Eduardo Marques de Sá
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ON THE DIAGONALS OF INTEGRAL MATRICES

E. MARQUES DE SÁ, Coimbra

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Throughout this note we deal mainly with matrices over a principal ideal domain, R. For the general concepts and results on matrices over R, we refer to [6]. The elements of R will be denoted by greek letters, and the symbols "$\alpha <: \beta$" and "$\gamma =: \delta$" will be used to mean that α *divides* β and γ *is associated* δ, respectively. Therefore, $\alpha =: \beta$ iff $\alpha <: \beta <: \alpha$. We let M be a multiplicatively closed subset of R, which constitutes a complete system of representatives of the classes of associated elements of R. The *invariant factors* of a matrix and the *gcd* (*greatest commun divisor*) of elements of R, will be taken from M.

Let A be an $n \times m$ matrix of rank r. Let $\alpha_1, \ldots, \alpha_r$ be the invariant factors of A, ordered so that $\alpha_1 <: \ldots <: \alpha_r$. It is well known that A is equivalent to the $n \times m$ matrix

$$\begin{bmatrix} \operatorname{diag}(\alpha_1, \ldots, \alpha_r), & 0 \\ 0, & 0 \end{bmatrix}$$

where the 0's, if they exist, are zero blocks of the appropriate sizes. For convenience, we shall eventually extend the sequence $\alpha_1 <: \ldots <: \alpha_r$, with a finite or an infinite tail of zeros, so that new sequences as $\alpha_1 <: \ldots <: \alpha_n$ or $\alpha_1 <: \alpha_2 <: \alpha_3 <: \ldots$ are obtained, with $\alpha_i = 0$ for $i > r$. The additional zeros are also considered as invariant factors.

As the first invariant factor, α_1, is the gcd of the elements of A, the following easy proposition characterizes the diagonals of the matrices equivalent to A.

Proposition. *Let A be an n-square matrix over R, $n \geqq 2$. Let $\delta_1, \ldots, \delta_n$ be elements of R, that are multiple of α_1. Then, A is equivalent to a matrix with diagonal $(\delta_1, \ldots, \delta_n)$.*

Proof. Let $\alpha_1 <: \alpha_2 <: \ldots$ be the invariant factors of A. If $n = 2$, it is easily seen that

$$\begin{bmatrix} \delta_1 & \alpha_1 \\ \delta_1(\delta_2/\alpha_1) - \alpha_2 & \delta_2 \end{bmatrix}$$

is equivalent to A. Therefore, the proposition is proved for $n = 2$. To prove the proposition for a general $n \geqq 3$, we proceed by induction, assuming that it holds for $n - 1$. Firstly, we apply the case $n = 2$ to prove that A is equivalent to

$$C = \begin{bmatrix} \delta_1, & \alpha_1 \\ \delta_1 - \alpha_2, & \alpha_1 \end{bmatrix} \dotplus \operatorname{diag}(\alpha_3, \ldots, \alpha_n),$$

Then, we take $(n - 1)$-square unimodular matrices, P and Q, such that the diagonal of $P \operatorname{diag}(\alpha_1, \alpha_3, \ldots, \alpha_n) Q$ is $(\delta_2, \ldots, \delta_n)$. The matrix $(1 \dotplus P) C (1 \dotplus Q)$ is equivalent to A and has the prescribed diagonal. □

Next, we characterize the diagonals of triangular matrices, equivalent to a given square matrix A.

Theorem. *Let A be an n-square matrix over R, with invariant factors $\alpha_1 <: \ldots <: \alpha_n$. Let $\delta_1, \ldots, \delta_n$ be n elements of R. Then, there exists a triangular matrix, equivalent to A, with diagonal $(\delta_1, \ldots, \delta_n)$, if and only if the following relations hold*:

$$\text{(1)} \qquad \alpha_1 \ldots \alpha_k <: \gcd\{\delta_{i_1} \ldots \delta_{i_k} : 1 \leqq i_1 < \ldots < i_k \leqq n\}, \quad k = 1, \ldots, n - 1,$$

$$\text{(2)} \qquad \alpha_1 \ldots \alpha_n =: \delta_1 \ldots \delta_n .$$

To prove our theorem we need the following basic result, that has been proved in [7] and [8], and that can be viewed as a consequence of a theorem in [1].

(3) *Let A and B be matrices over R, of dimensions $(n + p) \times (m + q)$ and $n \times m$, respectively $(p, q \geqq 0)$. Let $\alpha_1 <: \alpha_2 <: \ldots$ and $\beta_1 <: \beta_2 <: \ldots$ be their respective sequences of invariant factors. Then, A is equivalent to a matrix having B as a submatrix, if and only if the following relations hold*:

$$\text{(4)} \qquad \alpha_i <: \beta_i <: \alpha_{i+p+q}, \quad i = 1, 2, \ldots$$

Remark. R. C. THOMPSON [8] has already observed the striking analogy between the relations (4), and the interlacing inequalities for eigenvalues of hermitian matrices and for singular values of arbitrary complex matrices.

