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POWERS AND GEVREY'S REGULARITY FOR A SYSTEM OF DIFFERENTIAL OPERATORS

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The purpose of this paper is to give some results about powers and Gevrey regularity in the interior and up to boundary for a system of differential operators, which are, in particular, extensions of those of KOTAKE-NARASHIMAN [8] and NELSON [11].

I — POWERS AND G_S REGULARITY

First, we recall the definition (or characterization) of the analyticity of a function:

Definition I-1. A function u, C^∞ in an open set of $\mathbb{R}^n$, is analytic in Ω if, for every compact set K of Ω, there exists a constant $L = L_K > 0$ such that, for every $\alpha \in \mathbb{N}^n$, we have:

$$\|D^\alpha u\|_{L^2(K)} \leqq L^{|\alpha|+1}(|\alpha|!)$$

where we write, for $\alpha = (\alpha_1, \ldots, \alpha_n)$,

$$|\alpha| = \alpha_1 + \ldots + \alpha_n \quad \text{and} \quad D^\alpha = \mathrm{i}^{-|\alpha|}\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} .$$

We denote by $a(\Omega)$ the space of analytic functions in Ω.

In [8], Kotake and Narashiman characterize the analyticity with the help of powers of an elliptic operator in the following manner:

Theorem 0. *Let P be an elliptic differential operator of order $m \geqq 1$ with analytic coefficients in an open set Ω of $\mathbb{R}^n$. Then the following two propositions are equivalent*:

(i) $u \in a(\Omega)$;

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(ii) $u \in C^\infty(\Omega)$ *and, for every compact set K of* Ω, *there exists a constant* $L = L_K > 0$ *such that, for every* $k \in \mathbb{N}$, *we have*:

$$\|P^k u\|_{L^2(K)} \leqq L^{k+1}((mk)!) .$$

In [11], Nelson characterizes the analyticity with the help of powers of n real vector fields linearly independent in the following manner:

Theorem 0′. *Let* $P_1, \dots, P_n$ *be real vector fields with analytic coefficients and linearly independent at every point of an open set* Ω *of* $\mathbb{R}^n$. *Then the following two propositions are equivalent:*

(i) $u \in a(\Omega)$;

(ii) $u \in C^\infty(\Omega)$ *and, for every compact set K of* Ω, *there exists a constant* $L = L_K > 0$ *such that, for every* $1 \leqq i_j \leqq n$, $1 \leqq j \leqq k$ *and* $k \geqq 1$, *we have*:

$$\|P_{i_1} \dots P_{i_k} u\|_{L^2(K)} \leqq L^{k+1}(k!) .$$

The purpose of this paper is to extend these results concerning more general operators and Gevrey's classes of order $s \geqq 1$ in the interior and also up to the boundary.

We recall the definition of Gevrey's classes:

Definition 2. Let K be a compact set in $\mathbb{R}^n$ and S a real number $\geqq 1$. By *Gevrey's class of order S in K* we mean the space $G_S(K)$ of the restrictions over K of C^∞ functions u in a neighbourhood of K such that there exists a constant $L > 0$ such that, for every $\alpha \in \mathbb{N}^n$, we have:

$$\|D^\alpha u\|_{L^2(K)} \leqq L^{|\alpha|+1}(|\alpha|!)^S .$$

Let Ω be an open set of $\mathbb{R}^n$; by *Gevrey's class of order S in* Ω we mean the space $G_S(\Omega)$ of functions which are in $G_S(K)$ for every compact subset K of Ω.

If K is "smooth enough", we can replace the $L^2(K)$-norm by the $L^\infty(K)$-norm. For $S = 1$, we get of course analytic functions.

Let Ω be an open set in $\mathbb{R}^n$ with boundary $\partial\Omega$ and $P_j \equiv P_j(x; D)$, $j = 1, \dots, N$, differential operators of order $m_j \in \mathbb{N}$. Let the principal part of order m_j of P_j be denoted by $P'_j = P'_j(x; D)$; we introduce the following two conditions:

(A) for every $x \in \Omega$, the polynomials $P'_j(x; \xi)$ for $1 \leqq j \leqq N$ have no common non trivial real zero;

(B) for every $x \in \partial\Omega$, the polynomials $P'_j(x; \xi)$ for $1 \leqq j \leqq N$ have no common non trivial complex zero.

