Marco Antonio Teixeira

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ON LOCAL FIRST ORDER STRUCTURAL STABILITY OF PAIRINGS OF VECTOR FIELDS AND FUNCTIONS

MARCO ANTONIO TEIXEIRA, Campinas

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INTRODUCTION

In this work we shall study manifolds on which both a vector field and a real valued function are defined.

Consider now a foliation F on a manifold M. We say that two vector fields X and Y are F-equivalent if there is a homeomorphism $h : M \to M$ which is an equivalence between X and Y and preserves the leaves of F. A variation of the case codim $(F) = 1$ is to consider field-functions on M, that is, pairs (X, f) where X is a vector field and f is a real-valued function (both defined on M) and defining (X, f) to be equivalent to (Y, g) if there is a homeomorphism $h : M \to M$ which sends trajectories of X to trajectories of Y and level curves of f to level curves of g.

The concept of structural stability of (X, f) was introduced in [4]. In section 3 of that paper an example was given of a field-function (X, f) where f is the height function, which is structurally stable. In an idealized situation the magnetic field of the earth can be thought as a particular case of this example and the level curves of f turn out to be the parallels. As in the example the geographic poles do not coincide with the magnetic poles and it seems clear that the last situation is a stable phenomenon.

We restrict ourselves to the case of two dimensional manifolds.

We are interested in studying the set of points $p \in M$ where the trajectory of X and the level curve of f have non tranversal contact; such points are the singularities of (X, f). We find similar singularities when one studies the so called *Σ-gradient* vector fields, which are special vector fields on a compact Riemmanian manifold obtained as orthogonal projections of gradient vector fields on a given distribution of contact elements. The singularities of such fields in certain cases are the singularities of a class of vector fields coming from the mechanics named Appel fields (see more details in [5]).

The singularities of (X, f) were generically classified in [4]. The main aim of this paper is to classify the codimension-one singularities of a field-function; i.e. those singularities which generically occur in one parameter families of field-functions. The classification is done in terms of the local structural stability definition of a field-function. We shall show that there exist ten different topological types of codimension-one singularities. The unfolding of each one of these singularities is given.

In Section 1 we give some definitions and preliminary results. The main result is stated in Section 2. The remaining sections are devoted to defining codimension-one singularities, to give auxiliary results and to prove the main theorem. Figures and references appear at the end of the paper.

We shall finish the introduction by exhibiting an example.

In $\mathbb{R}^2$ consider the function $f(x, y) = x$, the field $X_\alpha = (\cos x \text{ sen } y - \alpha \text{ sen } x, \text{ sen } x \cos y)$ $\alpha \in \mathbb{R}$, the usual metric and the following equivalence relation:

$$(x_1, y_1) \sim (x_2, y_2) \Leftrightarrow \text{there exist } (m, n) \in Z^2 \text{ such that}$$

$$x_2 = x_1 + 2m\pi \quad \text{and} \quad y_2 = y_1 + 2n\pi .$$

In $T^2 = \mathbb{R}^2/\sim$ consider the quotient metric.

The singularities of (X_α, f) in $\mathbb{R}^2$ are given by the relation:

$$\cos x \text{ sen } t - \alpha \text{ sen } x = 0 .$$

In T^2 the set of singularities of (X_α, f) is:

a) 4 circles if $\alpha = 0$;

b) 2 circles if $\alpha \neq 0$.

So we are tempted to say that $\alpha = 0$ is a bifurcation point of the one parameter family (X_α, f) on T^2.

1. PRELIMINAIRES

Consider M a C^∞ two dimensional compact manifold without boundary.

Let X^r be the space of C^r vector fields on M with the C^r topology and F^{r+1} be the space of C^{r+1} real value functions with the C^{r+1} topology. We topologise $W = X^r \times F^{r+1}$ with the natural product topology; we shall assume $r > 2$.

We will fix on M a Riemannian metric of class C^∞.

Definition 1.1. Let $p, q \in M$. We say that $(X, f) \in W$ at p *is equivalent to* (Y, g) at q if there exist neighborhoods, U of p and V of q, in M and a homeomorphism h: $U \to V$ which maps trajectories of $X_{|U}$ onto trajectories of $Y_{|U}$ and level curves of $f_{|U}$ onto level curves of $g_{|V}$. From this the Local Structural Stability in W is given in a natural way. Denote by Σ^l the subset of W consisting of the locally structurally stable field-functions.

Consider $(X, f) \in W$ and $p \in M$. The folowing notation will be used in the text:

i) $X(f)(p)$ is the derivative of f along X at p;

ii) $L_f(p)$ is the level curve of f passing by p;

iii) $\gamma_X(p)$ is the trajectory of X passing by p;

iv) $\phi_X(x, t)$ is the solution of $\dot{x} = X(x)$ satisfying $\phi_X(x, 0) = x$; $\gamma_X(x) = \{\phi_X(x, t)/t \in \mathbb{R}\}$;

v) Df_p is the derivative of f at p;

vi) For a subset S of M, ∂S is the boundary of S, $f_{|S}$ is the restriction of f to S and $M - S$ is the set of points $q \in M$ such that $q \notin S$.

Definition 1.2. A point $p \in M$ is *a regular point of* $(X, f) \in W$ if $X(f)(p) \neq 0$. If $X(f)(p) = 0$ then p is a critical point or a singularity of (X, f). The critical set of (X, f) (denoted $C(X, f)$) is the set of the critical points $p \in M$ of (X, f).

Definition 1.3. A point $p \in C(X, f)$ is said to be *a critical point of* (X, f) of type:

I – if i) $X(p) = 0$; ii) p is a hyperbolic critical point of X; iii) the eigenvalues of DX_p are distinct; iv) p is a regular point of f; v) the eigenspaces of DX_p are transversal to $L_f(p)$ at p.

II – if i) $X(p) \neq 0$; ii) p is a non degenerate critical point of f; iii) $X(X(f)) \neq 0$.

III – if i) $X(p) \neq 0$ ii) p is a regular point of f; iii) $(X(X, f))(p) \neq 0$.

IV – if i) $X(p) \neq 0$; ii) p is a regular point of f; iii) $D(X(f))_p \neq 0$; iv) $X(X(f))(p) = 0$ but $X(X(X(f)))(p) \neq 0$.

We will refer to the critical point of (X, f) of type J as G_J-singularity of (X, f), $J =$ I, II, III and IV.

Definition 1.4. A point $p \in M$ is said to be a *generic point* of (X, f) if either it is a regular point of (X, f) or it is a G_J-singularity of (X, f), $J =$ I, II, III and IV.

Remark 1.5. Let p be a non degenerate node of X (this means that if λ_1, λ_2 are the eigenvalues of DX_p, with $X(p) = 0$, then $\lambda_1 . \lambda_2 > 0$ and $\lambda_1 \neq \lambda_2$).

We shall refer to a weak trajectory of X at p as that trajectory of X tangent to the eigenspace DX_p associated to the eigenvalue of larger absolute value.

In [4] was proved that:

a) $(X, f) \in \Sigma^l$ if and only if any point of M is a generic point of (X, f);

b) Σ^l is open and dense in W.

If X is a C^r vector field on M with $X(p) = 0$ then the determinant and the trace of DX_p will be denoted by $\Delta(X, p)$ and $\sigma(X, p)$ respectively.

2. STATEMENT OF RESULTS

For technical reasons a Q-singularity of $(X, f) \in W$ will be defined later.