Also remarkable is the resemblance between our Theorem and a result by H. WEYL [9] and A. HORN [4], that can be stated as follows: *Let A be an n-square complex matrix, with singular values $\alpha_1 \geqq \alpha_2 \geqq \ldots \geqq \alpha_n \geqq 0$. Let $\delta_1, \ldots, \delta_n$ be complex numbers. Then, there exists a triangular matrix, unitarily equivalent to A, with diagonal $(\delta_1, \ldots, \delta_n)$, if and only if the following relations hold*:

$$\alpha_1 \ldots \alpha_k \geqq \sup\{|\delta_{i_1} \ldots \delta_{i_k}| : 1 \leqq i_1 < \ldots < i_k \leqq n\}, \quad k = 1, \ldots, n - 1,$$

$$\alpha_1 \ldots \alpha_n = |\delta_1 \ldots \delta_n| .$$

Though not so impressive, the analogy between our relations (1)–(2) and the so called Hardy-Littlewood-Polya inequalities

$$a_1 \leqq \ldots \leqq a_n \,, \tag{5}$$

$$a_1 + \ldots + a_k \leqq \min\{d_{i_1} + \ldots + d_{i_k} : 1 \leqq i_1 < \ldots < i_k \leqq n\}\,, \quad k = 1, \ldots, n-1\,, \tag{6}$$

$$a_1 + \ldots + a_n = d_1 + \ldots + d_n\,, \tag{7}$$

is worth noting as well. These inequalities characterize the diagonals, $(d_1, \ldots, d_n)$, of the hermitian matrices with prescribed eigenvalues $a_1 \leqq \ldots \leqq a_n$ [3]. The claimed analogy is easily seen in the case when A is nonsingular. For, given a prime $\pi \in R$, let us define the integers a_i and d_i as being the multiplicites of π in the prime factorization of α_i and δ_i, respectively. Then, (5)–(7) follow from $\alpha_1 <: \ldots <: \alpha_n$, (1) and (2).

As a matter of fact, we shall need in the sequel the following lemma, involving both the interlacing inequalities for eigenvalues of hermitian matrices and the Hardy-Littlewood-Polya inequalities.

Lemma. *Let $n \geqq 2$, $a_1, \ldots, a_n$ and $d_1, \ldots, d_n$ be integers such that* (5), (6) *and* (7) *are satisfied. Then the following system of inequalities*:

$$x_1 + \ldots + x_k \leqq \min\{d_{i_1} + \ldots + d_{i_k} : 1 \leqq i_1 < \ldots < i_k \leqq n-1\}\,, \quad k = 1, \ldots, n-2\,, \tag{8}$$

$$x_1 + \ldots + x_{n-1} = d_1 + \ldots + d_{n-1}\,, \tag{9}$$

$$a_1 \leqq x_1 \leqq a_2 \leqq x_2 \leqq \ldots \leqq x_{n-1} \leqq a_n\,, \tag{10}$$

has an integral solution, $x_1, \ldots, x_{n-1}$.

Proof. From (5)–(7), and from [3], it follows that there exists an $n \times n$ hermitian matrix H, having $a_1, \ldots, a_n$ as eigenvalues and $(d_1, \ldots, d_n)$ as main diagonal. Applying the Cauchy interlacing inequalities [5], and applying again [3], we deduce that the eigenvalues, $\hat{x}_1, \ldots, \hat{x}_{n-1}$, of the hermitian matrix obtained from H by deleting the last row and column, constitute a solution of (8)–(10).

It remains to prove that one of the solutions of (8)–(10), is integral. For, let us rewrite the system (8)–(10) in the form:

$$\begin{bmatrix} -\infty \\ -\infty \\ \vdots \\ -\infty \\ e_{n-1} \end{bmatrix} \leqq \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 1 & 1 & 0 & \ldots & 0 \\ 1 & 1 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ 1 & 1 & 1 & \ldots & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \end{bmatrix} \leqq \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ \vdots \\ e_{n-1} \end{bmatrix}, \quad \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \end{bmatrix} \leqq \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \end{bmatrix} \leqq \begin{bmatrix} a_2 \\ a_3 \\ a_4 \\ \vdots \\ a_n \end{bmatrix}, \tag{11}$$

where $e_k = \min\{d_{i_1} + \ldots + d_{i_k} : 1 \leqq i_1 < \ldots < i_k \leqq n-1\}$, $k = 1, \ldots, n-1$.