First of all, we have the following theorem on powers in Gevrey's classes $G_S(\Omega)$, which generalizes the Kotaké-Narashiman and Nelson's theorems:

Theorem 1. *If the operators P_j, $j = 1, \ldots, N$, have coefficients in $G_S(\Omega)$ and satisfy the condition* (A), *then the following two propositions are equivalent*:

(i) $u \in G_S(\Omega)$;

(ii) $u \in C^\infty(\Omega)$ *and for every compact subset K of Ω, there exists a constant $L = L_K > 0$ such that, for every $1 \leqq i_j \leqq N$, $1 \leqq j \leqq k$ and $k \geqq 1$, we have*:

$$\|P_{i_1} \ldots P_{i_k} u\|_{L^2(K)} \leqq L^{k+1}((\sum_{j=1}^{k} m_{i_j})!)^S .$$

We have also the following result which is a result on powers in Gevrey's classes $G_S(\bar{\Omega})$:

Theorem 2. *If Ω is a bounded open set in $\mathbb{R}^n$ with Lipschitzian boundary, if the operators P_j for $1 \leqq j \leqq N$ have coefficients in $G_S(\bar{\Omega})$ and satisfy the conditions* (A) *and* (B), *then the following two propositions are equivalent*:

(i) $u \in G_S(\bar{\Omega})$;

(ii) $u \in C^\infty(\bar{\Omega})$ *and there exists a constant $L > 0$ such that, for every $1 \leqq i_j \leqq N$, $1 \leqq j \leqq k$ and $k \geqq 1$, we have*:

$$\|P_{i_1} \ldots P_{i_k}\|_{L^2(\Omega)} \leqq L^{k+1}((\sum_{j=1}^{k} m_{i_j})!)^S .$$

We recall that an open set Ω in $\mathbb{R}^n$ with Lipschitzian boundary $\partial\Omega$ is an open set such that, for every point $x_0 \in \partial\Omega$, there exists a real number $r > 0$, a system of local coordinates $(x_1, \ldots, x_n)$ and a Lipschitzian function $h = h(x_1, \ldots, x_{n-1})$ such that

$$\Omega \cap B(x_0, r) = \{(x_1, \ldots, x_n);\ x_n > h(x_1, \ldots, x_{n-1})\} \cap B(x_0, r)$$

where $B(x_0, r)$ is a ball with center x_0 and radius r.

The implications (i) $\Rightarrow$ (ii) are always true and easy to prove. The method used to prove the implication (ii) $\Rightarrow$ (i) in Theorem 2 (as well as in Theorem 1) is an adaptation of that of Kotaké-Narashiman [8] using the tools of Morrey-Nirenberg [10].

First, we may consider only operators with the same order m. In fact, for $j = 1, \ldots, N$, we put $\hat{m}_j = \prod_{i \neq j} m_i$ and $Q_j = P_j^{\hat{m}_j}$. The operators $Q_j = Q_j(x; D)$ for $1 \leqq j \leqq N$ have the order $m = \prod_{j=1}^{N} m_j$ and satisfy the conditions (A) and (B) if and only if the operators P_j for $j = 1, \ldots, N$ satisfy the conditions (A) and (B). Moreover, if $u \in C^\infty(\bar{\Omega})$ and if there exists a constant $L > 0$ such that, for every $1 \leqq i_j \leqq N$, $1 \leqq j \leqq k$ and $k \geqq 1$, we have:

$$\|P_{i_1} \ldots P_{i_k} u\|_{L^2(\Omega)} \leqq L^{k+1}((\sum_{j=1}^{k} m_{i_j})!)^S ,$$

then we have also:

$$\|Q_{i_1} \dots Q_{i_k} u\|_{L^2(\Omega)} \leqq L'^{k+1}((km)!)^S$$

with $L' = (\max(L, 1))^m$.

Thus, in the following we assume that all the operators P_j have the same order m.

The starting point of the proof is a global a priori estimate which is given in ARONSZAJN [2], SMITH [12] (cf. also Bolley-Camus [3]):

Proposition I-1. *Under the assumptions of Theorem 2, for every $k \geqq 1$ there exists a constant $L > 0$ such that, for every $u \in C^\infty(\overline{\Omega})$, we have:*

$$\|u\|_{H^k(\Omega)} \leqq C \,.\, \Big\{ \sum_{j=1}^{N} \|P_j u\|_{H^{k-m}(\Omega)} + \|u\|_{L^2(\Omega)} \Big\} \,.$$

By localization, we are going to deduce two other a priori estimates.

Proposition I-2. *Under the assumptions of Theorem 2, for every $x \in \overline{\Omega}$, for every open neighbourhoods W and W' of x in $\overline{\Omega}$, W' being relatively compact in W, there exists a constant $A > 0$ such that, for every $u \in C^\infty(W)$, we have:*

$$\|u\|_{H^m(W')} \leqq A \,.\, \Big\{ \sum_{j=1}^{N} \|P_j u\|_{L^2(W)} + \|u\|_{L^2(W)} \Big\} \,.$$

Proof. By Proposition I-1, there exists a constant $C > 0$ such that, for every $u \in C^\infty(W)$ and $1 \leqq k \leqq m$, we have:

$$\|u\|_{H^k(W)} \leqq C \,.\, \Big\{ \sum_{j=1}^{N} \|P_j u\|_{H^{-m+k}(W)} + \|u\|_{L^2(W)} \Big\} \,.$$

We are going to deduce Proposition I-2 from this estimate by proving by induction on p, for $1 \leqq p \leqq m$, that there exists a constant $C_p > 0$ and a function $\Phi_p \in C_0^\infty(W)$ equal to 1 on $\overline{W}'$ such that, for every function $u \in C^\infty(W)$, we have:

(p) $$\|u\|_{H^m(W')} \leqq C_p \,.\, \Big\{ \sum_{j=1}^{N} \|P_j u\|_{L^2(W)} + \|u\|_{L^2(W)} + \|\Phi_p u\|_{H^{m-p}(W)} \Big\} \,.$$

For $p = 1$, we consider a function $\Phi_0 \in C_0^\infty(W)$ equal to 1 on $\overline{W}'$; then, if $u \in C^\infty(W)$, the preceding estimate written with $k = m$ implies

$$\|u\|_{H^m(W')} \leqq \|\Phi_0 u\|_{H^m(W)} \leqq C \,.\, \Big\{ \sum_{j=1}^{N} \|P_j(\Phi_0 u)\|_{L^2(W)} + \|\Phi_0 u\|_{L^2(W)} \Big\} \,.$$

However, $P_j(\Phi_0 u) = \Phi_0 P_j u - [P_j, \Phi_0]\, \Phi_1 u$ where $\Phi_1 \in C_0^\infty(W)$ is equal to 1 on the support of Φ_0 and $[P_j, \Phi_0]$ means the commutator of P_j and Φ_0. Hence,

$$\|P_j(\Phi_0 u)\|_{L^2(W)} \leqq C_1' \,.\, \big\{ \|P_j u\|_{L^2(W)} + \|\Phi_1 u\|_{H^{m-1}(W)} \big\}$$

for $1 \leqq j \leqq N$; then we get (1).

Suppose (p) is true and show $(p + 1)$ if $p + 1 \leqq m$.

From the preceding estimate written with $k = m - p$, we get for every $u \in C^\infty(W)$:

$$\|\Phi_p u\|_{H^{m-p}(W)} \leqq C \cdot \{\sum_{j=1}^{N} \|P_j(\Phi_p u)\|_{H^{-p}(W)} + \|\Phi_p u\|_{L^2(W)}\} .$$

Writing $P_j(\Phi_p u) = \Phi_p P_j u + [P_j, \Phi_p] \Phi_{p+1} u$ where $\Phi_{p+1} \in C_0^\infty(W)$ is equal to 1 on the support of Φ_p. Hence,

$$\|P_j(\Phi_p u)\|_{H^p(W)} \leqq C'_{p+1} \cdot \{\|P_j u\|_{L^2(W)} + \|\Phi_{p+1} u\|_{H^{m-(p+1)}(W)}\}$$

for $1 \leqq j \leqq N$, which yields $(p + 1)$.

In particular, the inequality (m) is exactly the inequality of Proposition I-2.

In the second step, we establish an other a priori estimate localized for some particular open sets W and W'. To this end, we need some notations: let x be a point in $\bar{\Omega}$, $0 \leqq \varrho < R < R_1$;

$$W = \Omega \cap B(x; R_1), \qquad \hat{W} = \bar{\Omega} \cap B(x; R_1),$$

$$W_\varrho = \Omega \cap B(x; R - \varrho), \qquad \hat{W}_\varrho = \bar{\Omega} \cap B(x; R - \varrho).$$

Then we have the following refined a priori estimate:

Proposition I-3. *Under the assumptions of Theorem 2, for every $x \in \bar{\Omega}$ and $0 < R < R_1$ there exists a constant $C > 0$ such that, for every $u \in C^\infty(W)$, for every $\alpha \in \mathbb{N}^n$ with $|\alpha| \leqq m$, ϱ and $\varrho' > 0$ with $\varrho + \varrho' < R$ and $\varrho \leqq 1$, we have:*

$$\varrho^m \|D^\alpha u\|_{L^2(W_{\varrho+\varrho'})} \leqq C \cdot \{\varrho^m \sum_{j=1}^{N} \|P_j u\|_{L^2(W_{\varrho'})} + \sum_{|\beta| \leqq m-1} \varrho^{|\beta|} \|D^\beta u\|_{L^2(W_{\varrho'})}\} .$$

Proof. We consider a function $\varphi \in C_0^\infty(\hat{W}_{\varrho'})$ such that $0 \leqq \varphi \leqq 1$, $\varphi = 1$ on $W_{\varrho+\varrho'}$, $\|D^\alpha \varphi\|_{L^\infty(W_0)} \leqq C_\alpha \varrho^{-|\alpha|}$ where C_α is a constant which depends on α and not on x, ϱ and ϱ'.

We apply Proposition I-1 to the function φu for $u \in C^\infty(\hat{W})$:

$$\|D^\alpha(\varphi u)\|_{L^2(W_0)} \leqq A \cdot \{\sum_{j=1}^{N} \|P_j(\varphi u)\|_{L^2(W_0)} + \|u\|_{L^2(W_0)}\}$$

for $|\alpha| \leqq m$.