In this paper we shall prove the following theorem:

Theorem A. *Let $p \in M$ be a Q-singularity of (X, f) then:*

1. *There exist neighborhoods B of (X, f) in W and N of p in M and a C^{r-1} function $G : B \to R$ such that $G(Y, g) = 0$ if and only if (Y, g) has a Q-singularity in N as unique non generic critical point in N;*

2) *Following 1) if $G(Y, g) = 0$ then (X, f) at p is germ equivalent to each (Y, g) at some $q \in N$;*

3) *There are 10 different topological types of Q-singularities; moreover, the universal unfolding of each one of these Q-singularities is given* (see the pictures);

4) *If p is neither a generic point of (X, f) nor a Q-singularity of (X, f) then there is a sequence (X_n, f_n) in W tending to (X, f) such that p is a Q-singularity of (X_n, f_n).*

3. THE Q_1-SINGULARITY

Definition 3.1. A point p is a *Q_1-singularity of $(X, f) \in W$* if: i) $X(p) = 0$ and p is hyperbolic; ii) p is not a critical point of f and $L_f(p)$ is tangent to one of the eigenspaces of DX_p; iii) the eigenvalues of DX_p are real and distinct. Under the above conditions will be shown (Lemma 3.2) that $C(X, f)$ is a local submanifold of M (around p) and p is a critical point of $X(X(f))_{|W}$. We impose the following additional hypothesis: iv) p is a non degenerate critical point of $X(X(f))_{|W}$.

Lemma 3.2. *Suppose $(X, f) \in W$ and $p \in M$ satisfying the above conditions* 3.1.i, 3.1.ii *and* 3.2.ii. *Then there are neighborhoods U of 0 in R, N of p in M and a C^r imbedding $\alpha : U \to N$ such that $\alpha(0) = p$ and $q \in C(X\ f) \cap N$ if and only if $q \in \alpha(U)$. Furthermore, $C(X, f)$ is tangent to $L_f(p)$ at p.*

Proof. Consider a system of coordinates around p (say (x_1, x_2)) satisfying $x_1(p) = x_2(p) = 0$ and $f(x_1, x_2) = x_2$. Let (X^1, X^2) be the components of X in these coordinates.

By the submersion local form this system of coordinates can be obtained such that $\partial/\partial x_1$ is an eigenvector of DX_p (this implies, in particular that $(\partial X^2/\partial x_1)(0, 0) = 0$).

The following C^r real function

$$F(x_1\ x_2) = X(f)(x_1\ x_2)$$

satisfies

$$F(x_1, x_2) = X^2(x_1, x_2), \quad F(0\ 0) = 0 \quad \text{and} \quad DF(0, 0) \quad \text{is non singular}.$$

We may, by hypothesis, consider $\partial/\partial x_2$ an eigenvector of DX_p; this implies $(\partial X^1/\partial x_2)(0, 0) = 0$ and $(\partial X^2/\partial x_2)(0, 0) \neq 0$.

Now, a simple calculation shows that we can get a C^r real function $x_2 = \alpha(x_1)$ defined in a convenient-neighborhood of 0 in R, such that:

$$\alpha(0) = 0\,, \quad F(x_1, x_2) = 0 \quad \text{only if} \quad x_2 = \alpha(x_1) \quad \text{and} \quad \alpha'(0) = 0\,.$$

This ends the proof. □

Remark 3.2.a. Considering $X(X(f))$ given in the same coordinates above named is trivial to show that p is a critical point of $X(X(f))_{|C(X,f)}$.

Lemma 3.3. *Suppose that $p \in M$ is a Q_1-singularity of $(X, f) \in W$. The curve $\alpha(t)$ obtained in Lemma 3.2 satisfies:*

a) *if $t \neq 0$ then $\alpha(t)$ is a G_{III}-singularity of (X, f);*

b) *the contact between α and $L_f(p)$ at p is of 2nd order.*

Proof. It is clear that $X(X(f))(p) = 0$.

Consider $x = (x_1, x_2)$ the coordinates given in the proof of Lemma 3.2. We have:

$$X(X(f))(x_1, x_2) = \left(X^1 \frac{\partial X^2}{\partial x_1} + X^2 \frac{\partial X^2}{\partial x_2}\right)(x_1, x_2)\,.$$

So, the function $h(x_1) = X(X(f))(x_1, \alpha(x_1))$ satisfies:

i) $h(0) = h'(0) = 0$ and ii) $h''(0) \neq 0$.

The above condition ii) implies that:

a) if $x_1 \neq 0$ then $h(x_1) \neq 0$; this means if $q \in C(X, f) \cap \alpha(U)$ and $q \neq p$ then q is a G_{III}-singularity of (X, f);

b) $\alpha''(0) \neq 0$; this means that the contact between $C(X, f)$ and $L_f(p)$ at p is of 2nd order. □

Lemma 3.4. *Let $p \in M$ be a Q_1-singularity of $(X, f) \in W$. Then there exist neighborhoods B of (X, f) in W and N of p in M and a C^{r-2} function $q : B \to N$ such that*

a) $q(X, f) = p$,

b) $Y(g)(q(Y, g)) = 0$,

c) $Y(Y(g))(q(Y, g)) = 0$ *if and only if $q(Y, g)$ is a Q_1-singularity,*

d) *if $Y(Y(g))(q(Y, g)) \neq 0$ then there exist two points in $N \cap C(Y, g)$, q_1 and q_2 $(q_i \neq q(Y, g)\ i = 1, 2)$ such that, q_1 is a G_I-singularity of (Y, g), q_2 is a G_{IV}-singularity of (Y, g) and if $q \neq q_i$ $(i = 1, 2)$ then q is a G_{III}-singularity of (Y, g).*

Proof. Let (x_1, x_2) the same coordinates around p (say in U) given in the proof of Lemma 3.2.

Define a C^r function $F : W \times U \to R$ by:

$$F(Y, g, x_1, x_2) = Y(g)(x_1, x_2)\,.$$

Since $(\partial F/\partial x_2)(X, f, 0, 0) \neq 0$ we can get a C^r function $x_2 = \alpha(Y, g, x_1)$ which is solution of $F = 0$ and satisfies $\alpha(X, f, 0) = 0$. The function $x_1 \to (Y, g, x_1)$ describes the critical set of (Y, g).

Define the function $H: B \times I \to R$ by

$H(Y, g, x_1) = Y(Y(g))(x_1, \alpha(x_1, Y, g))$ where B and I are convenient neighborhoods of (X, f) (in W) at $x_1 = 0$ (in R) respectively. We have:

$$\frac{\partial H}{\partial x_1}(X, f, 0) = 9 \quad \text{and} \quad \frac{\partial^2 H}{\partial x_1^2}(X, f, 0) \neq 0$$

$$\left(\text{suppose without loss of generality that } \frac{\partial^2 H}{\partial x_1^2}(X, f, 0) > 0\right).$$

So, we find a function $x_1 = \beta(Y, g)$ which is solution of $(\partial H/\partial x_1)(Y, g, x_1) = 0$ and satisfies $\beta(X, f) = 0$. If B is chosen small enough then for each $(Y, g) \in B$ the point $\beta(Y, g)$ is the minimum of the function

$$x_1 \to H(Y, g, x_1).$$

So:

i) if $H(Y, g, \beta(Y, g)) > 0$ then for every $x_1 \in \mathrm{I}$, $H(Y, g, x_1) > 0$;

ii) $H(Y, g, x_1) = 0$ if and only if $x_1 = \beta(Y, g)$;

iii) if $H(Y, g, \beta(Y, g)) < 0$ then there exist two points $\bar{x}, \bar{\bar{x}}$ in I, $\bar{x} < \beta(Y, g) < \bar{\bar{x}}$ such that $H(Y, g, \bar{x}) = H(Y, g, \bar{\bar{x}}) = 0$, $(\partial H/\partial x_1)(Y, g, \bar{x}) > 0$ and $(\partial H/\partial x_1) \cdot (Y, g, \bar{\bar{x}}) > 0$.

