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ON COMPACT EMBEDDING THEOREMS IN WEIGHTED SOBOLEV SPACES

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INTRODUCTION

In preparing this paper about compact embedding theorems concerning Sobolev weighted spaces (SW-spaces) I have been guided by some recent papers in which such theorems are successfully used.

Some papers (see for instance [5], [6], [13]) deal with the discreteness of the spectrum of certain linear partial differential operators in unbounded domains in R^n. Others build up an existence theory for some boundary value problems with non-linear perturbating terms [6], [9].

The usefulness of employing the SW-spaces in the studies of singular boundary value problems is well known (for related references see [2]).

The results obtained in the above mentioned papers in the fields of spectral theory and nonlinear analysis stimulate further investigations on compact embeddings.

Here we want to establish a general compact embedding criterion (Theorem 2.1) which may be transferred also to Orlicz-Sobolev weighted spaces. Such a criterion becomes a very useful tool when we can use Gagliardo-Nirenberg interpolation inequalities, as shown in Theorem 3.4.

1. PRELIMINARIES

Let Ω_1 and Ω_2 be two non void open sets in R^n; by

$$\Omega_1 \subset\subset \Omega_2$$

we mean that Ω_1 is bounded and $\bar{\Omega}_1 \subset \Omega_2$. If B_1 and B_2 are two B-spaces,

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$$B_1 \hookrightarrow B_2$$

means that B_1 is (isometric to) a subspace of B_2 and that the natural injection of B_1 into B_2 is a continuous map. We shall write

$$B_1 \hookrightarrow\hookrightarrow B_2$$

if $B_1 \hookrightarrow B_2$ and the natural injection of B_1 into B_2 is a compact map, that is, if every B_1-bounded sequence has a B_2-convergent subsequence.

Let us recall now some fundamental results (see [1], [11]).

Theorem 1.1 (Sobolev embeding theorem). *Let Ω be a domain in R^n with the cone property; let j, m be non-negative integers and $p \in [1, +\infty[$; if $mp < n$ then*

$$W^{j+m,p}(\Omega) \hookrightarrow W^{j,q}(\Omega) \quad \textit{for} \quad p \leqq q \leqq \frac{np}{n - mp}.$$

Theorem 1.2 (Rellich-Kondrashov theorem). *Let Ω be a domain in R^n with the cone property; let j, m be non-negative integers and $p \in [1, +\infty[$; if $mp < n$ then for every bounded subdomain Ω_0 of Ω we have*

$$W^{j+m,p}(\Omega_0) \hookrightarrow\hookrightarrow W^{j,q}(\Omega_0) \quad \textit{for} \quad p \leqq q < \frac{np}{n - mp}.$$

Let us now recall the following results.

Proposition 1.3. *If Ω is an unbounded domain in R^n with a finite volume and if $m \geqq 1$ and $q > p \geqq 1$, then the embedding*

$$W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$$

cannot hold.

The proof may be found in [1]. For the sake of simplicity we prove this proposition in a very simple special case.

Let

$$\Omega = \{(x_1. x_2) \in R^2 | \, x_1 \in]1, +\infty[\text{ and } 0 < x_2 < f(x_1)\}$$

where f is a positive element of $C^0(]1, +\infty[)$ such that

$$|\Omega| = \int_1^{+\infty} f(t) \, \mathrm{d}t < +\infty.$$

We assume that f has a finite order zero at infinity, i.e. there exists $\alpha \in R_+$ such that

$$\int_1^{+\infty} f(t) \, t^\alpha \, \mathrm{d}t = +\infty \quad \text{and} \quad \int_1^{+\infty} f(t) \, t^\beta \, \mathrm{d}t < +\infty \quad \text{for every} \quad \beta < \alpha.$$

Given $q > p \geqq 1$ we consider the function u defined in Ω by

$$u(x) = |x|^{\alpha/q}, \quad x \in \Omega .$$

It is easy to see that $u \in W^{m,p}(\Omega)$ for any $m \in N$; but we have $u \notin L^q(\Omega)$.