On the other hand, if we put

$$P_j = P_j(x; D) = \sum_{|\lambda| \leqq m} a_{j\lambda}(x) D^\lambda ,$$

we have

$$P_j(\varphi u) - \varphi P_j u = \sum_{\substack{\beta < \lambda \\ |\lambda| \leqq m}} a_{j\lambda} \binom{\lambda}{\beta} D^{\lambda-\beta} D^\beta u .$$

However, there exist constants $C_{j,\lambda,\beta} > 0$, independent of ϱ, such that

$$\left\| a_{j\lambda} \binom{\lambda}{\beta} D^{\lambda-\beta} \varphi \right\|_{L^\infty(W_0)} \leqq C_{j,\lambda,\beta} \varrho^{-|\lambda-\beta|} .$$

Then

$$\|D^\alpha(\varphi u)\|_{L^2(W_0)} \leqq A' \cdot \{\sum_{j=1}^{N} \|P_j u\|_{L^2 W_{\varrho'})} + \sum_{\substack{|\beta|<|\lambda| \\ |\lambda| \leqq m}} \varrho^{-|\lambda|+|\beta|} \|D^\alpha u\|_{L^2(W_{\varrho'})}\}$$

and, since $\varrho \leqq 1$, we have

$$\|D^\alpha(\varphi u)\|_{L^2(W_0)} \leqq A' \cdot \{\sum_{1 \leqq j}^{N} \|P_j u\|_{L^2(W_{\varrho'})} + \sum_{\substack{|\beta|<|\lambda| \\ |\lambda| \leqq m}} \varrho^{-m+|\beta|} \|D^\beta u\|_{L^2(W_{\varrho'})}\},$$

which yields the inequality of Proposition I-3.

We now use induction on this inequality to obtain an estimate of one derivative of u in terms of some powers of $P_j u$:

Proposition I-4. *Under the assumptions of Theorem* 2, *for every* $x \in \bar{\Omega}$, $0 < R < R_1$ *there exists a constant* $A \geqq 1$ *such that, for every* ϱ *with* $0 < \varrho < \min(1, R)$, *every* $u \in C^\infty(W)$, *every* $\alpha \in \mathbb{N}^n$ *with* $|\alpha| \leqq km$ *and* $k \geqq 1$, *we have*:

$$\varrho^{|\alpha|S} \|D^\alpha u\|_{|L^2(W_{|\alpha|\varrho})} \leqq A^{|\alpha|+1} \cdot \{\sum_{\nu=1}^{k} \varrho^{(\nu-1)mS} \sum_{\substack{1 \leqq i_j \leqq N \\ 1 \leqq j \leqq \nu}} \|P_{i_1} \dots P_{i_\nu} u\|_{L^2(W)} + \|u\|_{L^2(W)}\}.$$

Proof. The coefficients $a_{j\nu}$ of the operators P_j being in the class $G_S(\bar{\Omega})$, there exists a constant $B > 0$ such that, for every $\alpha \in \mathbb{N}^n$, we have:

$$\sum_{j=1}^{N} \sum_{|\lambda| \leqq m} \|D^\alpha a_{j\lambda}\|_{L^\infty(W_0)} \leqq B^{|\alpha|+1}(\alpha!)^S;$$

then

$$\sum_{j=1}^{N} \sum_{|\lambda| \leqq m} \|D^\alpha a_{j\lambda}\|_{L^\infty(W_0)} \leqq B^{|\alpha|+1}(\alpha!)^S \varrho^{-|\alpha|S}.$$

We put

$$S_k(u) = S_k(u; \varrho) = \sum_{\nu=1}^{k} \varrho^{(\nu-1)mS} \sum_{\substack{1 \leqq i_j \leqq N \\ 1 \leqq j \leqq \nu}} \|P_{i_1} \dots P_{i_\nu} u\|_{L^2(W)} + \|u\|_{L^2(W)}.$$

Then we have

$$\sum_{j=1}^{N} \varrho^{mS} S_k(P_j u) \leqq S_{k+1}(u)$$

and

$$S_k(u) \leqq S_{k+1}(u).$$

We now prove the inequality in Proposition I-4 by induction on k. First, the inequality from Proposition I-2 gives

$$\|D^\alpha u\|_{L^2(W_0)} \leqq A \cdot \{\sum_{j=1}^{N} \|P_j u\|_{L^2(W)} + \|u\|_{L^2(W)}\}$$

for $|\alpha| \leqq m$.

We can choose $A \geqq 1$ and since $\varrho \leqq 1$, we have the inequality from Proposition I-4 for $k = 1$.