Observe that i) can never occur since each Y close enough to X(in X^r) has a critical point p_y, close to p (in N) which satisfies: $Y(Y(g))(p_y) = 0$. Now the conclusion of the lemma is immediate if we consider $q(Y, g) = (\beta(Y, g), \alpha(\beta(Y, g), Y, g))$. □

Lemma 3.5. *Let $p \in M$ be a Q_1-singularity of $(X, f) \in W$. Then there exist neighborhoods B of (X, f) in W, N of p in M and a C^{r-2} function $G: B \to R$ satisfying:*

a) *$G(Y, g) = 0$ if and only if (Y, g) has a Q_1-singularity $p(Y, g)$ in N; any point in $C(Y, g) \cap N$ different from $p(Y, g)$ is a G_{III}-singularity;*

b) *if $G(Y, g) \neq 0$ then $C(Y, g) \cap N$ contains one point q which is a G_{I}-singularity of (Y, g), one point q_2 which is a G_{IV}-singularrity. Furthermore, if $q \in C(Y. g) \cap N$, $q \neq q_i$, $i = 1, 2$ then is a G_{III}-singularity of (Y, g);*

c) $DG(X, f) \neq 0$.

Proof. Each Y close enough to X in X^r has a critical point p_y which is of the same type as p and the correspondence $Y \to p_y$ is C^r. Observe that $Y(Y(g))(p_y) = 0$; so if p_y coincides with $q(Y, g)$ obtained in Lemma 3.4) then p_y is a Q_1-singularity of (Y, g).

The following function permits us to finish the proof:

$$G(Y, g) = \frac{\partial}{\partial x_1} (Y(Y(g)) (p_y)$$

(we are considering the coordinates given in Lemma 3.4) . □

Remark 3.6 – Lemmas 3.4 and 3.5 describe the unfolding of a Q_1-singularity.

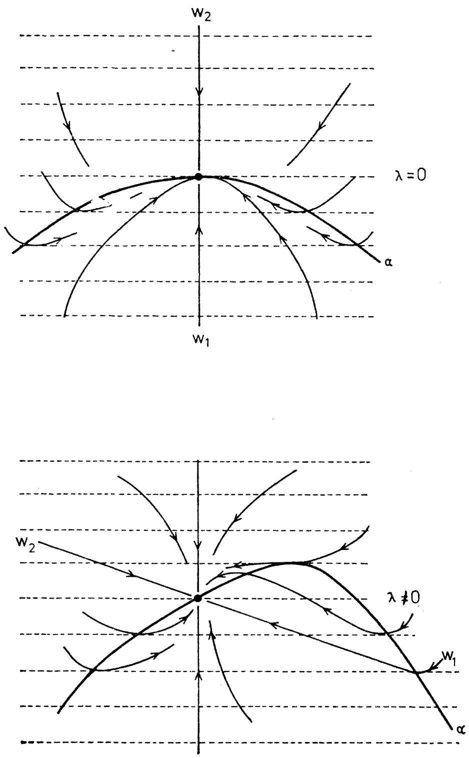

Fig. 1. Unfolding of a Q_1-singularity (case one)*)

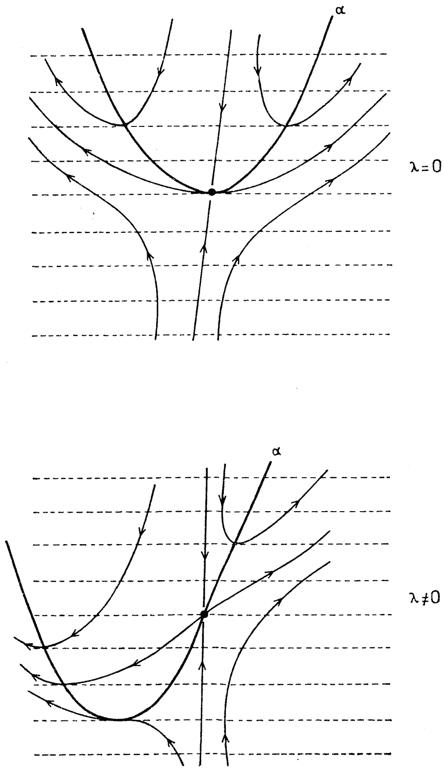

Fig. 2. Unfolding of a Q_1-singularity (case two)

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*) If (Y, g) is a field function, broken lines in the following figures, represent the level curves of g, unbroken lines represent the trajectories of Y, and thick unbroken lines represent the crittical set of (Y, g).

We are considering a one-parameter family of field-functions $((Y, g), \lambda)$ such that $((Y, g), 0)$ contains some Q-singularity. As in [2] and [3] we are assuming that this family has some transver sality condition with respect to λ.

Remark 3.7. In Lemma 3.5, the neighborhoods β and N can be choosen such that if $G(Y, g) = 0$ then (X, f) at p is germ equivalent to (Y, g) at p_y (see the technique used in [4]).

4. THE Q_2-SINGULARITY

Definition 4.1. A point $p \in M$ is *a Q_2-singularity of* $(X, f) \in W$ if: i) $X(p) = 0$; ii) If λ_1, λ_2 are the eigenvalues of DX_p then $\lambda_1 = \lambda_2 = \lambda \neq 0$ and rank $[DX_p - \lambda I] = 1$; iii) p is not a critical point of f and $L_f(p)$ is transverse to that eigenspace of DX_p at p.

Let $p \in M$ be a Q_2-singularity of $(X, f) \in W$.

Lemma 4.2. *There exist neighborhoods U of 0 in R, N of p in M and a C^r imbedding $\alpha : U \to N$ such that*: a) $\alpha(0) = p$ *and* $q \in C(X, f) \cap N$ *only if* $q \in \alpha(U)$; b) $C(X, f)$ *is transversal to* $L_f(p)$ *at* p; c) *if* $q \in \alpha(U)$ *and* $q \neq p$ *then* q *is a* G_{III}-*singularity of* (X, f).

Proof. Consider a system of coordinates $x = (x_1, x_2)$ around p satisfying $x_1(p) = x_2(p) = 0$ and $X(x_1, x_2) = (\lambda x_1 + a x_2, \lambda x_2)$ with $\lambda \neq 0$, $a \neq 0$.

The following C^r real function $F(x_1, x_2) = X(f)(x_1, x_2)$ satisfies $F(0, 0) = 0$, $(\partial F/\partial x_1)(0, 0) = \lambda(\partial f/\partial x_1)(0, 0)$ and

$$\frac{\partial F}{\partial x_2}(0, 0) = \left(a \frac{\partial f}{\partial x_1} + \lambda \frac{\partial f}{\partial x_2}\right)(0, 0).$$

Because $(\partial f/\partial x_1)(0, 0) \neq 0$ (this follows from condition iii) of definition 4.1) we can get a C^r function $x_1 = \alpha(x_2)$, defined in convenient neighborhood of 0 in R, such that $\alpha(0) = 0$ and $F(x_1, x_2) = 0$ only if $x_1 = \alpha(x_2)$.

It is easily shown that $C(X, f)$ is transverse to $L_f(p)$ at p.

The function $h(x_2) = X(X(f))(\alpha(x_2). x_2)$ satisfies $h'(0) \neq 0$. This finishes the proof. □

Corollary 4.3. *There exist neighborhoods B of (X, f) in W and N of p in M and a C^{r-1} function $q : B \to N$ such that*: a) *The element $q(Y, g)$ is the unique solution of $Y(g) = Y(Y(g)) = 0$ for each $(Y, g) \in B$*; b) $Y(q(Y, g)) = 0$.

Proof. Part a) follows immediately from 4.2.

Since p is a hyperbolic critical point of X, N and B can be obtained such that for each $(Y, g) \in B$, Y contains) a single critical point $q_y \in N$ which satisfies $Y(g)(q_y) = Y(Y(g))(q_y) = 0$. Now part a) implies part b). □

Remark 4.4. The next lemma describes the universal unfold ing of a Q_2-singularity (see figure 3).

