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A SET FUNCTION WITHOUT σ-ADDITIVE EXTENSION HAVING FINITELY ADDITIVE EXTENSIONS ARBITRARILY CLOSE TO σ-ADDITIVITY

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In [1, Satz 1.13] and its preceding proof we have shown the following decomposition theorem for (positive, finitely additive) measures[1]) with respect to their extensibility to σ-additive measures:

Theorem 1. *Let $\mathfrak{R}$ and $\mathfrak{S}$ be rings of subsets of a set X with $\mathfrak{S} \subset \mathfrak{R}$ and let μ be a finite measure on $\mathfrak{S}$.*

Then μ has a unique decomposition $\mu = \mu_\sigma + \mu_\varphi$ such that μ_σ is the restriction of a σ-additive measure on $\mathfrak{R}$ whereas μ_φ is a measure on $\mathfrak{S}$ having only purely finitely additive extensions to $\mathfrak{R}$ [2]). Moreover, μ_σ is the largest measure on $\mathfrak{S}$ dominated by μ which admits an extension to a σ-additive measure on $\mathfrak{R}$.

As a consequence of Theorem 1 we get immediately:

Corollary. *In the situation of the Theorem, suppose that for arbitrary $\varepsilon > 0$ there is a measure $\sigma_\varepsilon \leqq \mu$ on $\mathfrak{S}$ admitting a σ-additive extension to $\mathfrak{R}$ such that*

$$\sigma_\varepsilon(S) + \varepsilon \geqq \mu(S) \quad \textit{for every} \quad S \in \mathfrak{S} .$$

Then μ is the restriction of a σ-additive measure on $\mathfrak{R}$.

In this note we want to investigate whether the conclusions of Theorem 1 and its Corollary remain valid if $\mathfrak{S}$ is an arbitrary system of sets contained in the ring $\mathfrak{R}$ and μ is a finite premeasure on $\mathfrak{S}$ in the sense of the following definition.

Definition. Let $\mathfrak{S}$ be a system of subsets of a set X. A mapping $\mu : \mathfrak{S} \to [0, \infty]$ is called a *premeasure* if μ admits an extension to a measure on the ring generated by $\mathfrak{S}$.

[1]) By a *measure* μ we mean in this note a positive, finitely additive, extended real-valued set function on a ring with $\mu(\emptyset) = 0$.

[2]) A measure ν on $\mathfrak{R}$ is called *purely finitely additive*, if the only σ-additive measure on $\mathfrak{R}$ dominated by ν is the measure identical 0.

Remark. If $\mathfrak{S}$ is closed under finite intersections, then, for any finite premeasure μ on $\mathfrak{S}$, there is only one extension of μ to the ring generated by $\mathfrak{S}$. Hence Theorem 1 and its Corollary may be generalized to this situation.

If $\mathfrak{S}$ is not closed under finite intersections, one may have the suspicion that the generalization of Theorem 1 to this situation does not hold (Theorem 1 is a band type decomposition theorem, whereas there is no lattice structure assumed for $\mathfrak{S}$), but one is tempted to believe that at least the Corollary is also true for premeasures instead of measures. However, the following result shows that this conjecture is wrong even in a nice measure-theoretic setting.

Theorem 2. *There are a compact metrizable space X, a system $\mathfrak{S}$ of closed subsets of X generating the Borel σ-algebra $\mathfrak{R} = \mathfrak{B}(X)$ on X with $X \in \mathfrak{S}$, and a finite premeasure μ on $\mathfrak{S}$ having the following properties*:

(1) *μ does not admit an extension to a σ-additive measure on $\mathfrak{R}$;*

(2) *for arbitrary $\varepsilon > 0$ there are premeasures σ_ε and φ_ε on $\mathfrak{S}$ with*

$$\mu = \sigma_\varepsilon + \varphi_\varepsilon \quad \textit{and} \quad 0 < \varphi_\varepsilon(X) \leqq \varepsilon$$

such that σ_ε is the restriction of a σ-additive measure on $\mathfrak{R}$, whereas each measure on $\mathfrak{R}$ extending φ_ε is purely finitely additive.