Let $\alpha \in \mathbb{N}^n$ be such that $km < |\alpha| \leqq (k+1)m$ and assume the inequality from Proposition I-4 to be proved for every $\beta \in \mathbb{N}^n$ such that $|\beta| \leqq |\alpha| - 1$. We put $\alpha = \alpha_0 + \alpha'$ with $|\alpha_0| = m$. We use the inequality of Proposition I-3 with $(|\alpha| - 1)\varrho$ instead of ϱ', α_0 instead of α and $D^{\alpha'}u$ instead of u, which yields

$$\varrho^{|\alpha|S} \| D^\alpha u \|_{L^2(W_{|\alpha|\varrho})} \leqq C \,.\, \{\varrho^{|\alpha|S} \sum_{j=1}^{N} \| P_j(D^{\alpha'}u) \|_{L^2(W_{(|\alpha|-1)\varrho})} + \\ + \sum_{|\beta| \leqq m-1} \varrho^{|\alpha|S - m + |\beta|} \| D^{\beta + \alpha''} u \|_{L^2(W_{(|\alpha|-1)\varrho})}\} \,.$$

However, we have

$$D^{\alpha'}(P_j u) - P_j(D^{\alpha'}u) = \sum_{|\lambda| \leqq m} \sum_{\gamma \leqq \alpha'} \binom{\alpha'}{\gamma} D^{\alpha' - \gamma} a_{j\lambda} D^{\gamma + \lambda} u \,,$$

$$\sum_{j=1}^{N} \| D^{\alpha' - \gamma} a_{j\lambda} \|_{L^2(W_{km\varrho})} \leqq B^{|\alpha' - \gamma| + 1} ((\alpha' - \gamma)!)^S (mk\varrho)^{-|\alpha' - \gamma|S}$$

and

$$\binom{\alpha'}{\gamma} \frac{((\alpha' - \gamma)!)^S}{mk^{|\alpha' - \gamma|S}} \leqq \left(\binom{\alpha'}{\gamma} \frac{(\alpha' - \gamma)!}{(mk)^{|\alpha' - \gamma|}} \right)^S \leqq \left(\frac{|\alpha'|}{mk} \right)^{|\alpha' - \gamma|S} \leqq 1$$

since $|\alpha'| = |\alpha| - m \leqq km$.

Hence,

$$D^{\alpha'}(P_j u) - P_j(D^{\alpha'}u)\|_{L^2(W_{(|\alpha|-1)\varrho})} \leqq \sum_{|\lambda| \leqq m} \sum_{\gamma < \alpha''} B^{|\alpha' - \gamma| + 1} \varrho^{-|\alpha' - \gamma|S} \| D^{\gamma + \lambda} u \|_{L^2(W_{(|\alpha|-1)\varrho})}$$

and thus, for $km < |\alpha| \leqq (k+1)m$, we have:

$$\varrho^{|\alpha|S} \| D^\alpha u \|_{L^2(W_{|\alpha|\varrho})} \leqq C \,.\, \{\varrho^{|\alpha|S} \sum_{j=1}^{N} \| D^{\alpha'} P_j u \|_{L^2(W_{|\alpha'|\varrho})} + \\ + \sum_{|\beta| < m} \varrho^{|\alpha|S - m + |\beta|} \| D^{\beta + \alpha'} u \|_{L^2(W_{|\beta + \alpha'|\varrho})} + \\ + \sum_{|\lambda| \leqq m} \sum_{\gamma < \alpha'} \varrho^{(m + |\gamma|)S} B^{|\alpha' - \gamma| + 1} \| D^{\gamma + \lambda} u \|_{L^2(W_{(m + |\gamma|)\varrho})}\} \,.$$

We can now apply the induction assumption to estimate each term on the right hand side of this inequality; the first term is

$$\leqq \varrho^{mS} A^{|\alpha'| + 1} \sum_{j=1}^{N} S_k(P_j u) \leqq A^{|\alpha'| + 1} S_{k+1}(u) \,,$$

the second term is

$$\leqq \sum_{|\beta| \leqq m} A^{|\beta + \alpha'| + 1} S_{k+1}(u) \,,$$

and the third term is

$$\leqq \sum_{|\lambda| \leqq m} \sum_{\gamma < \alpha'} B^{|\alpha' - \gamma| + 1} A^{m + |\gamma| + 1} S_{k+1}(u) \,.$$

Then we have

$$\varrho^{|\alpha|S}\|D^{\alpha}u\|_{L^2(W_{|\alpha|\varrho})} \leqq A^{|\alpha|+1}S_{k+1}(u)\{CA^{-m} + C\sum_{|\beta|<m} A^{-1} + $$
$$+ \sum_{|\lambda|\leqq m}\sum_{\gamma<\alpha'} B^{|\alpha'-\gamma|+1}A^{-|\alpha'-\gamma|}\} .$$

However,

$$C\sum_{|\lambda|\leqq m}\sum_{\gamma<\alpha'} B^{|\alpha'-\gamma|+1}A^{-|\alpha'-\gamma|} \leqq C\,.\,m^n B^2 A^{-1}\sum_{|\beta|\geqq 0}(BA^{-1})^{|\beta|} .$$

We can choose A large enough, independent of α and ϱ, in order to make the term between the brackets $\leqq 1$, which completes the proof of Proposition I-4.