It is necessary to complete Proposition 1.3 by observing that if Ω has a finite measure and is unbounded, it cannot have the cone property. Actually, Theorem 1.1 fails to be true if Ω has not the cone property (see [1] Chap. V). In fact, if Ω has too sharp cusps (for instance exponential cusps) a proposition holds which is similar to Proposition 1.3; when Ω has finite order cusps, then the assertion of Sobolev Theorem 1.1 still holds provided it is modified in the folowing way:

Given $\lambda \in [1, +\infty[$ and an integer $k \leqq n - 1$, let us consider

$$Q_{k,\lambda} = \{x \in R^n |\, x_1^2 + \ldots + x_k^2 < x_{k+1}^2, x_{k+1} > 0, \ldots, x_n > 0 ,$$
$$x_1^2 + \ldots + x_k^2 + (x_{k+1}^2 + \ldots + x_n^2)^\lambda < 1\} .$$

$Q_{k,\lambda}$ is a standard cusp and $Q_{k,1}$ a standard cone. It is easy to see that $Q_{n-1,1}$ is a usual cone and $Q_{n-1,\lambda}$ is a conical domain; the greater λ is, the sharper is this domain at its vertex. $Q_{k,\lambda}$ for $\lambda > 1$ has not the cone property. Now let

$$\Omega = \bigcup_{G \in \Gamma} G , \quad G \subset R^n$$

where Γ has the finite intersection property, and for every $G \in \Gamma$ there exists a C^1-homeomorphism ψ_G from G into a standard cusp $Q_{k,\lambda}$. We assume that

$$\sup_{G \in \Gamma} k(\lambda - 1) = \nu < +\infty ,$$

and that there exists $A \in R_+$ such that

$$A^{-1} \leqq |\psi'_G| \leqq A \quad \text{for every} \quad G \in \Gamma .$$

Theorem 1.4. *Under the above hypotheses and provided $mp < n + \nu$ we have*

$$W^{m,p}(\Omega) \mathrel{\subset\!\!\!\!\!\rightarrow} L^q(\Omega) \quad \textit{for} \quad p \leqq q \leqq \frac{(n + \nu)\, p}{n + \nu - mp} .$$

In the sequel, ν will be called *the measure of nonregularity* of Ω.

Let us recall now that an unbounded domain Ω is quasibounded if it fulfils the following condition:

Condition 1.5.

$$\lim_{x \in \Omega, |x| \to +\infty} \operatorname{dist}(x, \partial\Omega) = 0 .$$

It is easy to prove

Proposition 1.6. *If Ω is neither bounded nor quasibounded, then the embedding*

$$W^{m,p}(\Omega) \subset\subset L^q(\Omega)$$

cannot hold whatever the integer $m \geqq 1$ and the numbers $p, q \in]1, +\infty[$ may be.

Proof. If Condition 1.5 is not fulfilled then

$$\limsup_{x \in \Omega, |x| \to +\infty} \operatorname{dist}(x, \partial\Omega) > 0 .$$

In this case it is possible to find a disjoint sequence of open balls $(B_k)_{k \in N}$ with the same radius, all contained in Ω. Given a non-zero function $u_1 \in \mathscr{D}(B_1)$, we define $u_k \in \mathscr{D}(B_k)$ as a translate of u_1 for every $k \in N$.

It follows that

$$\|u_k\|_{W^{m,p}(\Omega)} = \|u_1\|_{W^{m,p}(B_1)} \quad \text{for every} \quad k \in N ,$$

$$\|u_h - u_k\|_{L^q(\Omega)} = 2^{1/q}\|u_1\|_{L^q(B_1)} \quad \text{for every} \quad h, k \in N , \quad h \neq k .$$

Therefore the sequence $(u_k)_{k \in N}$ is bounded in $W^{m,p}(\Omega)$, but it has no subsequence which L^q-converges. The proof is thus complete.

We infer: if Ω is unbounded the compact embedding $W^{1,p}(\Omega) \subset\subset L^q(\Omega)$ for $q > p$ may be true only if Ω is quasibounded and of infinite measure. The quasiboundedness of Ω implies the unboundedness of the boundary of Ω. An example of such a domain is given by

$$\Omega = \{(x, y) \in R^2 |\ |xy| < 1\} .$$

Nevertheless, let us observe that quasibounded domains may be rather complicated, as the following example shows:

Example 1.7.

$$\Omega = R^2 \setminus \bigcup_{k=1}^{\infty} S_k$$

where

$$S_k = \{(x_1, x_2) \in R^2 |\ |x| \geqq k \text{ and } \arg(x_1 + ix_2) = n\pi 2^{-k},\ n = 1, \ldots, 2^{k+1}\} .$$

We have to mention at this point that

$$W^{1,p}(\Omega) \subset\subset L^p(\Omega)$$

if and only if the following condition is fulfilled (see [7], Theorem 2.8):

Condition 1.8.

$$\lim_{|x| \to \infty} |B(x, 1) \cap \Omega| = 0 .$$

Condition 1.8 implies that Ω, if it is unbounded, it is quasibounded. Moreover, the unbounded quasibounded domain from Example 1.7 does not satisfy Condition 1.8; nevertheless we have

$$\overset{\circ}{W}{}^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$$

as is proved in [1], Theorem 6.13.