The example which will prove Theorem 2 is a modification of [2, Beispiel 4.11]. In order to make it more comprehensible, let us first point out some special technical features of its construction:

(i) The space X to be constructed contains a countable subset $A = \{\alpha_1, \alpha_2, \dots\}$ which belongs to $\mathfrak{S}$, whereas all proper subsets of A belong to $\mathfrak{R} \setminus \mathfrak{S}$.

(ii) Although the premeasure μ of the example, of course, does not have a unique decomposition in the sense of Theorem 1, every premeasure μ' on $\mathfrak{S}' = \mathfrak{S} \cup \mathfrak{P}(A)$ ($\mathfrak{P}(A)$ denoting the power set of A) which extends μ has this property.

(iii) There is another countable set $B \in \mathfrak{R} \setminus \mathfrak{S}$ disjoint to A on which only purely finitely additive parts of the extensions of μ are concentrated.

(iv) For any measure ν on $\mathfrak{R}$ extending μ the value $\nu(B)$ is uniquely determined by the restriction of ν to $\mathfrak{P}(A)$. This uniqueness is made possible by a further set Γ and its subsets which controll the interaction between A and B. More explicitely, for any extension ν of μ to $\mathfrak{R}$ and arbitrary $k \in \mathbb{N}$ the discrete (and therefore σ-additive) mass $\nu(\{\alpha_k\})$ is "reduced" to the purely finitely additive mass $\nu(B_k) = 2^{-k} \cdot \nu(\{\alpha_k\})$ of a certain subset $B_k \subset B$ corresponding to $\{\alpha_k\} \subset A$.

Due to this reduction of mass as well as the sufficiently rich choice of extensions of μ to premeasures on $\mathfrak{S}'$, properties (1) and (2) of Theorem 2 can be satisfied.

(v) Finally a single point δ is introduced which enables us to choose $X \in \mathfrak{S}$ and which takes care of the compactness of the topology to be defined on X.

Now let us turn to the example in detail:

Example (cf. the diagram below and the list of values for the given set functions on page 379).

Let $X = A \cup B \cup \Gamma \cup \Delta$ be the union of four disjoint subsets A, B, Γ, Δ, where $\Delta = \{\delta\}$ is a one-point set, $A = \{\alpha_k : k \in \mathbb{N}\}$ and $B = \{\beta_{kj} : k, j \in \mathbb{N}\}$ are countable, and $\Gamma = \prod_{k=1}^{\infty} \Gamma_k$ is the countable cartesian product of finite sets $\Gamma_k = \{\gamma_{k0}, \gamma_{k1}, \dots, \gamma_{k2^k}\}$ $(k \in \mathbb{N})$. In addition we introduce the sets $B_k = \{\beta_{kj} : j \in \mathbb{N}\} \subset B$ $(k \in \mathbb{N})$ and $G_{kj} = p_k^{-1}(\gamma_{kj}) \subset \Gamma$ $(k \in \mathbb{N},\ j = 0, 1, \dots, 2^k)$, where $p_k : \Gamma \to \Gamma_k$ denotes the projection mapping.

We endow the countable sets A and B_k $(k \in \mathbb{N})$ with some compact Hausdorff topology. Then we introduce the direct sum topology on $B = \bigcup_{k=1}^{\infty} B_k$ and topologize $B \cup \Delta$ such that it becomes the Alexandroff compactification of B. On Γ we define the product of the discrete topologies on the finite sets Γ_k $(k \in \mathbb{N})$. Finally we endow $X = A \cup \Gamma \cup (B \cup \Delta)$ with the direct sum of the compact topologies defined above. Hence X is a metrizable compact space and its subsets $A, \Gamma, B_k, G_{kj}, \Gamma \setminus G_{k0} = \bigcup_{i=1}^{2^k} G_{ki}$ $(k \in \mathbb{N}, j = 0, \dots, 2^k)$ are compact and therefore closed.