Now we can present the result about the powers "locally up to the boundary":

Proposition I-5. *Under the assumptions of Theorem 2, if $x \in \bar{\Omega}$ and $u \in C^{\infty}(\bar{\Omega}) \cap \cap B(x; R_2)$ is such that, for every open neighbourhood U of x in $\bar{\Omega}$ with U relatively compact in $\bar{\Omega} \cap B(x; R_2)$, there exists a constant $L = L_U > 0$ such that, for every $1 \leqq i_j \leqq N$, $1 \leqq j \leqq k$ and $k \geqq 1$, we have:*

$$\|P_{i_1} \dots P_{i_k}u\|_{L^2(U)} \leqq L^{k+1}(km!)^S ,$$

then $u \in G_S(\bar{\Omega} \cap B(x; R_2))$.

Proof. We fix $R' < R_2$ and put $U' = \Omega \cap B(x; R_2)$. We want to show that $u \in \in G_S(\bar{U}')$. We choose R_1 and R such that $R' < R_1 < R_2$ and keeping the notation used in Proposition I-4, we have

$$\|P_{i_1} \dots P_{i_k}u\|_{L^2(W)} \leqq L^{k+1}(km!)^S ,$$

hence

$$S_k(u) \leqq \sum_{\nu=1}^{k} \varrho^{(\nu-1)mS}N^{\nu}L^{\nu+1}((\nu m)!)^S + L$$

for every ϱ such that $0 < \varrho < \text{Min}\,(1, R)$.

We choose $\varrho = (R - R')/km$, $R - R'$ being small enough; then we get

$$((\nu m)!)^S\,\varrho^{(\nu-1)mS} \leqq (km)^{mS}$$

for $\nu \leqq k$.

Therefore, there exists a constant $B_1 > 0$ such that

$$S_k(u) \leqq \sum_{\nu=1}^{k} N^{\nu}\,L^{\nu+1}(km)^{mS} + L \leqq B_1^{k+1}$$

for $k \geqq 1$.

By Proposition I-4, there exists a constant $B_2 > 0$ such that, for $\alpha \in \mathbb{N}^n$ with $|\alpha| \leqq km$ and $k \geqq 1$, we have:

$$\|D^{\alpha}u\|_{L^2(W_{R-R'})} \leqq B_2^{k+1}k^{kS} .$$

In particular, if we apply this formula for $|\alpha| = k$, we get, for every $\alpha \in \mathbb{N}^n$:

$$\|D^\alpha u\|_{L^2(U')} \leqq B_2^{|\alpha|+1}|\alpha|^{|\alpha|S},$$

which yields $u \in G_S(\overline{U}')$.

Theorem 2, the assertion (ii) $\Rightarrow$ (i), is proved.

Remark I-1. In the case when $\overline{\Omega}$ is a C^∞ compact manifold with boundary, the condition (B) can be replaced, in Theorem 2, by the following condition:
(B′) for every $x \in \partial\Omega$, the polynomials $P_j'(x; \xi)$ for $1 \leqq j \leqq N$ have no common non trivial complex zero with imaginary part orthogonal to $\partial\Omega$ in x.

Remark I-2. By the same method, the inequalities of coerciveness given in AGMON [1] allow to obtain some similar results about powers in the classes $G_S(\overline{\Omega})$ for boundary value problems associated with some systems $(P_1, \ldots, P_N; B_1, \ldots, B_p)$ where P_j are differential operators and B_j are differential operators at the boundary; the case when the system of P_j is reduced to a single operator is that which was studied by LIONS-MAGENES [9] while the case when the system of B_j is empty is the case that we have studied here.

II — G_S-REGULARITY

The following corollary about the $G_S(\overline{\Omega})$-regularity is a consequence of Theorem 1:

Corollary II-1. *Under the assumptions of Theorem* 1, *the following two propositions are equivalent*:

(i) $u \in G_S(\Omega)$;

(ii) $u \in C^\infty(\Omega)$ *and* $P_j u \in G_S(\Omega)$ *for* $1 \leqq j \leqq N$.

From Theorem 2 we get the following corollary about the $G_S(\overline{\Omega})$-regularity:

Corollary II-2. *Under the assumptions of Theorem* 2, *the following two propositions are equivalent*:

(i) $u \in G_S(\overline{\Omega})$;

(ii) $u \in C^\infty(\overline{\Omega})$ *and* $P_j u \in G_S(\overline{\Omega})$ *for* $1 \leqq j \leqq N$.

Remark II-1. Using the results on regularity given by SMITH [11] (cf. also Bolley-Camus [3]), we can replace $u \in C^\infty(\overline{\Omega})$ by $u \in \mathscr{D}'(\Omega)$ in Corollary II-2. In the same way, we can replace $u \in C^\infty(\Omega)$ by $u \in \mathscr{D}'(\Omega)$ in Corollary I-1, using the ellipticity of the operator $\sum_{j=1}^{N} P_j^* P_j$ in Ω.

It is easy to see that neither the condition (A) for Corollary II-1 nor the conditions (A) and (B) (or (B′)) for Corollary II-2 are necessary.

When the operators $P_j = P_j(D)$ have constant coefficients, we introduce the following condition:
(C) The set of complex common roots ξ of the polynomials $P_j(\xi)$, for $1 \leqq j \leqq N$, is finite.

