymmetric semigroup S and let $xy \in P$ for some $x, y \in S$. Since S is a pseudosymmetric semigroup, P is a pseudosymmetric ideal. Therefore $xsy \in P$ for all $s \in S$. Hence $(x)(y) \subseteq P$. So either $(x) \subseteq P$ or $(y) \subseteq \subseteq P$. Thus either $x \in P$ or $y \in P$. So P is a completely prime ideal.

Remark 3. If A is an ideal in a pseudosymmetric semigroup S, then $\{x \in S \mid x^n \in A$ for some positive integer $n\}$ is a completely semiprime ideal of S.

Remark 4. If A is an ideal in a pseudosymmetric semigroup S, then $\sqrt{A} = \{x \in S \mid x^n \in A$ for some positive integer $n\}$ by Theorem II.3.7 [2], where $\sqrt{A}$ is the radical of A, that is the intersection of all prime ideals of S containing A.

Lemma 2. *In a pseudosymmetric semigroup S, a maximal ideal M is prime if and only if $M = \sqrt{M}$.*

Proof. If a maximal ideal M of S is a prime ideal, then it is completely prime and hence completely semiprime. So $M = \sqrt{M}$. Conversely if $M = \sqrt{M}$, then by Remark 4, M is a prime ideal of S.

Definition 3. An ideal P of a semigroup S is said to be *left (right) primary* provided $xy \in P$ for some $x, y \in S$ and $y \notin P$ $(x \notin P)$ imply $x^n \in P (y^n \in P)$ for some positive integer n.

Definition 4. An ideal P of a semigroup S is said to be *primary* provided it is both the left and right primary ideal.

In a commutative semigroup left primary, right primary and primary ideals coincide. But in an arbitrary semigroup they are different. This is shown by the following example.

Example 1. Let $S = \{a, b, c\}$ with the multiplication given by the following table.

	a	b	c
a	a	a	a
b	a	a	a
c	a	b	c

Now in S the ideal $\{a\}$ is left primary but not right primary.

Definition 5. An ideal A of a semigroup S is *semiprimary* provided $\sqrt{A}$ is a completely prime ideal.

Definition 6. A semigroup S is said to be (*left, right, semi*)*primary* provided every ideal of S is (left, right, semi) primary.

Lemma 3. *Every left (right) primary ideal of a pseudosymmetric semigroup S is a semiprimary ideal.*

Proof. Let A be any left primary ideal of a pseudosymmetric semigroup S and let $xy \in \sqrt{A}$ for some $x, y \in S$. Now $(xy)^n \in A$ for some n. Let k be the least positive integer such that $(xy)^k \in A$. If $k = 1$, then $xy \in A$. Now if $y \in A$, then $y \in \sqrt{A}$. If $y \notin A$, then since A is a left primary ideal, we have $x^m \in A$ for some positive integer m and hence $x \in \sqrt{A}$. Suppose $k > 1$. Consider $x \,.\, y(xy)^{k-1} = (xy)^k \in A$. If $y(xy)^{k-1} \notin$ $\notin A$, then $x^m \in A$ for some positive integer m and hence $x \in \sqrt{A}$. If $y(xy)^{k-1} \in A$, then since $(xy)^{k-1} \notin A$, we have $y^m \in A$ for some positive integer m and hence $y \in \sqrt{A}$. Thus $\sqrt{A}$ is a completely prime ideal.

Remark 5. In a semiprimary semigroup, every completely semiprime ideal is completely prime.

Remark 6. A pseudosymmetric semigroup S is semiprimary if and only if every ideal A of S satisfies the following condition: $xy \in A$ implies either $x^n \in A$ or $y^n \in A$ for some positive integer n.

Lemma 4. *Let A be an ideal in a pseudosymmetric semigroup S. If $\sqrt{A}$ is a maximal ideal of S, then A is a semiprimary ideal.*

The proof of this lemma is easy and will be omitted.

Lemma 5. *Let A be an ideal in a duo semigroup with identity. If $\sqrt{A} = M$, where M is a maximal ideal, then A is a primary ideal.*

Proof. S has a unique maximal ideal, which is the union of all proper ideals. Hence M is the unique maximal ideal and every element of S not belonging to M is a unit. Let $xy \in A$ and $x \notin A$. If no power of y is in A, then $y \notin \sqrt{A} = M$. Thus y has an inverse, say y^{-1}. Now $x = xy \,.\, y^{-1} \in A$, a contradiction. So A is a right primary ideal. Similarly we can show that A is a left primary ideal and hence A is a primary ideal.