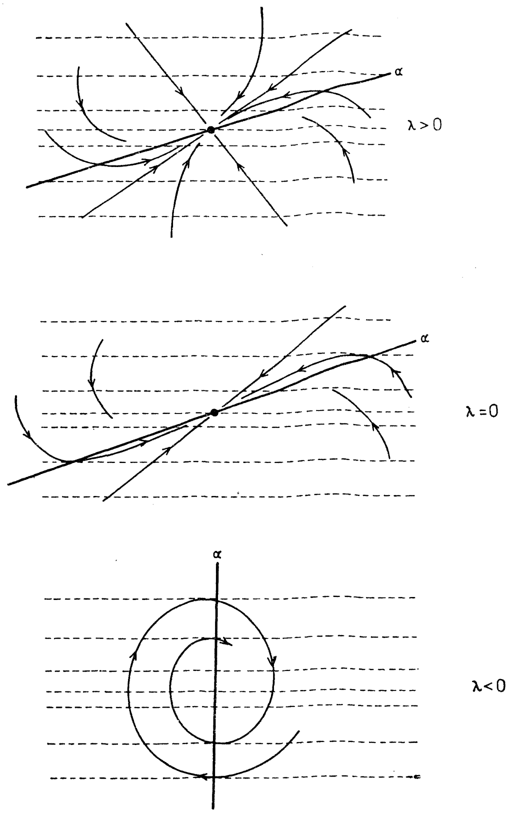

Fig. 3. Unfolding of a Q_2-singularity

Lemma 4.5. *There exist neighborhoods B of (X, f) in W and N of p in M and a C^r function $G : B \to R$ such that*:

a) *$G(Y, g) = 0$ if and only if (Y, g) has a Q_2-singularity in N;*

b) *If $G(Y, g) \neq 0$ then $C(Y, g) \cap N$ contains a point which is a G_{I}-singularity of (Y, g) and other points of it are G_{III}-singularities of (Y, g).*

c) *$DG(X, f) \neq 0$.*

Proof. It Y is a field close to X in X^r, let p_y be the critical point of Y in a neighborhood N of p in M.

The required function is given by:

$$G(Y, g) = \sigma^2(Y, p_y) - 4\,\Delta(Y, p_y)\,.$$

Now, it is not difficult to prove the present Lemma. □

Remark 4.6. In Lemma 4.5, the neighborhoods B and N can be choosen such that if $G(Y, g) = 0$ then (X, f) at p is germ equivalent to (Y, g) at P_y.

5. THE Q_3-SINGULARITY

Definition 5.1. A point $p \in M$ is a *Q_3-singularity of* $(X, f) \in W$ if: i) $X(p) = 0$; ii) p is a saddle-node [2] of X; iii) p is not a critical point of f and $L_f(p)$ is transversal to the eigenspaces of DX_p at p. Under the above conditions will be shown that $C(X, f)$ is a local submanifold of M (around p) and p is a critical point of $X(X, (f))_{|C(X,f)}$. We impose the following additional hypothesis: iv) p is a non degenerate critical point of $X(X(f))_{|C(X,f)}$.

Let $p \in M$ be a Q_3-singularity of $(X, f) \in W$.

Lemma 5.2. *There are neighborhoods U of* 0 *in R and N of p in M and a C^r imbedding α: $U \to N$ such that*: a) $\alpha(0) = p$ *and* $q \in C(X, f) \cap N$ *only if* $q \in \alpha(U)$; b) α *is tangent to that eigenspace of DX_p associated to the eigenvalue $\lambda_1 = 0$ at p. Denote this eigenspace by T_1*; c) *the contact, between α and T_1 at p, is of* 2^{nd} *order*; d) *if $q \in \alpha(U)$ and $q \neq p$ then q is a G_{III}-singularity of (X, f).*

Proof. Let (x_1, x_2) be a system of coordinates around p such that $x_1(p) = x_2(p) = 0$ and $(\partial/\partial x_1)(p) \in T_i$; $i = 1, 2$ (see [2], page 15).

Take the function $F(x_1, x_2) = X(f)(x_1, x_2)$; it satisfies $(\partial F/\partial x_2)(0, 0) \neq 0$. Applying the Implicit Function Theorem there exists a C^r function $x_2 = \alpha(x_1)$ defined around $x_1 = 0$ which is solution of $F(x_1, x_2) = 0$ and satisfying $\alpha(0) = \alpha'(0) = 0$.

We now look at the following function

$$H(x_1) = X(X(f))(x_1, \alpha(x_1)).$$

It satisfies $H(0) = H'(0) = 0$ and $H''(0) \neq 0$. From this fact and by a simple calculation we deduce that $\alpha''(0) \neq 0$ and if $x_1 \neq 0$ then $H(x_1) \neq 0$ (this shows 5.2.c and 5.2.d. respectively). □

Lemma 5.3. *There exist neighborhoods B of (X, f) in W and N of p in M and a C^r function $G : B \to R$ such that*:

a) $G(Y, g) = 0$ *if and only if (Y, g) has a Q_3-singularity in N; the other points in $C(Y, g) \cap N$ are G_{III}-singularities*;

b) *if $G(Y, g) < 0$ then $C(Y, g) \cap N$ contains two points q_1 and q_2 which are G_I-singularity of (Y, g) and the other points of it are G_{III}-singularities*;

c) *if $G(Y, g) > 0$ then $C(Y, g) \cap N$ contains only G_{III}-singularities*

d) $DG(X, f) \neq 0$.

Proof. Consider the same coordinates around p given in preceeding proof.

As before we can get a function $q : B \to N$ given by $q(Y, g) = (\beta(Y, g), \alpha(\beta(Y, g), Y, g))$ where

$$x_2 = \alpha'(x_1, Y, g) \quad \text{is the solution of}$$

$$Y(g)(x_1, x_2) = 0 \quad \text{provided that} \quad \frac{\partial}{\partial x_2} Y(g)(x_1, x_2) \neq 0$$

and,

$$x_1 = \beta(Y, g) \quad \text{is a non degenerate critical point of}$$

$$H(x_1, Y, g) = Y(Y(g))(x_1, \alpha(x_1, Y, g)) \quad \text{for} \quad (Y, g)$$

belonging to some neighborhood of (X, f) in W (see proof of Lemma 5.2).

Note that $\beta(X, f) = \alpha(0, X, f) = 0$.

Suppose, for simplicity, that $(\partial^2 H/\partial x_1^2)(\beta(Y, g), Y, g) > 0$.

For each (Y, g) close enough to (X, f) in W, we have:

a) if $H(\beta(Y, g), Y, g) = 0$ then $H(x_1, Y, g) > 0$ for every x_1;

b) if $H(\beta(Y, g), Y, g) < 0$ then there are points a_1, a_2 close to $\beta(Y, g)$ in R with $a_1 < \beta < a_2$ and satisfying $H(a_1, Y, g) = H(b_1, Y, g) = 0$, $(\partial H/\partial x_1)(a_1, Y, g) < 0$ and $(\partial H/\partial x_1)(a_2, Y, g) > 0$.

c) if $H(\beta(Y, g), Y, g) > 0$ then $H(x_1, Y. g) > 0$ for every x_1 running in a neighborhood of $x_1 = 0$ in R.

Now, observe that

$$\frac{\partial \alpha}{\partial x_1}(q(Y, g), Y, g) = 0\,;$$

so using the both following equations: $Y(g)(q(Y, g) = 0$, $H(\beta(Y, g), Y, g) = 0$ we conclude that if $H(\beta(Y, g), Y, g) = 0$ then

$$Y(q(Y, g)) = \frac{\partial \alpha}{\partial x_1}(q(Y, g), Y, g) = 0 \quad \text{and} \quad \frac{\partial^2 \alpha}{\partial x_1^2}(q(Y, g), Y, g) \neq 0\,.$$

Furthermore, it can be easily show (by picking canonical coordinades around $q(Y, g)$) that $q(Y, g)$ is a Q_3-singularity of (Y, g).

