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Czechoslovak Mathematical Journal, Vol. 29 (1979), No. 4, 584–587

Persistent URL: http://dml.cz/dmlcz/101640

Czechoslovak Mathematical Journal, 29 (104) 1979, Praha

TOLERANCES AND CONVEXITY

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(Received March 20, 1978)

If R is a congruence on a lattice L and x, y are comparable elements of L such that $(x, y) \in R$, then arbitrary two elements of the interval bounded by the elements x, y are in R. We shall show that this property holds even without the requirement of the transitivity of R and thus we shall characterize compatible tolerances among the compatible relations on a lattice. A further well-known property of a congruence is that each class of a congruence on a lattice is a convex sublattice of this lattice. (Theorem 89 in [6].) In this paper it is proved that this result can be generalized for blocks of a tolerance which are a generalization of congruence classes (see [1], [2], [4]).

A binary relation R on a set A is called *a tolerance*, if it is reflexive and symmetric. Let $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ be an algebra, let R be a binary relation on A. A relation R is called *compatible with* $\mathfrak{A}$, if R is a support of a subalgebra of the direct product $\mathfrak{A} \times \mathfrak{A}$, i.e. if it has Substitution Property [5] for all operations of the algebra $\mathfrak{A}$.

Theorem 1. *Let L be a lattice, let R be a reflexive compatible relation on L. Then the following two assertions are equivalent:*

(a) *R is a compatible tolerance on L.*

(b) *If $(a, b) \in R$, then $(x, y) \in R$ for any two elements x, y of L fulfilling $a \wedge b \leqq x \leqq a \vee b$, $a \wedge b \leqq y \leqq a \vee b$.*

Proof. (a) $\Rightarrow$ (b). Let $(a, b) \in R$. The reflexivity of R implies $(b, b) \in R$ and by the compatibility of R we obtain $(a \wedge b, b) \in R$. Analogously $(a \wedge b, a) \in R$, therefore $(a \wedge b, a \vee b) \in R$. Let $a \wedge b \leqq x \leqq a \vee b$, $a \wedge b \leqq y \leqq a \vee b$. As $(x, x) \in R$, we have $((a \wedge b) \vee x, (a \vee b) \vee x) \in R$ which means $(x, a \vee b) \in R$. Analogously we obtain $(y, a \vee b) \in R$ and the symmetry of R implies $(a \vee b, y) \in R$. Then $(x, y) = (x \wedge (a \vee b), (a \vee b) \wedge y) \in R$.

(b) $\Rightarrow$ (a) is evident, because (b) immediately implies the symmetry of R.

Definition. Let R be a binary relation on a set A (i.e. $R \subseteq A \times A$). A non-empty subset B of A will be called a *block of the relation* R, if

(i) $B \times B \subseteq R$;

(ii) if $B \subseteq C$ and $C \times C \subseteq R$, then $B = C$.

Therefore, a block B of a relation R on A is such a non-empty subset of A that the restriction of R onto B is a universal relation and B is maximal with respect to this property.

Lemma 1. *Let L be a lattice, let a, b, c, z be elements of L such that $a \leqq c \leqq b$. Let T be a compatible tolerance on L such that $(a, b) \in T$, $(a, z) \in T$, $(b, z) \in T$. Then also $(c, z) \in T$.*

Proof. From $(a, b) \in T$, $(a, z) \in T$ we have $(a, b \vee z) \in T$; analogously $(b, z) \in T$, $(z, z) \in T$ imply $(b \vee z, z) \in T$ and, by the symmetry, $(z, b \vee z) \in T$. Then by the compatibility $(b \vee z, a \wedge z) \in T$. The elements c and z belong to the interval $\langle a \wedge z, b \vee z \rangle$ and by Theorem 1 we have $(c, z) \in T$.

Lemma 2. *Let T be a compatible tolerance on an idempotent algebra $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ and let B be a block of T. Then $\langle B, \mathscr{F}_B \rangle$, where $\mathscr{F}_B$ is the restriction of $\mathscr{F}$ onto B, is a subalgebra of $\mathfrak{A}$.*

This was proved in [2], Theorem 4.