Let us now observe that among other advantages, the use of weighted Sobolev spaces offers certain compact embeddings which hold with weaker conditions on Ω or without the condition of boundedness. Actually, the following assertions may be derived from our Theorems 2.1, 2.2 and from the results by R. A. ADAMS mentioned above.

Theorem 1.9. *Let Ω be a non-empty domain of R^n with a finite measure of non-regularity ν; let p_0, p, q be real numbers with*

$$1 \leqq p_0 < n + \nu\,, \quad p_0 \leqq p < q = \frac{(n + \nu)\, p_0}{n + \nu - p_0}\,.$$

If μ is a continuous positive function on Ω such that $\mu \in L^\alpha(\Omega)$ with $q/(q - p) \leqq \leqq \alpha < +\infty$, then

$$W^{1,p_0}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega, \mu)\,.$$

Theorem 1.10. *If $1 \leqq p_0 < n$ and $p_0 \leqq p < np_0/(n - p_0)$, then*

$$W^{1,p_0}(R^n) \hookrightarrow\hookrightarrow L^p(R^n, (1 + |x|)^{-\beta})$$

for every $\beta > 0$.[1])

2. SOME EMBEDDING THEOREMS

We shall limit our considerations to SW-spaces of order one, $W^{1,p}(\Omega, \mu_0, \mu_1)$; μ_0, μ_1, ϱ are positive continuous functions on Ω; $u \in W^{1,p}(\Omega; \mu_0, \mu_1)$ means that

$$\|u\|_{W^{1,p}(\Omega,\mu_0,\mu_1)} = (\|u\|^p_{L^p(\Omega,\mu_0)} + \|\nabla u\|^p_{L^p(\Omega,\mu_1)})^{1/p} < +\infty\,.$$

Theorem 2.1. *Let $p \in [1, n[$ and $p \leqq q < np/(n - p)$; then*

$$W^{1,p}(\Omega, \mu_0, \mu_1) \hookrightarrow\hookrightarrow L^q(\Omega, \varrho) \tag{2.1}$$

if and only if for every bounded sequence $(f_n)_{n\in N}$ of elements of $W^{1,p}(\Omega, \mu_0, \mu_1)$ the set functions

$$E \mapsto \int_E |f_n(x)|^q\, \varrho(x)\, \mathrm{d}x\,, \quad n = 1, 2, \ldots \tag{2.2}$$

are uniformly absolutely continuous.

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[1]) These results can be improved by a more general theorem by V. BENCI and D. FORTUNATO (see [4], Theorem 2.8 and [6], Theorem 2.7).

Proof of the "if" part. We can select a subsequence of $(f_n)_{n\in N}$, say $(\phi_n)_{n\in N}$, which converges almost everywhere in Ω to a function f belonging to $L^p_{loc}(\Omega)$.

This follows by standard procedures, by using the embeddings

$$W^{1,p}(K) \subset\subset L^q(K)$$

for $K \subset\subset \Omega$ regular enough.

Further, the uniform absolute continuity of the set functions (2.2) implies

$$\lim_{n\to\infty} \int_\Omega |\phi_n(x) - f(x)|^q \varrho(x)\, dx = 0 .$$

The "if" part is proved.