Then we have the following necessary and sufficient condition of $G_S(\Omega)$-regularity:

Theorem II-1. *Let Ω be a bounded open set in $\mathbb{R}^n$ with Lipschitzian boundary, and let P_j be operators with constant coefficients, $1 \leqq j \leqq N$; then the following two propositions are equivalent*:

(i) *The space* $\{u \in \mathcal{D}'(\Omega);\ P_j u \in G_S(\bar{\Omega}),\ 1 \leqq j \leqq N\}$ *is the space* $G_S(\bar{\Omega})$;
(ii) *the operators* P_j, $1 \leqq j \leqq N$, *satisfy the condition* (C).

The proof given in the case of the space $C^\infty(\Omega)$ in Bolley-Camus [3] can be applied to the space $G_S(\bar{\Omega})$. We recall it here.

Proof. We assume that (i) is true. We introduce the space

$$Y(\Omega) = \{u \in \mathcal{D}'(\Omega);\ P_j u = 0,\ 1 \leqq j \leqq N\}\,.$$

We denote by $Y^0(\Omega)$ and $Y^1(\Omega)$ the space $Y(\Omega)$ equipped with the $L^2(\Omega)$-norm and $H^1(\Omega)$-norm, respectively. The identity map from $Y^1(\Omega)$ into $Y^0(\Omega)$ being continuous and these spaces being Banach spaces, the two norms, $L^2(\Omega)$-norm and $H^1(\Omega)$-norm, are equivalent on $Y(\Omega)$. Thus, there exists a constant $C > 0$ such that, for every $u \in$ $\in Y(\Omega)$, we have:

$$\|u\|_{H^1(\Omega)} \leqq C\,.\,\|u\|_{L^2(\Omega)}\,.$$

The unit ball of $Y^0(\Omega)$ is then compact and therefore $Y(\Omega)$ is of finite dimension.

But if $\xi \in \mathbb{C}^n$ satisfies $P_j(\xi) = 0$ for $1 \leqq j \leqq N$, the function $u(x) = e^{i\langle x,\xi\rangle}$ satisfies $P_j u = 0$ for $1 \leqq j \leqq N$. Then, necessarily, the set of complex common roots of the polynomials is finite.

We now assume that (ii) is true. Let $\xi^1, \ldots, \xi^\nu$ be the complex common roots of the polynomials P_j for $1 \leqq j \leqq N$. For each $1 \leqq j \leqq n$, we consider the polynomial

$$Q_j(\xi) = \prod_{i=1}^{\nu} (\xi_j - \xi_j^i)$$

where we have put $\xi = (\xi_1, \ldots, \xi_n)$.

Then we have $Q_j(\xi^i) = 0$ for $1 \leqq i \leqq \nu$; that is, the polynomials Q_j, $1 \leqq j \leqq n$, vanish on the set of complex common roots of the polynomials P_j, $1 \leqq j \leqq N$. From the "Nullstellensatz" (see e.g. Van der Warden [13]), there exists an integer $\varrho \geqq 1$ such that the polynomials Q_j^ϱ for $1 \leqq j \leqq n$ belong to the ideal spanned by the polynomials P_l, $1 \leqq l \leqq N$; that is, there exist polynomials A_{jl} such that

$$Q_j^\varrho(\xi) = \sum_{l=1}^{N} A_{jl}(\xi)\, P_l(\xi)\,, \quad 1 \leqq j \leqq n\,.$$

The polynomials Q_j^ϱ are polynomials of order $\nu\varrho$ the principal part of which is equal to $\xi_j^{\nu\varrho}$; these principal parts have only 0 as a complex common root, that is, they satisfy the conditions (A) and (B). Hence, if $u \in \mathscr{D}'(\Omega)$ and $P_j u \in G_S(\bar{\Omega})$ for $1 \leqq j \leqq N$, then $Q_j^\varrho u \in G_S(\bar{\Omega})$ for $1 \leqq j \leqq n$. By Smith [12], Bolley-Camus [3] we find $u \in C^\infty(\bar{\Omega})$ and Corollary II-2 yields $u \in G_S(\bar{\Omega})$.

Theorem II-1, in particular, implies the following sufficient condition of $G_S(\Omega)$-regularity:

Corollary II-3. *Let P_j be differential operators, $1 \leqq j \leqq N$, with constant coefficients and satisfying the condition* (C); *then the following two propositions are equivalent*:

(i) $u \in G_S(\Omega)$;

(ii) $u \in C^\infty(\Omega)$ *and* $P_j u \in G_S(\Omega)$ *for* $1 \leqq j \leqq N$.

Remark II-2. It follows from the preceding theorems that, if the polynomials $P_j \equiv P_j(\xi)$, $1 \leqq j \leqq N$ (with constant coefficients), have principal parts without complex common roots different from 0, that is they satisfy the condition (B), then they have only a finite number of complex common roots, that is they satisfy the condition (C): this is a "classical" result in algebraic geometry.