Now it is obvious that

$$\mathfrak{S} = \{X, A, \Gamma\} \cup \{\{\beta_{kj}\} : k, j \in \mathbb{N}\} \cup$$
$$\cup \{\{\alpha_k\} \cup (\Gamma \setminus G_{k0}) : k \in \mathbb{N}\} \cup \{B_k \cup G_{kj} : k \in \mathbb{N},\ j = 1, \dots, 2^k\}$$

is a system of closed subsets of X.

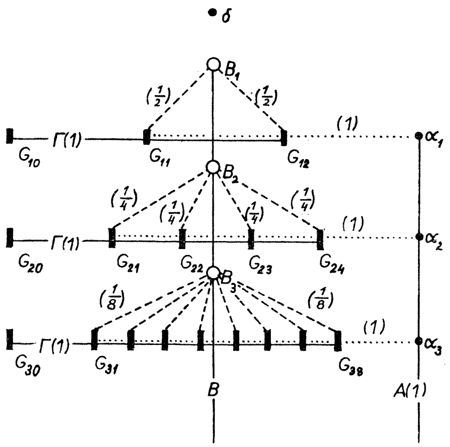

Diagram of X with given values for μ (in brackets). Note that the set Γ appears several times, by the identities $\Gamma = \bigcup_{j=0}^{2^k} G_{kj}$ $(k \in \mathbb{N})$.

	X	A, Γ	$\{\alpha_k\} \cup (\Gamma \setminus G_{k0})$ $(k \in \mathbb{N})$	$B_k \cup G_{kj}\,(j = 1, \ldots, 2^k)$		$\{\beta_{kj}\}$ $(k, j \in \mathbb{N})$
				$k = k_\varepsilon$	$k \in \mathbb{N} \setminus \{k_\varepsilon\}$	
μ	3	1	1	2^{-k_ε}	2^{-k}	0
σ_ε	$3 - 2^{-k_\varepsilon}$	1	1	0	2^{-k}	0
φ_ε	2^{-k_ε}	0	0	2^{-k_ε}	0	0

List of values for the premeasures μ, σ_ε, φ_ε $(\varepsilon > 0)$ on $\mathfrak{S}$.

We define $\mu : \mathfrak{S} \to \mathbb{R}_+$ by

$\mu(X) = 3$,

$\mu(S) = 1$ for $S \in \{A, \Gamma\} \cup \{\{\alpha_k\} \cup (\Gamma \setminus G_{k0}) : k \in \mathbb{N}\}$,

$\mu(S) = 2^{-k}$ for $S \in \{B_k \cup G_{kj} : k \in \mathbb{N},\ j = 1, \ldots, 2^k\}$,

$\mu(S) = 0$ for $S \in \{\{\beta_{kj}\} : k,\ j \in \mathbb{N}\}$.

Let us prove that $\mathfrak{S}$ and μ have the required properties:

(a) Obviously, *$X \in \mathfrak{S}$ and $\mathfrak{S}$ is a system of closed Borel subsets of X.*

(b) *$\mathfrak{R}$ is the σ-algebra generated by $\mathfrak{S}$*: For $k \in \mathbb{N}$ we have $\{\alpha_k\} = A \cap (\{\alpha_k\} \cup (\Gamma \setminus G_{k0}))$. Therefore, the σ-algebra $\mathfrak{A}$ generated by $\mathfrak{S}$ contains all one-point subsets of the countable set $A \cup B \cup \Delta$ and hence the power set $\mathfrak{P}(A \cup B \cup \Delta)$. Similarly, also $G_{kj} = \Gamma \cap (B_k \cup G_{kj}) \in \mathfrak{A}$ $(k \in \mathbb{N},\ j = 1, \ldots, 2^k)$ and $G_{k0} = \Gamma \setminus \bigcup_{j=1}^{2^k} G_{kj} \in \mathfrak{A}$ $(k \in \mathbb{N})$. Hence $\mathfrak{A}$ contains the (relatively) open subsets of the countable-product space Γ. From this and the definition of the topology on X we get $\mathfrak{R} \subset \mathfrak{A}$, hence, by (a), $\mathfrak{R} = \mathfrak{A}$.