Proof of the "only if" part. To this end we employ the following results:

Cafiero theorem (see [8]). *A sequence of completely additive set-functions $(\mu_n)_{n\in N}$ on a σ-field J is uniformly additive if and only if to every sequence $(I_k)_{k\in N}$ of disjoint sets in J and every $\varepsilon > 0$ there exist k_0 and ν such that*

$$n > \nu \Rightarrow \mu_n(I_{k_0}) < \varepsilon .$$

Caccioppoli theorem (see [8]). *A uniformly additive family of absolutely continuous set functions on a measure space (J, μ) is uniformly absolutely continuous.*

We suppose now that (2.1) holds and that there exists a sequence $(f_n)_{n\in N}$ bounded in $W^{1,p}(\Omega, \mu_0, \mu_1)$ for which the set functions (2.2) are not uniformly additive. In this case there would exist, by the quoted Cafiero theorem, a sequence of measurable disjoint subsets of Ω, say $(I_k)_{k\in N}$, an $\varepsilon_0 > 0$ and a subsequence of $(f_n)_{n\in N}$ (denoted again by $(f_n)_{n\in N}$) such that

$$\int_{I_n} |f_n(x)|^q \varrho(x)\, dx \geqq \varepsilon_0 \quad \text{for every} \quad n \in N .$$

But this leads to a contradiction via the compactness in $L^q(\Omega, \varrho)$ of the sequence $(f_n)_{n\in N}$.

In fact, let $(f_{n_k})_{k\in N}$ be a subsequence of $(f_n)_{n\in N}$ convergent to a function g in $L^q(\Omega, \varrho)$; we have

$$\varepsilon_0 \leqq \int_{I_{n_k}} |f_{n_k}(x)|^q \varrho(x)\, dx \leqq 2^{q-1} \int_\Omega |f_{n_k}(x) - g(x)|^q \varrho(x)\, dx + 2^{q-1} \int_{I_{n_k}} |g(x)|^q \varrho(x)\, dx .$$

It is enough to observe that the right hand side tends to zero, because

$$\lim_{k\to\infty} \int_{I_{n_k}} |g(x)|^q \varrho(x)\, dx = 0 .$$

Theorem 2.2. *Let Ω be a non-void subset of R^n; let p and q be two integers such that $1 \leqq p < q < +\infty$; let μ and ϱ be two functions in $C^0(\Omega)$, which are positive at each point of Ω. Then*

$$L^q(\Omega, \varrho) \subset\!\!\!\rightarrow L^p(\Omega, \mu) \tag{2.3}$$

if and only if

$$\int_\Omega (\mu(x)\, \varrho^{-p/q}(x))^{q/(q-p)}\, dx < +\infty ,$$

that is

$$\mu^{q/(q-p)} \varrho^{-p/(q-p)} \in L^1(\Omega) . \tag{2.4}$$

Proof. Let us suppose that (2.4) holds. If $\phi \in \mathscr{D}(\Omega)$, the Hölder inequality yields

$$\|\phi\|^p_{L^p(\Omega,\mu)} \leqq \|\mu \varrho^{-p/q}\|_{L^{q/(q-p)}(\Omega)} \|\phi\|^p_{L^q(\Omega,\varrho)}$$

and (2.3) follows because of continuity.

Vice versa, let us suppose that (2.3) holds; then a positive constant exists such that

$$\|f\|_{L^p(\Omega,\mu)} \leqq c\|f\|_{L^q(\Omega,\varrho)} \quad \text{for every} \quad f \in L^q(\Omega, \varrho) .$$

Let E be any compact subset of Ω and set

$$\gamma(E) = \|\mu \varrho^{-p/q}\|_{L^{q/(q-p(E))}} . \tag{2.5}$$

Next we fix f in such a way that

$$|f(x)|^p \varrho^{p/q}(x) = (\mu(x)\, \varrho^{-p/q}(x))^{q/(q-p)-1} \quad \text{for every} \quad x \in E . \tag{2.6}$$

An easy calculation yields

$$\|f\|_{L^p(E,\mu)} = (\gamma(E))^{1/p} \|f\|_{L^q(E,\varrho)} .$$

On the other hand, if we denote the characteristic function of the set E by χ_E, we have

$$\|\chi_E f\|_{L^p(\Omega,\mu)} = \|f\|_{L^p(E,\mu)} = (\gamma(E))^{1/p} \|f\|_{L^q(E,\varrho)} \leqq c\|\chi_E f\|_{L^q(\Omega,\varrho)}$$

which implies $(\gamma(E))^{1/p} \leqq c$, and finally we obtain

$$\int_E (\mu(x)\, \varrho^{-p/q}(x))^{q/(q-p)}\, dx \leqq c^{qp/(q-p)} . \tag{2.7}$$

Since the right hand side does not depend on E, (2.4) follows from (2.7), and the theorem is completely proved.