Using the same notation as in [2. p. 224], (11) is a system of the type:

$$b \leqq Ax \leqq b' , \quad c \leqq x \leqq c' ,$$

where b, b', c and c' are vectors, whose entries are integers ($\pm\infty$ allowed), and A is the triangular matrix of 0's and 1's, appearing in (11). Then, [2, Theorem 2] asserts that every vertex of the polyhedron

$$Q(b, b', c, c') = \{x : b \leqq Ax \leqq b', \ c \leqq x \leqq c'\}$$

has all integral coordinates, if and only if A has the *unimodular property* (i.e., every minor determinant of A equals 0, $+1$ or -1). It is easy to prove that our triangular matrix A has the unimodular property (apply, for instance, Theorem 5 of [2], with $V_2 = \emptyset$). The proof of the Lemma is now complete. □

Proof of the Theorem. It is a well known fact that the invariant factors of A can be given by:

$$\alpha_1 \dots \alpha_k = \gcd \{\text{minors of } A, \text{ of order } k\} , \quad k = 1, \dots, n .$$

If the α_k's are the invariant factors of a triangular matrix T, with diagonal $(\delta_1, \dots, \delta_n)$, then conditions (1)–(2) must hold. In fact, $\delta_{i_1} \dots \delta_{i_k}$ is the determinant of a $k \times k$ principal submatrix of T, for $1 \leqq k \leqq n$, $1 \leqq i_1 < \dots < i_k \leqq n$. Thus, the "only if" part of the Theorem is proved.

The converse will be easier to prove, if we show first that the ordering of the δ's is irrelevant. For, let T be a lower triangular matrix, with diagonal $(\delta_1, \dots, \delta_n)$, and let s be an integer, $1 \leqq s < n$. Consider the following block decomposition of T:

$$T = \begin{bmatrix} T_1 & 0 & 0 \\ F & T_2 & 0 \\ G & H & T_3 \end{bmatrix} , \quad \text{with} \quad T_2 = \begin{bmatrix} \delta_s & 0 \\ \gamma & \delta_{s+1} \end{bmatrix} ,$$

where the diagonal blocks are square. As

$$T_2' = \begin{bmatrix} \delta_{s+1} & 0 \\ \gamma & \delta_s \end{bmatrix}$$

is equivalent to T_2, there exist 2×2 unimodular matrices, U and V, such that $UT_2V = T_2'$. Therefore, T is equivalent to the triangular matrix

$$T' = \begin{bmatrix} T_1 & 0 & 0 \\ UF & T_2' & 0 \\ G & HV & T_3 \end{bmatrix}$$

the diagonal of which is $(\delta_1, \dots, \delta_{s-1}, \delta_{s+1}, \delta_s, \delta_{s+2}, \dots, \delta_n)$. As any permutation, σ, is the product of transpositions, then T equivalent to a triangular matrix with diagonal $(\delta_{\sigma(1)}, \dots, \delta_{\sigma(n)})$.

Thus, we shall assume, from now on, that the δ's are ordered in such a way that $\delta_n = 0$ iff $\delta_1 \delta_2 \ldots \delta_n = 0$.

The sufficiency of conditions (1)–(2) will be proved by induction on n. The case $n = 1$ is trivial. Now, assume that the Theorem is true for $(n-1) \times (n-1)$ matrices $(n - 1 \geqq 1)$, and let $\alpha_1 <: \ldots <: \alpha_n$ and $\delta_1, \ldots, \delta_n$ satisfy (1)–(2). Our problem would be solved if we could find $n - 1$ elements of R – $\chi_1, \ldots, \chi_{n-1}$ – such that

$$(12) \quad \chi_1 \ldots \chi_k <: \gcd \{\delta_{i_1} \ldots \delta_{i_k} : 1 \leqq i_1 < \ldots < i_k \leqq n-1\}, \quad k = 1, \ldots, n-2,$$

$$(13) \qquad \chi_1 \ldots \chi_{n-1} =: \delta_1 \ldots \delta_{n-1},$$

$$(14) \qquad \alpha_1 <: \chi_1 <: \alpha_2 <: \chi_2 <: \ldots <: \chi_{n-1} <: \alpha_n .$$