A straightforward computation shows that the points $q_1(Y, g) = (a_1, \alpha(a_1, Y, g))$, $q_2(Y, g) = (a_2, \alpha(a_2, Y, g))$ are G_{I}-singularities of (Y, g).

If $H(\beta(Y, g), Y, g) > 0$ then $C(Y, g) \cap N$ contains only G_{III}-singularities where N is a convenient neighborhood of p in M.

The required function is given by

$$G(Y, g) = H(\beta(Y, g), Y, g)\,. \quad \square$$

Remark 5.4. In Lemma 5.3, the neighborhoods B and N can be chosen such that if $G(Y, g) = 0$ then (X, f) at p is germ equivalent to (Y, g) at $q(Y, g)$.

Remark 5.5. Lemma 5.3 describes the universal unfolding of a Q_3-singularity (see Figure 4).

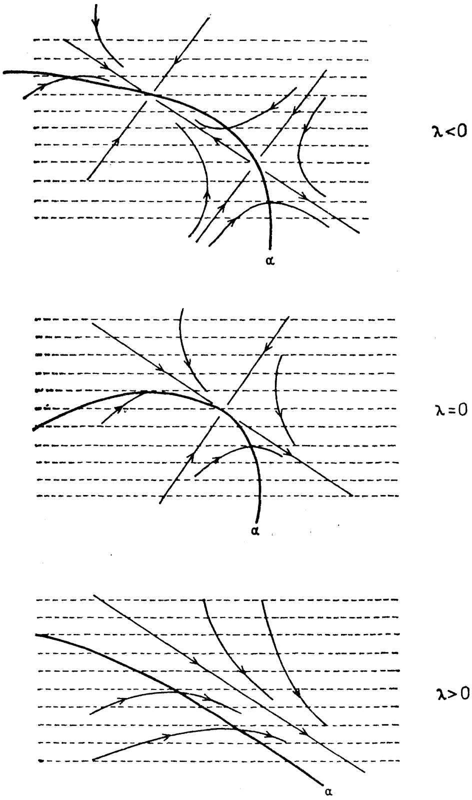

Fig. 4. Unfolding of a Q_3-singularity

6. THE Q_4-SINGULARITY

Definition 6.1. A point $p \in M$ is a *Q_4-singularity of* $(X, f) \in W$ if: i) $X(p) = 0$, ii) p is a composed focus [2] of X; iii) p is not a critical point of f.

Let $p \in M$ be a Q_4-singularity of $(X, f) \in W$.

Lemma 6.2. *There are neighborhoods U of 0 in R and N of p in M and a C^r imbedding $\alpha : U \to N$ such that:* a) $\alpha(0) = p$ *and* q $C(X, f) \cap N$ *only if* $q \in \alpha(U)$; b) *if* $q \in \alpha(U)$ *and* $q \neq p$ *then* q *is* G_{III}*-singularity of* (X, f); c) α *is transverse to* $L_f(p)$ *at* p.

Proof. As p is a composed focus of X, let $x = (x_1, x_2)$ be a system of coordinates around p in M, satisfying $x(p) = 0$,

$$\frac{\partial X^1}{\partial x_2}(p) = a = -\frac{\partial X^2}{\partial x_1}(p) \neq 0 \quad \text{and} \quad \frac{\partial X^1}{\partial x_1}(p) = \frac{\partial X^2}{\partial x_2}(p) = 0 .$$

(see [2], page 24) .

The function $F(x_1, x_2) = X(f)(x_1, x_2)$ satisfies

$F(0, 0) = 0$, $(\partial F/\partial x_1)(0, 0) = -a(\partial f/\partial x_2)(0, 0)$, $(\partial F/\partial x_2)(0, 0) = a(\partial f/\partial x_1)(0, 0)$. Since p is a regular point of f we may assume $(\partial f/\partial x_1)(0, 0) \neq 0$; so let $x_2 = \alpha(x_1)$ be the solution of $F(x_1, x_2) = 0$, satisfying $\alpha(0) = 0$.

As before, by using the function

$H(x_1) = X(X(f))(x_1, \alpha(x_1))$ the present proof is immediately concluded. □

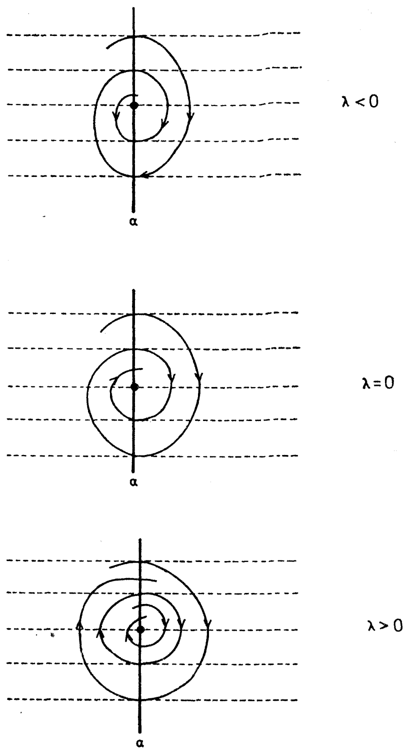

Fig. 5. Unfolding of a Q_4-singularity

Corollary 6.3. *There exist neighborhoods B of* (X, f) *in W and N of p in M and a* C^{r-1} *function* $q: B \to N$ *such that*: a) $Y(g)(q(Y, g)) = Y(Y(g))(q(Y, g)) = 0$ *and* b) $Y(q(Y, g)) = 0$, *for each* $(Y, g) \in B$.

Proof. It follows immediately from 6.2 and Lemma 3.12 in [2], page 25. □

Lemma 6.4. *There exist neighborhoods B of* (X, f) *in W, N of p in M and a* C^r *function* $G: B \to R$ *satisfying*:

a) $G(Y, g) = 0$ *if and only if has a* Q_4*-singularity in N*;

b) *if* $G(Y, g) \neq 0$ *then* $C(Y, g) \cap$ *contains one point which is a* G_I*-singularity of* (Y, g) *and the other points of it are* G_{III}*-singularities*

c) $DG(X, f) \neq 0$.

The Lemma 6.4 is demonstrated by already known methods.

Remark 6.5. Remark 5.4 holds if p is a Q_4-singularity.

Remark 6.6. Lemma 6.3, describes the universal unfolding of a Q_4-singularity (see Figure 5).

7. THE Q_5-SINGULARITY

Definition 7.1. A point $p \in M$ is a Q_5*-singularity of* $(X, f) \in W$ if: i) $X(p) \neq 0$, ii) p is a degenerate critical point of f but the reare coordinates (x_1, x_2) around p in M such that $x_1(p) = x_2(p) = 0$ and $f(x_1, x_2) = \frac{1}{3}\varepsilon_1 x_1^3 + \frac{1}{2}\varepsilon_2 x_2^2$ where $\varepsilon_i = \pm 1$, $i = 1, 2$.

Let $p \in M$ be a Q_5-singularity of $(X, f) \in W$.

Lemma 7.2. *There are neighborhoods U of* 0 *in R, N of p in M and a* C^r *imbedding* $\alpha: U \to N$ *such that*: a) $\alpha(0) = p$ *and* $q \in C(X, f) \cap N$ *only if* $q \in \alpha(U)$; b) α *is transverse to* $L_f(p)$ *at p*; c) *if* $q \in \alpha(U)$ *and* $q \neq p$ *then q is* G_{III}*-singularity of* (X, f); d) α *can be choosen such that* $(f \circ \alpha)(t) = t^3$.