Remark. In particular, it follows that

$$L^q(\Omega, \varrho) \subset\!\!\!\rightarrow L^p(\Omega)$$

for $q > p \geqq 1$ if and only if

$$\varrho^{-p/(q-p)} \in L^1(\Omega)\,.$$

Theorem 2.3. *Let Ω be an open subset in R^n; let μ and ϱ be two positive and continuous functions in Ω and let $1 \leqq p < q$. If*

$$L^q(\Omega, \varrho) \hookrightarrow L^p(\Omega, \mu)$$

then for any sequence $(f_n)_{n\in N}$ of elements of $L^q(\Omega, \varrho)$ which is bounded in $L^q(\Omega, \varrho)$, the set functions

$$E \mapsto \int_E |f_n(x)|^p \, \mu(x)\, \mathrm{d}x$$

are uniformly absolutely continuous.

Proof. In fact, we have

$$\int_\Omega |f_n(x)|^q \, \varrho(x)\, \mathrm{d}x \leqq M^q \quad \text{for every} \quad n \in N\,,$$

which yields easily

$$\int_E |f_n(x)|^p \, \mu(x)\, \mathrm{d}x \leqq M^p \left(\int_E \mu^{q/(q-p)}(x)\, \varrho^{-p/(q-p)}(x)\, \mathrm{d}x\right)^{(q-p)/q}.$$

This implies the above theorem by virtue of the absolute continuity of the integral function of a summable function.

Theorem 2.4. *Let Ω satisfy the cone property; let*

$$\varrho \in C^1(\Omega) \cap C^0_+(\Omega)\,, \quad \varrho_0, \varrho_1 \in C^0_+(\Omega)$$

and let $p_0 \in [1, n[$ be such that

$$|\nabla\varrho| \leqq c\varrho_0^{1/p_0}\varrho_1^{1/(n-p_0)}\,, \; \varrho \leqq c\varrho_1^{n/(n-p_0)1}\,.$$

Then

$$W^{1,p_0}(\Omega, \varrho_0 \cdot \varrho_1) \hookrightarrow L^q(\Omega, \varrho)$$

provided $q \leqq np_0/(n - p_0)$.

Furthermore, let $\mu \in C^0_+(\Omega)$ and $p_0 \leqq p < q$ be such that

$$\mu^{q/(q-p)}\varrho^{-p/(q-p)} \in L^1(\Omega)\,;$$

then

$$W^{1,p_0}(\Omega, \varrho_0, \varrho_1) \hookrightarrow GL^p(\Omega, \mu)\,.$$

The proof easily follows from Sobolev Theorem 1.1 and Theorems 2.1 and 2.2.

The hypothesis $\varrho \in C^1(\Omega)$ may be substituted by $\varrho \in \mathrm{Lip}\,(\Omega)$. So we can have results concerning SW-spaces whose weight function depends on a distance, that is on $\mathrm{dist}\,(x, E)$ with $E \subset \partial\Omega$.

Finally, we want to point out that the above results extend to non-isotropic SW-spaces.

3. COMPACT EMBEDDINGS: SPECIAL DOMAINS

Let Ω be an unbounded set of R^n and $\delta \in C^0_+(\Omega)$ a positive continuous function divergent for $|x| \to +\infty$ we put

$$\begin{aligned}
\Omega_1 &= \{x \in \Omega | \; \delta(x) < 1\}\,, \\
\Omega_2 &= \{x \in \Omega | \; \delta(x) > 1\}\,, \\
\Omega_0 &= \Omega\,, \\
A_i(x_0) &= \{x \in \Omega_i | \; |x - x_0| < \delta(x_0)\}\,, \quad i = 0, 1, 2\,.
\end{aligned}$$

We assume that Ω and δ satisfy the following axioms by M. Troisi:

T_1) There exists $c_1 \in R_+$ such that

$$c_1^{-1}\,\delta(x_0) \leqq \delta(x) \leqq c_1\,\delta(x_0) \quad \text{for every} \quad x \in A_i(x_0)\,.$$

T_2) If $\chi_i(\cdot, x_0)$ is the characteristic function of the set $A_i(x_0)$, then the inequalities

$$c_2^{-1}\,\delta^n(x) \leqq \int_{\Omega_i} \chi_i(x, x_0)\,\mathrm{d}x_0 \leqq c_2\,\delta^n(x)$$

hold for every $x \in A_i(x_0)$, where c_2 is a positive constant independent of x.