To see that these conditions are what we need, suppose, for a moment, that we have found $\chi_1, \ldots, \chi_{n-1}$ satisfying (12)–(14). Then, by (14) and the result (3) we may extend the matrix

$$D = \begin{bmatrix} \chi_1 & & & 0 \\ & \chi_2 & & \\ & & \ddots & \\ 0 & & & \chi_{n-1} \end{bmatrix}, \quad \text{to a matrix} \quad A' = \left[\begin{array}{ccc|c} \chi_1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & \chi_{n-1} & 0 \\ \hline \xi_1 & \ldots & \xi_{n-1} & \xi_n \end{array}\right],$$

in such a way that A' has invariant factors $\alpha_1, \ldots, \alpha_n$. By induction, D is equivalent to a (lower) triangular matrix, with diagonal $(\delta_1, \ldots, \delta_{n-1})$. Therefore, A' is equivalent to a triangular matrix T, with diagonal $(\delta_1, \ldots, \delta_{n-1}, \xi_n)$. Moreover, we can choose $\xi_n = \delta_n$. For, if $\chi_{n-1} \neq 0$, then (2), (13) and $\chi_1, \ldots, \chi_{n-1} \neq 0$ imply that $\xi_n = \omega \delta_n$, where ω is a unit of R. The factor ω is removed, if we multiply the last column of T by $1/\omega$. If $\chi_{n-1} = 0$, then $\alpha_n = \delta_n = 0$. Let ξ denote the gcd $\{\xi_{n-1}, \xi_n\}$. The matrix $[\xi_{n-1}, \xi_n]$ is, then, equivalent to $[\xi, 0]$. Therefore, the last row of A' can be changed to $[\xi_1, \ldots, \xi_{n-2}, \xi, 0]$, without changing the invariant factors.

Thus, it remains to prove that, under conditions (1)–(2), there exist $\chi_1, \ldots, \chi_{n-1}$ subject to (12)–(14). We split the proof into two cases.

Case 1. When $\delta_n = 0$. Then, condition (2) implies that $\alpha_n = 0$. The relation $\alpha_1 \ldots \alpha_{n-1} <: \delta_1 \ldots \delta_{n-1}$ follows from (1). Therefore, there exists $\varrho \in R$, such that $\alpha_1 \ldots \alpha_{n-1} \varrho = \delta_1 \ldots \delta_{n-1}$. Define $\chi_1, \ldots, \chi_{n-1}$ by: $\chi_1 = \alpha_1, \ldots, \chi_{n-2} = \alpha_{n-2}$ and $\chi_{n-1} = \varrho \alpha_{n-1}$. It is clear that these χ's satisfy (12)–(14).

Case 2. When $\delta_n \neq 0$, i.e., $\delta_1 \ldots \delta_n \neq 0$. Then every α_i is nonzero, $i = 1, \ldots, n$. Let $\mathscr{P}$ be the set of the irreducible factors of the product $\alpha_1 \ldots \alpha_n \delta_1 \ldots \delta_n$. Given $\pi \in \mathscr{P}$, denote by $a_i(\pi)$ and $d_i(\pi)$ the exponents of π in the irreducible factorization of α_i and δ_i, respectively, $i = 1, \ldots, n$. By the Lemma, there exist integers $x_1(\pi), \ldots$

$\ldots, x_{n-1}(\pi)$, such that

$$x_1(\pi) + \ldots + x_k(\pi) \leqq \min \{d_{i_1}(\pi) + \ldots + d_{i_k}(\pi) : 1 \leqq i_1 < \ldots < i_k \leqq n-1\}, \quad k = 1, \ldots, n-2, \tag{15}$$

$$x_1(\pi) + \ldots + x_{n-1}(\pi) = d_1(\pi) + \ldots + d_{n-1}(\pi), \tag{16}$$

$$a_1(\pi) \leqq x_1(\pi) \leqq a_2(\pi) \leqq \ldots \leqq x_{n-1}(\pi) \leqq a_n(\pi). \tag{17}$$

If we denote the powers $\pi^{a_i(\pi)}$, $\pi^{d_i(\pi)}$ and $\pi^{x_j(\pi)}$, respectively by $\alpha_i(\pi)$, $\delta_i(\pi)$ and $\chi_j(\pi)$, for $1 \leqq i \leqq n$, $1 \leqq j < n$, then (15)−(17) can be written as

$$\chi_1(\pi) \ldots \chi_k(\pi) <: \gcd \{\delta_{i_1}(\pi) \ldots \delta_{i_k}(\pi) : 1 \leqq i_1 < \ldots < i_k \leqq n-1\}, \quad k = 1, \ldots, n-2, \tag{18}$$

$$\chi_1(\pi) \ldots \chi_{n-1}(\pi) = \delta_1(\pi) \ldots \delta_{n-1}(\pi), \tag{19}$$

$$\alpha_1(\pi) <: \chi_1(\pi) <: \alpha_2(\pi) <: \ldots <: \chi_{n-1}(\pi) <: \alpha_n(\pi). \tag{20}$$

If we define χ_i by $\chi_i = \prod_{\pi \in \mathcal{P}} \chi_i(\pi)$, $i = 1, \ldots, n-1$, it is a simple exercise to prove that (12)−(14) follow from (18)−(20).

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Author's address: Instituto de Matemática, Universidade de Coimbra, Portugal.