III — "REDUCED POWERS" AND G_S-REGULARITY

In [5], Damlakhi gives a refinement of Nelson's theorem (Theorem 0′) in the following sense:

Theorem [5]. *Let $P_1, \ldots, P_n$ be real vector fields with analytic coefficients and linearly independent at each point of an open set Ω; then the following two propositions are equivalent*:

(i) $u \in a(\Omega)$;

(ii) $u \in C^\infty(\Omega)$ *and, for every subset K of Ω, there exists a constant $L = L_K > 0$ such that, for every $k \geqq 1$ and $1 \leqq i \leqq n$, we have*:

$$\|P_i^k u\|_{L^2(K)} \leqq L^{k+1}(k!) .$$

In a similar way and in accordance with the preceding Chapters I and II, we are going to put forward the following two conjectures:

Conjecture 1. *Under the assumptions of Theorem* 1, *the following two propositions are equivalent*:

(i) $u \in G_S(\Omega)$;

(ii) $u \in C^\infty(\Omega)$ *and, for every compact subset* K *of* Ω, *there exists a constant* $L = L_K > 0$ *such that, for every* $k \geqq 1$ *and* $1 \leqq i \leqq N$, *we have*:

$$\|P_i^k u\|_{L^2(K)} \leqq L^{k+1}((km_i)!)^S .$$

Conjecture 2. *Under the assumptions of Theorem* 2, *the following two propositions are equivalent*:

(i) $u \in G_S(\bar{\Omega})$;

(ii) $u \in C^\infty(\Omega)$ *and there exists a constant* $L > 0$ *such that, for every* $k \geqq 1$ *and* $1 \leqq i \leqq N$, *we have*:

$$\|P_i^k u\|_{L^2(\Omega)} \leqq L^{k+1}((km_i)!)^S .$$

An affirmative answer is given in a particular case by DAMLAKHI [5] who uses to this end the notion of the analytic wave front set of a hyperfunction and the fundamental theorem of Sato, and also the idea of adding another variable t (in $\mathbb{R}$) and of considering the evolution operators $P_j = \partial/\partial t - iP_j$, $1 \leqq j \leqq N$.

Conjecture 1 is true also in the case of operators P_j of order 1, with complex and constant coefficients. The proof of this result is based on the following proposition:

Proposition III-1. *Let* $P_j = P_j(\xi)$ *be polynomials,* $j = 1, \ldots, N$, *of order* 1 *with complex and constant coefficients*; *we assume that their principal parts have no real common roots different from* 0. *Then, for every compact sets* K_1 *and* K_2 *in* R^n, K_1 *being included in the interior* K_2^0 *of* K_2, *there exists a constant* $C > 0$ *such that, for every* $u \in C^\infty(K_2)$ *and* $\alpha \in \mathbb{N}^n$, *we have*:

$$\|D^\alpha u\|_{L^2(K_1)} \leqq C^{|\alpha|+1} \sum_{i=1}^{N} \sum_{|\beta| \leqq |\alpha|} \sum_{j=0}^{|\alpha|-|\beta|} C^{|\beta|} |\alpha|^{|\beta|} \frac{|\alpha|!}{(|\alpha| - |\beta| - j)!\, j!\, \beta!} \cdot \|P_i^{|\alpha|-|\beta|-j} u\|_{L^2(K_2)} .$$

This proposition is obtained by using, in particular, the special function of truncation given in HÖRMANDER [7].

Another affirmative answer to Conjecture 2 has been given for $s = 1$, $\Omega = (]-1, +1[)^n$ and for the canonical system of the first partial derivatives by Damlakhi [5], via the spectral theory of Legendre's operator in n variables.

Conjecture 2 is also true "locally" in the half-space $\mathbb{R}^n_+ = \{(x, t); t \geqq 0\}$ for the case of a transversal operator P_1 of order 1 with constant and real coefficients and tangential operators $P_2, \ldots, P_N$ with complex and constant coefficients. The proof is based on the following a priori estimate: there exists a constant $C > 0$ such that, for all $u \in C_0^\infty(\overline{R_+^n})$, $u(x, t) = 0$ for $t \geqq 1$, $k \geqq 1$ and $\alpha \in \mathbb{N}^{n-1}$, we have:

$$\left\| D_x^\alpha P_1^k u \right\|_{L^2(\mathbb{R}_+^n)} \leqq C^{|\alpha|+k+1} \left\{ \left\| P_1^{|\alpha|+k+1} u \right\|_{L^2(\mathbb{R}_+^n)} + \right.$$

$$\left. + \sum_{j=2}^{N} \sum_{l=0}^{|\alpha|+k+1} \binom{l}{|\alpha| + k + 1} \left\| P_j^{|\alpha|+k+1-l} u \right\|_{L^2(\mathbb{R}_+^n)} \right\}.$$

We can prove such an inequality by using the inequalities given in CARTAN [4] and HARDY-LITTLEWOOD-POLYA [6].

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