Lemma 6. *In a pseudosymmetric semiprimary semigroup S, the prime ideals of S form a chain under the set inclusion.*

Proof. Let A and B be two prime ideals of S. Now $A \cap B = \sqrt{A} \cap \sqrt{B} =$ $= \sqrt{(A \cap B)}$. Thus $A \cap B$ is a completely semiprime ideal and hence by Remark 5, $A \cap B$ is a completely prime ideal. Suppose $A \nsubseteq B$ and $B \nsubseteq A$. Then there exist elements a and b in S such that $a \in A \setminus B$ and $b \in B \setminus A$. Now $ab \in A \cap B$ and $a, b \notin A \cap B$, a contradiction. So either $A \subseteq B$ or $B \subseteq A$.

Theorem 1. *In a duo semiprimary semigroup S,*

(i) *Prime ideals form a chain under the set inclusion.*

(ii) *Idempotents form a chain under the natural ordering.*

Proof. By Lemma 6, (i) is trivial. For (ii), let e and f be two idempotents of S. Now $\sqrt{(e)}$ and $\sqrt{(f)}$ are prime ideals by Remark 6. By (i), either $\sqrt{(e)} \subseteq \sqrt{(f)}$ or $\sqrt{(f)} \subseteq \sqrt{(e)}$. Suppose $\sqrt{(e)} \subseteq \sqrt{(f)}$. Now $e \in \sqrt{(e)} \subseteq \sqrt{(f)}$. This implies $e \in (f)$. Since S is a duo semigroup and f is an idempotent, $(f) = fS = Sf$. So $e = fs = tf$ for some $s, t \in S$. Thus $ef = fe = e$. The second case can be proved similarly. This completes the proof of the theorem.

Theorem 2. *Let S be a regular pseudosymmetric semigroup. Then the following statements are equivalent*:

(i) *Every ideal in S is prime.*

(ii) *S is a primary semigroup.*

(iii) *S is a left primary semigroup.*

(iv) *S is a right primary semigroup.*

(v) *S is a semiprimary semigroup.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (v) and (ii) $\Rightarrow$ (iv) $\Rightarrow$ (v) are clear. For (v) $\Rightarrow$ (i), suppose S is a semiprimary semigroup. Let A be an ideal of S. Since S is semiprimary, A is a semiprimary ideal. Suppose $x^2 \in A$. Since S is regular, there exists an element $a \in S$ such that $xax = x$. Since $x^2 \in A$, we have $xax \in A$ and hence $x \in A$. So A is a completely semiprime ideal. Since S is a semiprimary semigroup, A is completely prime by Remark 5. Thus (v) implies (i).

Remark 7. In any semigroup S, the following statements are equivalent:

(i) Principal ideals of S are linearly ordered.

(ii) The ideals of S are linearly ordered.

Theorem 3. *Let S be a regular duo semigroup. Then the following statements are equivalent*:

(i) *Every ideal in S is prime.*

(ii) *S is a primary semigroup.*

(iii) *S is a left primary semigroup.*

(iv) *S is a right primary semigroup.*

(v) *S is a semiprimary semigroup.*

(vi) *The idempotents form a chain under the natural ordering.*

(vii) *Principal ideals of S are linearly ordered.*

(viii) *The ideals of S are linearly ordered.*

Proof. We prove that (vi) implies (vii) and (viii) implies (i). Assume (vi). Let (a) and (b) be two principal ideals of S. Since S is a regular semigroup, $(a) = (e)$ and $(b) = (f)$ for some idempotents e and f in S. By (vi), either $ef = fe = e$ or $ef = fe = f$. Thus either $(e) \subseteq (f)$ or $(f) \subseteq (e)$. So either $(a) \subseteq (b)$ or $(b) \subseteq (a)$.

Assume (viii). Let A be any ideal of S and let $xy \in A$. By (viii), either $(x) \subseteq (y)$ or $(y) \subseteq (x)$. Suppose $(x) \subseteq (y)$. Now $x \in (y) = yS \cup y = Sy \cup y$. So $x = ys = ty$ where s and t are elements of S or empty symbols. Therefore $x^2 = xys \in A$. Since S is regular, there is an element $a \in S$ such that $x = xax$. Since $x^2 \in A$, we have $x = xax \in A$. Similarly, if $(y) \subseteq (x)$, then we can show that $y \in A$. So A is a prime ideal. Theorem 2, Theorem 1 and Remark 7 complete the proof of the theorem.

In the following theorem we introduce a wide class of (semi) primary semigroups.

Theorem 4. *Let S be a (pseudosymmetric) duo semigroup with identity. If prime ideals are maximal, then S is a (semi) primary semigroup.*

The proof of this theorem is similar to that of Theorem 2.5 in [5].

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Author's address: Department of Mathematics, Nagarjuna University, Nagarjuna Nagar — 522510, A.P. India.