Proof. Let $x = (x_1, x_2)$ be that system of coordinates around p given in Definition 7.1.

The function $F(x_1, x_2) = X(f)(x_1, x_2)$ satisfies

$$F(0, 0) = \frac{\partial F}{\partial x_1}(0, 0) = 0 \quad \text{and} \quad \frac{\partial F}{\partial x_2}(0, 0) = \varepsilon_2 X^2(0, 0).$$

A simple calculation shows that the hypothesis $X(X(f))(p) \neq 0$ implies $X^2(0, 0) \neq 0$. So applying the Implicit Function Theorem there is a function $x_2 = \alpha(x_1)$ which is a solution of $F(x_1, x_2) = 0$ and satisfying $\alpha(0) = \alpha'(0) = 0$.

If one looks at the function $h(x_1) = (f \circ \alpha)(x_1)$ we deduce that $h(0) = h'(0) = h''(0) = 0$ and $h'''(0) = (\partial^3 F / \partial x_1^3)(0, 0) \neq 0$.

It is clear that α is transverse to X at p.

Now, the conclusion of the proof is trivial. □

Corollary 7.3. *There are neighborhoods B of (X, f) in W and N of p in M such that the critical set of each $(Y, g) \in B$ is a C^{r-1} codimension one submanifold of N transverse to $Y_{|N}$.*

Corollary 7.4. *Following Corollary* 7.4 *there is a C^{r-1} imbedding $i : B \to N$ such that*: i) $i(Y, g) \in C(Y, g)$; ii) $i(X, f) = p$; iii) $i(Y, g)$ *is a non degenerate critical point of $h : C(Y, g) \cap N \to R$ where* $h(Y) = dg(T_y(C(Y, g))(T_y(C(Y, g))))$ *being the tangent space of $C(Y, g)$ at Y).*

The next lemma is demonstraded by already known techniques:

Lemma 7.5. *There are neighborhoods B of (X, f) in W and N of p in M and a C^{r-1} function $G : B \to R$ such that*:

a) $G(Y, g) = 0$ if and only if (Y, g) has a Q_5-singularity; $q(Y, g)$ in N; any point in $C(Y, g)$ different from q is a G_{III}-singularity;

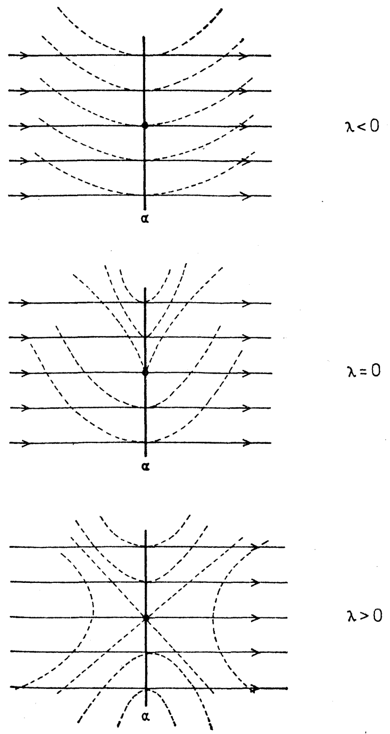

Fig. 6. Unfolding of a Q_5-singularity

b) If $G(Y, g) > 0$ then (Y, g) has a G_{II}-singularity in N; the other points in $C(Y, g) \cap N$ are G_{III}-singularities;

c) If $G(Y, g) < 0$ then $C(Y, g) \cap N$ contains only G_{III}-singularities;

d) $DG(X, f) \neq 0$.

Remark 7.6. Remark 5.4 holds if p is a Q_5-singularity.

Remark 7.7. Lemma 7.5 describes the universal unfolding of a Q_5-singularity (see Figure 6).

8. THE Q_6-SINGULARITY

Definition 8.1. A point $p \in M$ is a *Q_6-singularity of* $(X, f) \in W$ if: i) $X(p) \neq 0$, ii) p is a non degenerate critical point of f; iii) $X(X(f))(p) = 0$; iv) $X(X(X(f)))(p) \neq 0$.

Let $p \in M$ be a Q_6-singularity of $(X, f) \in W$.

Lemma 8.2. There are neighborhoods V of 0 in R and N of p in M and a C^r imbedding $\alpha : U \to N$ such that: a) $\alpha(0) = p$; b) $q \in C(X, f) \cap N$ only if $q \in \alpha(U)$; $C(X, f)$ is tangent to $L_f(p)$ at p and this contact is of 2nd order; d) if $q \in \alpha(U)$ and $q \neq p$ then q is a G_{III}-singularity; e) 0 is a regular point of $X(X(f))(\alpha(t))$.

Proof. Let $x = (x_1, x_2)$ be the system of coordinates around p satisfying $x(p) = 0$ and $f(x_1, x_2) = \frac{1}{2}\varepsilon_1 x_1^2 + \frac{1}{2}\varepsilon_2 x_2^2$ with $\varepsilon_i = \pm 1$.

The function $F(x_1, x_2) = X(f)(x_1, x_2)$ satisfies $F(0, 0) = 0$ and $DF(0, 0) = (\varepsilon_1 X^1, \varepsilon_2 X^2)$.

Assuming $X^2(p) \neq 0$ we can get a function $x_2 = \alpha(x_1)$ which is solution of $F(x_1, x_2) = 0$ such that $\alpha(0) = 0$ and

$$\alpha'(0) = \varepsilon_1 X^1(0, 0)/\varepsilon_2 X^2(0, 0) .$$

Note that $X(X(f))(p) = 0$ implies $\varepsilon_1 \neq \varepsilon_2$ and $X^1(0, 0) = \pm X^2(0, 0)$; so $\alpha'(0) = \pm 1$.

A straightforward calculation shows that $\alpha''(0) \neq 0$ and 0 is a regular point of $X(X(f))(x_1, \alpha(x_1))$ provided that $X(X(X(f)))(p) \neq 0$.

Now, the conclusion of this proof is trivial. □

Lemma 8.3. *There are neighborhoods B of (X, f) in W and N of p in M and a C^{r-1} function $q : B \to N$ such that:* a) $q(X, f) = p$; b) $Y(g)(q(Y, g)) = Y(Y(g))(q(Y, g)) = 0$ *and* $Y(Y(Y(g)))(q(Y, g)) \neq 0$; c) *if $q(Y, g)$ is a critical point of q then $g(Y, g)$ is a Q_6-singularity of (Y, g)*; d) *if $q(Y, g)$ is not a critical point of q then $q(Y, g)$ is a G_{IV}-singularity of (Y, g)*; e) *If $y \in C(Y, g) \cap N$ and $y \neq q(y, g)$ then y is a G_{III}-singularity of (Y, g).*

The proof of this Lemma is similar to that of Lemma 3.4.

Lemma 8.4. *There are neighborhoods B of (X, f) in W and N of p in M and a C^{r-1} function $G : B \to R$ such that:*

a) $G(Y, g) = 0$ *if and only if* (Y, g) *has a* Q_6*-singularity in* N; *the other points in* $C(Y, g) \cap N$ *are* G_{III}*-singularities of* (Y, g).

b) *If* $G(Y, g) \neq 0$ *then* $C(Y, g) \cap N$ *contains one point* q_1 *which is a* G_{II}*-singularity of* (Y, g), *one point* q_2 *which is a* G_{IV}*-singularity of* (Y, g); *furthermore if* $y \in C(Y, g) \cap N$ *with* $q_i \neq y$, $i = 1, 2$, *then* y *is a* G_{III}*-singularity of* (Y, g).

c) $DG(X, f) \neq 0$.

This proof is easy; it is enough to take the function $G(Y, g) = Y(Y(g))(p_g)$ where p_g is the critical point of g close to p in M and (Y, g) belongs to some suitable neighborhood of (X, f) in W. □

Remark 8.5. Remark 5.4 holds if p is a Q_6-singularity of (X, f).