It is easy to see that if Ω is unbounded, has the cone property and if we put

$$\delta(x) = \tfrac{1}{2}\,\mathrm{dist}\,(x, M) \quad \text{with} \quad \emptyset \neq M \subset \partial\Omega\,,$$

then $\Omega_i \neq \emptyset$, $i = 1, 2$ and T_1, T_2 are satisfied.

More generally, instead of T_2 we may introduce the following axiom:

T_2') For every $x_0 \in \Omega_i$ the set $A_i(x_0)$ has the cone property with a cone $\Gamma(x_0)$ such that the interior of $\Gamma(x_0)$ and the ratio $h(x_0)/\delta(x_0)$ are independent of x_0, where $h(x_0)$ is the height of $\Gamma(x_0)$.

We can prove that if Ω has the cone property and $\delta(x)$ is the distance from a subset of $\partial\Omega$, then T_2' is satisfied. Furthermore, it is easy to see that, because of T_1,

$$T_2' \Rightarrow T_2\,.$$

Let $p \in]0, +\infty[$, $s \in R$ and let E be a measurable subset of R^n; we put

$$|u|_{p,E} = \left(\int_E |u(x)|^p\,\mathrm{d}x\right)^{1/p}.$$

Given $\delta \in C^0_+(E)$, $L^p_s(E)$ is defined by

$$u \in L^p_s(E) \Leftrightarrow \delta^s u \in L^p(E)\,,$$

and

$$\|u\|_{p,s,E} = \|\delta^s u\|_{L^p(E)} = |\delta^s u|_{p,E}\,.$$

Now we show some examples of (Ω, δ) which satisfy axioms T_1 and T_2.

Let $M_1, \ldots, M_m$ be m arbitrary subsets of $\partial\Omega$, and let f be a uniformly Lipschitz-continuous function in R^m_+. By introducing a multiplicative constant we may assume that

$$f(v') - |f(v'')| \leqq \frac{1}{2\sqrt{m}} |v' - v''| \quad \text{for every} \quad v', v'' \in R^m_+ .$$

It is easy to verify that if Ω has the cone property and we put

$$\delta(x) = f(\text{dist}(x, M_1), \text{dist}(x, M_2), \ldots, \text{dist}(x, M_m)),$$

then (Ω, δ) satisfy T_1, T_2.

Lemma 3.1. (Lemmas 1.1 and 1.2, [12]). *If (Ω, δ) satisfies T_1 and T_2, $p \in]0, +\infty[$ and $s \in R$, then there exist $c_3, c_4 \in R_+$ such that*

$$\text{(3.1)} \qquad \int_{\Omega_i} (\delta^s(x) |u|_{p,A_i(x)})^p \, dx \geqq c_3 |u|^p_{p,s+n/p,\Omega_i}$$

for every function u for which

$$\delta^s(x) |u|_{p,A_i(x)} \in L^p(\Omega_i) .$$

Furthermore,

$$\text{(3.2)} \qquad \int_{\Omega_i} (\delta^s(x) |u|_{p,A_i(x)})^q \, dx \leqq c_4 |u|^q_{p,s+n/q,\Omega_i}$$

for every $q \geqq p$ and $u \in L^p_{s+n/q(\Omega)}$.

Proof. In order to prove (3.1) we have to estimate from below the integral

$$\int_{\Omega_i} \delta^{sp}(x) \, dx \int_{A_i(x)} |u(y)|^p \, dy = \int_{\Omega_i \times \Omega_i} \delta^{sp}(x) |u(y)|^p \chi_i(y, x) \, dx \, dy .$$

Using T_1 and then T_2, we obtain

$$\int_{\Omega_i \times \Omega_i} \delta^{sp}(x) |u(y)|^p \chi_i(y, x) \, dx \, dy \geqq c_1^{-|s|p} \int_{\Omega_i \Omega \times i} |u(y)|^p \delta^{sp}(y) \chi_i(y, x) \, dx \, dy =$$

$$= c_1^{-p|s|} \int_{\Omega_i} |u(y)|^p \delta^{sp}(y) \, dy \int_{\Omega_i} \chi_i(y, x) \, dx \geqq c_2^{-1} c_1^{-|s|p} \int_{\Omega_i} |u(y)|^p \delta^{sp+n}(y) \, dy .$$