Now we turn to the properties of μ:

(c) *If ν is any measure on $\mathfrak{R}$ extending μ, then the restriction of ν to the Borel subsets of B is purely finitely additive*: Indeed, B is countable and $\nu(\{\beta\}) = \mu(\{\beta\}) = 0$ for all $\beta \in B$.

(d) *If ν is any measure on $\mathfrak{R}$ extending μ, then*

$$\nu(B_k) = 2^{-k}\,\nu(\{\alpha_k\})\ (k \in \mathbb{N}):$$

For $k \in \mathbb{N}$ and $j = 1, \ldots, 2^k$ we have

$$\nu(\Gamma \setminus G_{k0}) = \sum_{j=1}^{2^k} \nu(G_{kj}) = \sum_{j=1}^{2^k} (\nu(B_k \cup G_{kj}) - \nu(B_k)) = 1 - 2^k\,\nu(B_k)\,.$$

On the other hand,

$$\nu(\Gamma \setminus G_{k0}) = \nu(\{\alpha_k\} \cup (\Gamma \setminus G_{k0})) - \nu(\{\alpha_k\}) = 1 - \nu(\{\alpha_k\}) \quad \text{for every} \quad k \in \mathbb{N},$$

hence

$$\nu(\{\alpha_k\}) = 2^k \, \nu(B_k).$$

So far we have proved properties of measures on $\mathfrak{R}$ extending μ without having discussed the existence of such a measure. Now we shall prove that μ is in fact a premeasure.

(e) *For every sequence* $\varrho = (\varrho_k)_{k \in N}$ *of positive real numbers with* $\sum_{k=1}^{\infty} \varrho_k = 1$, *there is a measure* ν_ϱ *on* $\mathfrak{R}$ *extending* μ *such that* $\nu_\varrho(\{\alpha_k\}) = \varrho_k$ $(k \in \mathbb{N})$ *and such that the purely finitely additive part of* ν_ϱ *is concentrated on* B: Define π_ϱ on $\Gamma \cap \mathfrak{R}$ (the Borel subsets of Γ) to be the product of the normed discrete measures

$$\varrho_k \varepsilon_{\gamma_{k0}} + \sum_{j=1}^{2^k} 2^{-k}(1 - \varrho_k)\, \varepsilon_{\gamma_{kj}} \quad (k \in \mathbb{N})$$

on Γ_k. Moreover, let τ_k $(k \in \mathbb{N})$ be some normed measure on $B_k \cap \mathfrak{R}$ such that $\tau_k(\{\beta_{kj}\}) = 0$ for all $j \in \mathbb{N}$ (each such τ_k is obviously purely finitely additive). Then it is easy to check that

$$\nu_\varrho : R \mapsto \sum_{k=1}^{\infty} \varrho_k \, \varepsilon_{\alpha_k}(R) + \sum_{k=1}^{\infty} 2^{-k} \varrho_k \tau_k(B_k \cap R) + \pi_\varrho(\Gamma \cap R) + \left(1 - \sum_{k=1}^{\infty} 2^{-k} \varrho_k\right) \varepsilon_\delta(R)$$

s a measure on $\mathfrak{R}$ extending μ with $\nu_\varrho(\{\alpha_k\}) = \varrho_k$ $(k \in \mathbb{N})$. Obviously, ν_ϱ is σ-additive on $\complement B$ and purely finitely additive on B.