Remark 8.6. Lemma 8.4 describes the universal unfolding of a Q_6-singularity (see Figure 7).

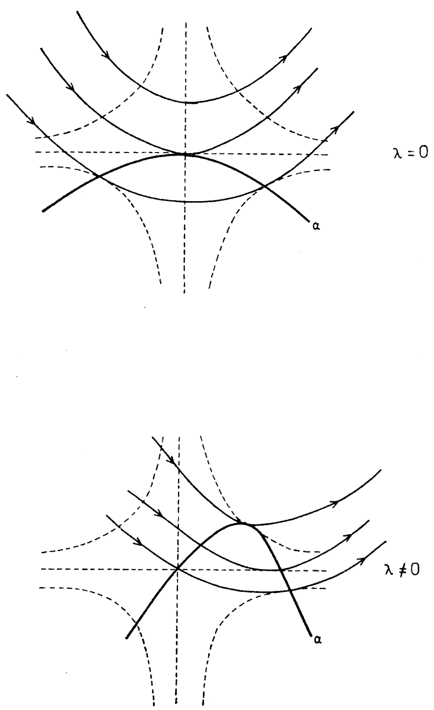

Fig. 7. Unfolding of a Q_6-singularity

Definition 9.1. A point $p \in M$ is a *Q_7-singularity of* $(X,f) \in W$ if: i) $X(p) \neq 0$; ii) p is a regular point if f; iii) $X(f)(p) = X(X(f))(p) = X(X(X(f)))(p) = 0$ and $X(X(X(X(f))))(p) \neq 0$; iv) $D[X(f)]_p \neq 0$.

Let $p \in M$ be a Q_7-singularity of $(X,f) \in W$.

Lemma 9.2. *There are neighborhoods U of* 0 *in R and N of p in M and a C^r imbedding $\alpha : U \to N$ such that*: a) $\alpha(0) = p$, b) *α is tangent to $L_f(p)$ at p and this contact is of* 3rd *order*; c) *if $q \in \alpha(U)$ and $q \neq p$ then q is a G_{III}-singularity of (X,f).*

Proof. First consider $x = (x_1, x_2)$ a system of coordinates around p such that $X(x_1, x_2) = (1, 0)$. We deduce that

$$X(f)(x_1, x_2) = \frac{\partial f}{\partial x_1}(x_1, x_2), \quad Dp(X(f)) = \frac{\partial^2 f}{\partial x_1^2}(0,0)\,\frac{\partial^2 f}{\partial x_1^2}(0,0),$$

$$X(X(f))(0,0) = \frac{\partial^2 f}{\partial x_1^2}(0,0) = 0.$$

The last equation and Definition 9.1 imply that $(\partial^2 f/\partial x_1\, \partial x_2)(0,0) \neq 0$; so we can get a function $x_2 = \alpha(x_1)$ which is solution of $X(f)(x_1, x_2) = 0$ and satisfies

$$\alpha(0) = \alpha'(0) = 0 \quad \text{and} \quad \alpha''(0) = -\frac{\partial^3 f}{\partial x_1^3}(0,0)\,\frac{\partial^2 f}{\partial x_1, \partial x_2}(0,0).$$

Observe that the condition $X(X(X(f)))(p) = 0$ implies $\alpha''(0) \neq 0$ and the condition $X(X(X(X(f))))(p) \neq 0$ implies $\alpha'''(0) \neq 0$.

One can show that $x_1 = 0$ is a non degenerate critical point of the following function.

$$h(x_1) = X(X(f))(x_1, \alpha(x_1))$$

This ends the proof. □

As before one shows the next lemma:

Lemma 9.3. *There are neighborhoods B of (X,f) in W and N of p in M and a C^{r-1} function $G : B \to R$ such that*:

a) *$G(Y, g) = 0$ if and only if (Y, g) has a Q_7-singularity $q(Y, g)$ in N; the other points in $C(Y, g) \cap N$ are G_{III}-singularities;*

b) *if $G(Y, g) < 0$ then $C(Y, g) \cap N$ contains two points q_1 and q_2, which are G_{IV}-singularities of (Y, g); the other elements of $C(Y, g) \cap N$ are G_{III}-singularities;*

c) *if $G(Y, g) > 0$ then $C(Y, g) \cap N$ contains only G_{III}-singularities;*

d) $DG(X,f) \neq 0$.

Remark 9.4. Remark 5.4 holds if p is a Q_7-singularity.

Remark 9.5. Lemma 9.3 describes the universal unfolding of a Q_7-singularity (see Figure 8).

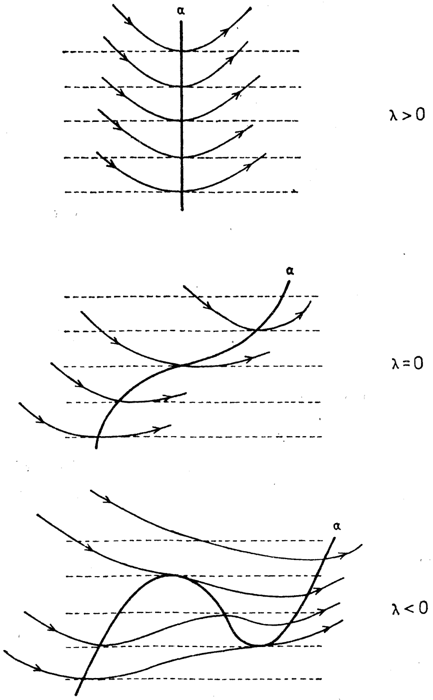

Fig. 8. Unfolding of a Q_7-singularity

10. THE Q_8-SINGULARITY

Definition 10.1. A point $p \in M$ is a Q_8-singularity of $(X, f) \in W$ if: i) $X(p) \neq 0$; ii) p is a regular point of f; iii) $X(F)(p) = 0$; iv) p is a non degenerate critical point of $X(f)$; v) $X(X(X(f)))(p) \neq 0$.

Remark 10.2. Here one has to distinguish two cases: i) p is a saddle of $X(f)$ and ii) p is a extremum of $X(f)$.

Remark 10.3. Supposing 10.1 i) and 10.2 i) valid then the condition "p is a critical point of $X(p)$" implies that $X(X(f))(p) = 0$.

Preliminair Calculation. Let $p \in M$ be a Q_8-singularity of $(X, f) \in W$.

Taking $x = (x_1, x_2)$ the same coordinates, around p (say in U), given in the proof of Lemma 8.2 define a function

$$H : U \times W \to R \quad \text{by} \quad H(x_1, x_2, Y, g) = Y(Y(g))(x_1, x_2).$$

We have:

$$H(0, 0, X, f) = 0, \quad DH(0, 0, X, f) = (\varepsilon_1 X^1, \varepsilon_2 X^2)(0, 0).$$

As $X(p) \neq 0$ we may assume $X^2(0, 0) \neq 0$; so there is a C^r real function $x_2 = \alpha(x_1, Y, g)$ which is solution of $H = 0$ and such that $\alpha(0, X, f) = 0$, $(\partial\alpha/\partial x_1) \cdot (0, X, f) = -\varepsilon_1 X^1(0, 0)/\varepsilon_2 X^2(0, 0)$.