Thus (3.1) has been proved. We put

$$I = \int_{\Omega_i} \delta^{sp}(x) \, dx \left(\int_{A_i(x)} |u(y)|^p \, dy \right)^{q/p}, \quad h(x) = \delta^{s(q-p)}(x) \left(\int_{A_i(x)} |u(y)|^p \, dy \right)^{q/p-1} .$$

We have

$$I = \int_{\Omega_i} \delta^{sp}(x)\, h(x)\, dx \int_{A_i(x)} |v(y)|^p\, dy = \int_{\Omega_i \times \Omega_i} \delta^{sp}(x)\, h(x)\, \chi_i(y, x)\, |v(y)|^p\, dx\, dy \leqq$$

$$\leqq c_1^{|s|p} \int_{\Omega_i \times \Omega_i} \delta^{sp}(y) |v(y)|^p\, h(x)\, \chi_i(y, x)\, dx\, dy =$$

$$= c_1^{|s|p} \int_{\Omega_i} \delta^{sp}(y)\, |v(y)|^p\, dy \int_{\Omega_i} h(x)\, \chi_i(y, x)\, dx \leqq$$

$$\leqq c_1^{|s|p} \int_{\Omega_i} \delta^{sp}(y)\, |v(y)|^p\, dy \left(\int_{\Omega_i} (h(x))^{q/(q-p)}\, dx \right)^{(q-p)/q} \left(\int_{\Omega_i} \chi_i(y, x)\, dx \right)^{p/q} \leqq$$

$$\leqq c_1^{|s|p} c_2^{p/q}\, I^{1-(p/q)} \int_{\Omega_i} \delta^{sp+(np/q)}(y)\, |u(y)|^p\, dy\, .$$

Hence (3.2) easily follows.

For $p = q$ we deduce from (3.1) and (3.2) that

$$\left(\int_{\Omega_i} (\delta^{s-(n/p)}(x)\, |u|_{p,A_i(x)})^p\, dx \right)^{1/p} \quad \text{is equivalent to } \|u\|_{p,s,\Omega_i}\, .$$

This means that we can estimate the weight norm of u by a weight norm of a certain mean value of u. Really, if we put

$$(M_p^\delta u)(x) = \left(\delta^{-n}(x) \int_{A_i(x)} |u(y)|^p\, dy \right)^{1/p}$$

we can assert that *under the hypotheses of Lemma* 3.1 *the two norms* $\|u\|_{p,s}$ *and* $\|M_p^\delta u\|_{p,s}$ *are equivalent.*

Theorem 3.1 ([12], Theorem 21). *Let* (Ω, δ) *satisfy* T_1 *and* T_2, $p \in [1, +\infty[$, *let r be a positive integer,* $s \in R$; *then there exists a constant C such that*

$$|\partial^k u|_{p,s-k+r,\Omega_i} \leqq C(|\partial^r u|^{k/r}_{p,s,\Omega_i} |u|^{1-k/r}_{p,s-r,\Omega_i} + |u|_{p,s-r,\Omega_i}) \tag{3.3}$$

holds for $k = 0, 1, \ldots, r - 1$ *and for every* $u \in \mathcal{D}(\Omega)$ *with the properties*

$$u \in L^p_{s-r}(\Omega_i)\, , \quad \partial^\alpha u \in L^p_s(\Omega_i) \quad \textit{for} \quad |\alpha| = r\, .$$

We will not prove this theorem here in detail, but we wish to give an idea of its proof which is useful for other more general cases.

It starts from the well know Gagliardo-Nirenberg interpolation inequalities for the domain

$$J(x) = \{\xi \in R^n |\, (\xi - x)\, \delta(x) = y - x \text{ with } y \in A_i(x)\}\, .$$

Next we obtain (3.3) by transforming the integrals over $A_i(x)$ and using Lemma 3.1.

It is important to observe that if we use a weight such as δ and, for instance, define the Sobolev weight space $W_s^{r,p}(\Omega)$ by means of

$$\|u\|_{W_s{}^{r,p}(\Omega)} = \sum_{k=0}^{r} \|\partial^k u\|_{p,s-k+r,\Omega},$$

the behaviour of the elements of $W_s^{r,p}(\Omega)$ at infinity and at a point of the boundary at which δ is zero are closely connected. We can improve this result by using the weight function

$$\varrho_{s,\lambda}(x) = \frac{\delta^{s+\lambda}}{1+\delta^{2\lambda}} \sim \begin{cases} \delta^{s-\lambda}, & |x| \to +\infty\,; \\ \delta^{s+\lambda}, & |x| \to x_0 \in \{y \mid \delta(y) = 0\}\,. \end{cases}$$