Now it is not difficult to show that μ has the required properties (1) and (2) of Theorem 2:

(1) Suppose, ν is a σ-additive extension of μ to $\mathfrak{R}$. Then, by (c), $\nu(B) = 0$, hence, by (d), $\nu(\{\alpha_k\}) = 0$ for every $k \in \mathbb{N}$. Therefore, the σ-additivity of ν implies $\nu(A) = 0$ which contradicts $\nu(A) = \mu(A) = 1$. Therefore, μ is not the restriction of a σ-additive measure on $\mathfrak{R}$.

(2) For $\varepsilon > 0$ choose $k_\varepsilon \in \mathbb{N}$ such that $\varepsilon \geqq 2^{-k_\varepsilon}$ and let $\varrho^\varepsilon = (\varrho_k^\varepsilon)_{k \in N}$ be the sequence with $\varrho_k^\varepsilon = 1$ for $k = k_\varepsilon$ and $\varrho_k^\varepsilon = 0$ for $k \neq k_\varepsilon$. Then, by (e), the extension $\nu_{\varrho^\varepsilon}$ of μ to $\mathfrak{R}$ is defined. Let σ_ε and φ_ε be the restrictions to $\mathfrak{S}$ of $1_{\complement B} \nu_{\varrho^\varepsilon}$ and $1_B \nu_{\varrho^\varepsilon}$, respectively ($1_{\complement B}$ and 1_B denoting the characteristic functions of $\complement B$ and B).

Evidently, σ_ε and φ_ε are premeasures and σ_ε is the restriction of a σ-additive measure on $\mathfrak{R}$. Moreover,

$$\varphi_\varepsilon(X) = \nu_{\varrho^\varepsilon}(B) = \nu_{\varrho^\varepsilon}(B_{k_\varepsilon}) = \varphi_\varepsilon(B_{k_\varepsilon} \cup G_{k_\varepsilon j}) = 2^{-k_\varepsilon} \leqq \varepsilon \quad (j = 1, \ldots, 2^{k_\varepsilon})$$

and

$$\varphi_\varepsilon(S) = 0 \quad \text{for} \quad S \in \mathfrak{S} \setminus \{X, B_{k_\varepsilon} \cup G_{k_\varepsilon j} : j = 1, \ldots, 2^k\}.$$

If λ is any measure on $\mathfrak{R}$ extending φ_ε, then $\lambda(\Gamma) = \varphi_\varepsilon(\Gamma) = 0$, hence, by the monotonicity of λ, $\lambda(G_{k_\varepsilon 1}) = 0$. This implies

$$\lambda(B_{k_\varepsilon}) = \varphi_\varepsilon(B_{k_\varepsilon} \cup G_{k_\varepsilon 1}) = 2^{-k_\varepsilon} = \varphi_\varepsilon(X) = \lambda(X)\,.$$

Hence λ is concentrated on $B_{k_\varepsilon} \subset B$ and therefore purely finitely additive, as the one-point subsets of B have measure 0.

Remarks. (1) In Theorem 2 one may replace the Borel σ-algebra $\mathfrak{R} = \mathfrak{B}(X)$ by the algebra generated by $\mathfrak{S}$. This follows immediately from the above example or from the standard extension theorem for σ-additive measures on rings.

(2) By a modification of the topology on X (replacing the compact topology defined on $A \cup B \cup \Delta = X \setminus \Gamma$ by the discrete topology), it is possible to make $\mathfrak{S}$ a system of closed-open subsets of the second countable locally compact space X. From this modification one can easily get a proof of [2, Bemerkung 6.10] by transforming the extension problem for set functions to measures into an existence problem for simultaneous preimage measures (using methods closely related to those described in [2, Bemerkung 1.7]).

However, it follows from [2, Korollar 6.5 or Satz 6.6] (by a similar argument) that in Theorem 2 it is impossible to require $\mathfrak{S}$ to be a system of closed-open sets and at the same time X to be compact and metrizable.

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