We consider the function

$$h(x_1, Y, g) = Y(g)(X_1, \alpha(x_1, Y, g)).$$

We have:

$$\frac{\partial h}{\partial x_1}(0, X, f) = 0 \quad \text{and} \quad \frac{\partial^2 h}{\partial x_1^2}(0, X, f) = \varepsilon_1 + \varepsilon_2 \left(\frac{\partial \alpha}{\partial x_1}(0, X, f)\right).$$

But we know that $X(X(X(f)))(p) \neq 0$; and this implies, in particular, that $(\partial^2 h/\partial x_1^2)(0, X, f) \neq 0$. Hence, as before, we can get a function $x_1 = \beta(Y, g)$ which is solution of $(\partial h/\partial x_1)(x_1, Y, g) = 0$ with $\beta(X, f) = 0$. Moreover, for each (Y, g), close enough (X, f) in W, $\beta(Y, g)$ is a non degenerate critical point of $h(x_1, Y, g)$ (say a minimum). This means the following:

a) if $h(\beta(Y, g), Y, g) = 0$ then $h(x_1, Y, g) = 0$ only if $x_1 = \beta(Y, g)$;

b) if $h(\beta(Y, g), Y, g) > 0$ then for any x_1 (and (Y, g) fixed) $h(x_1, Y, g) > 0$;

c) if $h(\beta(Y, g), Y, g) < 0$ then there exists two points $\beta_1, \beta_2, \beta_1 < \beta(\omega) < \beta_2$ such that $h(\beta_1, Y, g) = h(\beta_2, Y, g) = 0$ but

$$\frac{\partial h}{\partial x_1}(\beta_1, Y, g) > 0 \quad \text{and} \quad \frac{\partial h}{\partial x_1}(\beta_2, Y, g) < 0;$$

this implies that the point (β_i, Y, g), is a regular point of $Y(g)(x_1, x_2)$, $Y(Y(g)) \cdot (\beta, \alpha(\beta_i, Y, g) = 0$ and $Y(Y(Y(g)))(\beta_i, \alpha(\beta_i, Y, g)) \neq 0$, $i = 1, 2$. This corresponds to say that those points are G_{IV}-singularities of (Y, g).

We separate our study in two cases.

Case i) $\varepsilon_1 = \varepsilon_2$.

The above case a) says that the point $q(Y, g) = (\beta(Y, g), \alpha(\beta(Y, g), Y, g)$ is a Q_8-singularity of (Y, g); moreover the conex component of $C(Y, g)$ containing $q(Y, g)$ is a simple point.

The case c) says that $C(Y, g)$ is, locally around p, a closed curve where $q = (\beta_1, \alpha(\beta_1, Y, g)) \in C(Y, g)$ and $q_2 = (\beta_2, \alpha(\beta, Y, g))$ are G_{IV}-singularities of (Y, g) and the others points in the curve are G_{III}-singularities of (Y, g).

Case ii) $\varepsilon_1 \neq \varepsilon_2$.

The above case a) says that the point $q(Y, g) = (\beta(Y, g), \alpha(\beta(Y, g), Y, g)$ is a Q_8-singularity of (Y, g); moreover the conex component of $C(Y, g)$ containing $q(Y, g)$ is locally topologically equivalent to $x_1^2 = x_2^2$. If $q \in C(Y, g) \cap N$ (where N is a suitable neighborhoods of p in M) then q is a G_{III}-singularity of (Y, g).

The case b) says that $C(Y, g)$ is given, locally around p, by two disjoint branches of curves such that any point in it is a G_{III}-singularity.

The case c) says that $C(Y, g)$ is still given, locally around p, by two disjoint branches of curves, C_1 and C_2) such that $q_i = (\beta_i, \alpha(\beta_i, Y, g)) \in C_i$, $i = 1, 2$, are G_{IV}-singularities of (Y, g); the others points in $C(Y\ g) \cap N$ distinct from q_i $(i = 1, 2)$ are G_{III}-singularities of (Y, g) where N is some neighborhood of p in M.

The preceding calculation permits us to prove the following result.

Lemma 10.4. *Let $p \in M$ be a Q_8-singularity of $(X, f) \in W$. Then there exist neighborhoods B of (X, f) in W and N of p in M and a C^{r-1} function $G : B \to R$, such that:*

a) *$G(Y, g) = 0$ if and only if (Y, g) has a Q_8-singularity $q(Y, g) \in N$. Moreover either $C(Y, g) \cap N$ is a single point or it is homeomorphic to the curve $S = \{(x, y) \in$*

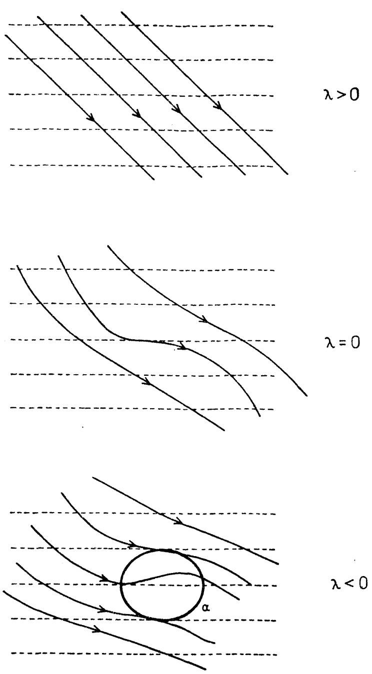

Fig. 9. Unfolding of a Q_8-singularity (case one)

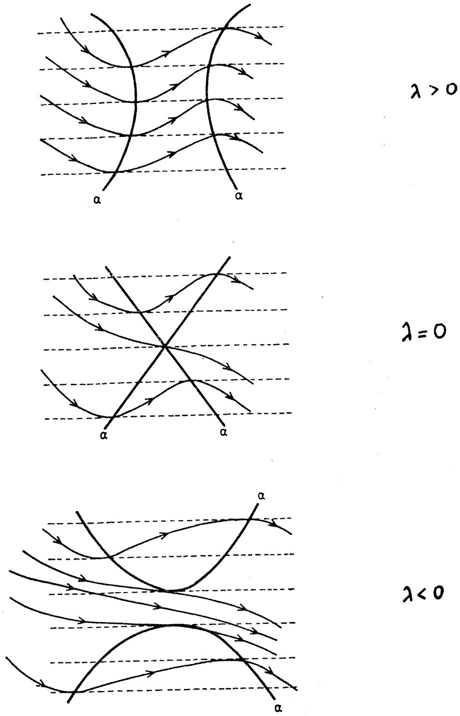

Fig. 10. Unfolding of a Q_8-singularity (case two)

$\in R^2; x^2 - y^2 = 0\}$; *if* $q \in C(Y, g) \cap N$ *and* $q \neq q(Y, g)$ *then* q *is a* G_{III}*-singularity of* (Y, g);

b) *if* $G(Y, g) > 0$ *then etiher* $C(Y, g) \cap N = \emptyset$ *or* $C(Y, g) \cap N$ *contains only* G_{III}*-singularity of* (Y, g);

c) *if* $G(Y, g) < 0$ *then* $C(Y, g) \cap N$ *contains two points of* (Y, g) *which are* G_{IV}*-singularities of* (Y, g); *the other points of* $C(Y, g) \cap N$ *are* G_{III}*-singularities of* (Y, g);

d) $DG(X, f) \neq 0$.

Proof. It is enough to define the following function $G(Y, g) = h(\beta(Y, g), Y, g)$ where the functions h and β are given in the above calculation. □

Remark 10.5. Remark 5.4 holds if p is a Q_8-singularity.

Remark 10.6. Lemma 10.4 describes de universal unfolding of a Q_8-singularity.

11. PROOF OF THEOREM A

We are calling by *Q-singularity of* (X, f) any one of the Q_i-singularity, $i = 1, 2, \ldots, 8$ defined in the preceding sections.

Proof of Theorem A. Part 1) and Part 3) follow from 3.6, 4.5, 5.3, 6.4, 7.5, 8.4, 9.3 and 10.4.

Part 2) follows from 3.7, 4.6, 5.4, 6.5, 7.6, 8.5, 9.4 and 10.5.

Finally, Part 4) is trivial.

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Author's address: Departmento de Matemática, Universidade Estadual de Campinas, 13100-Campinas (SP) — Brasil.