We consider the seminorms

$$|u|_{p,s,\lambda} = |\varrho_{s,\lambda} u|_{p,\Omega}\,.$$

Then the following theorems hold:

Theorem 3.2. *Under the hypotheses of Theorem* 3.1 *there exists aconstant c such that*

$$|\partial^k u|_{p,s,\lambda} \leqq c\big(|u|^{1-k/r}_{p,s-r,\lambda}\,|\partial^r u|^{k/r}_{p,s,\lambda} + |u|_{p,s-r,\lambda}\big) \tag{3.4}$$

for $k = 0, 1, \ldots, r-1$.

Theorem 3.3. *Under the hypotheses of Theorem* 3.1 *and for every* $a \in [k/r, 1[\,\cap\,[k/r, k/r + n/pr]$ *we have the estimate*

$$|\partial^k u|_{pn/(n-(ar-k)p),s-(1-a)r,\lambda} \leqq c\big(|\partial^r u|^{a}_{p,s,\lambda}|u|^{1-a}_{p,s-r,\lambda} + |u|_{p,s-r,\lambda}\big)\,. \tag{3.5}$$

We now define the space $W^{r,p}_{s,\lambda}(\Omega)$ in the following way:

$$u \in W^{r,p}_{s,\lambda}(\Omega) \Leftrightarrow \big(u \in L^p_{s-r,\lambda}(\Omega) \quad \text{and} \quad \partial^\alpha u \in L^p_{s-k+r,\lambda}(\Omega) \quad \text{for} \quad |\alpha| = k\big)\,,$$

$$\|u\|_{W^{r,p}{}_{s,\lambda}(\Omega)} = \Big(\sum_{k=0}^{r} |\partial^k u|^p_{p,s-k+r,\lambda}\Big)^{1/p}\,.$$

The following embeddings hold:

$$W^{r,p}_{s,\lambda}(\Omega) \hookrightarrow W^{r,p}_{s+k-r,\lambda+\tau}(\Omega) \quad \text{for every} \quad \tau \geqq 0\,, \tag{3.6}$$

$$W^{r,p}_{s,\lambda}(\Omega) \hookrightarrow W^{r-k,p}_{s-k,\lambda+\tau}(\Omega) \quad \text{for every} \quad \tau \geqq 0\,. \tag{3.6'}$$

Theorem 3.4. *Let* (Ω, δ) *satisfy* T_1 *and* T_2; *then given* $p \in\,]1, +\infty[$, *real numbers* s, λ *and an integer* r,

$$W^{r,p}_{s,\lambda}(\Omega) \hookrightarrow\hookrightarrow W^{r-k,p}_{s-k,\lambda+\tau}(\Omega) \quad \textit{for every} \quad \tau > 0 \tag{3.7}$$

holds for $k = 1, 2, \ldots, r$.

Proof. It suffices to consider the case $k = r$ because of (3.6′). Using Theorem 2.1 we have to prove that for every sequence $(u_n)_{n\in N}$ bounded in $W^{r,p}_{s,\lambda}(\Omega)$, the set functions

$$E \mapsto \left|\varrho_{s-r,\lambda+\tau} u_n\right|^p_{p,E}, \quad n = 1, 2, \ldots$$

are uniformly absolutely continuous.

Assume that

$$\left\|u_n\right\|_{W_{s,\lambda}{}^{r,p}} \leqq M, \quad n = 1, 2, \ldots$$

and put

$$K_h = \left\{x \in \Omega \middle| \, \delta(x) \leqq h\right\}, \quad K'_h = \Omega \setminus K_h .$$

We have

$$\left|\varrho_{s-r,\lambda+\tau} u_n\right|^p_{p,K'_h} \leqq \left(\frac{h}{1 + h^2}\right)^{\tau p} M^p . \tag{3.8}$$

Let us put now

$$E_h = E \cap K_h ;$$

then the Hölder-Schwarz inequality and interpolation inequalities (3.5) yield

$$\left|\varrho_{s-r,\lambda+\tau} u\right|^p_{p,E_h} \leqq c\left|E_h\right|^{ar/n} \tag{3.9}$$

where a is a positive real number less than both n/pr and τ/r. From (3.8) and (3.9) our assertion follows.

For further details see Sec. 6 of [3][2]).

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