Theorem 2. *Each block of a compatible tolerance on a lattice L is a convex sublattice of the lattice L.*

Proof. By Lemma 2, a block B of a compatible tolerance T on L is a sublattice of L; thus it remains to prove its convexity. Let $a \in B$, $b \in B$, $c \in L$ and $a \leqq c \leqq b$. Then $(a, b) \in T$ and for all $z \in B$ we have $(a, z) \in T$, $(b, z) \in T$. Therefore by Lemma 1 we have $(c, z) \in T$ for each $z \in B$ and thus $c \in B$.

Remark. For blocks of tolerances on semilattices, an analogous assertion does not hold. Let S be a semilattice with the operation $\circ$. For two elements x, y of S we shall write $x \leqq y$ if and only if $x \circ y = x$. A convex subsemilattice of a semilattice S is such a subsemilattice C of S that if $a \in C$, $b \in C$, $a \leqq x \leqq b$, then $x \in C$.

The set of all compatible tolerances on a given algebra $\mathfrak{A}$ forms a complete lattice with respect to the set inclusion [3]. If R is a binary relation on $\mathfrak{A}$, there exists a compatible tolerance T on $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ which is the least one containing R. This tolerance T is said to be generated by the relation R.

Lemma 3. *Let S be a semilattice with the operation $\circ$, let the ordering $\leqq$ be that from Remark. Let a, b, c be elements of S, $a < c < b$ and let R be the relation* *on S such that $R = \{(a, b)\}$. Then for the compatible tolerance T on S generated by the relation R we have $(b, c) \notin T$.*

Proof. Let T be the compatible tolerance on S generated by the relation R and suppose $(b, c) \in T$. Each compatible tolerance on S containing R evidently contains all pairs (x, y), where $x = x_1 \circ \ldots \circ x_n$, $y = y_1 \circ \ldots \circ y_n$, n being a positive integer, and for each $i = 1, \ldots, n$ either $(x_i, y_i) \in R$ or $(y_i, x_i) \in R$ or $x_i = y_i$; this follows from the symmetry and the compatibility. On the other hand, all such pairs evidently form a compatible tolerance on S; as T is the least compatible tolerance on S containing R, it is equal to the described tolerance. Therefore there exists a positive integer n and elements $b_1, \ldots, b_n$, $c_1, \ldots, c_n$ of S such that $b = b_1 \circ \ldots \circ b_n$, $c = c_1 \circ \ldots \circ c_n$ and for each $i = 1, \ldots, n$ either $(b_i, c_i) \in R$ or $(c_i, b_i) \in R$ or $b_i = c_i$. If $(b_i, c_i) \in R$ for some i, then $b_i = a < b$, which is impossible, because $b = b_1 \circ \ldots \circ b_n$ implies $b_i \geqq b > a$ for each i. If $(c_i, b_i) \in R$, then $c_i = a < c$, which is an analogous contradiction. Therefore only the case $b_i = c_i$ for each i remains. But then $b = c$, which is again a contradiction.

Lemma 4. *Let R be a binary relation on a set A, let $C \subseteq A$ and $C \times C \subseteq R$. Then there exists a block B of R such that $C \subseteq B$.*

Proof follows from Zorn's Lemma, because C fulfils the condition (i) from the definition of a block.

Theorem 3. *Let S be a semilattice. Then the following two assertions are equivalent:*

(a) *Each block of an arbitrary compatible tolerance on S is a convex subsemilattice of the semilattice S.*

(b) *The semilattice S contains no chain (with respect to $\leqq$) of the length* 3.

Proof. (b) $\Rightarrow$ (a). By Lemma 2, a block B of a compatible tolerance T on S is a subsemilattice of the semilattice S. If $a \leqq c \leqq b$, $a \in B$, $b \in B$, then by (b) we have $a = c$ or $b = c$, therefore $c \in B$ and B is convex.

(a) $\Rightarrow$ (b). Let S contain a chain of the length at least 3; then there exist elements a, b, c of S such that $a < c < b$. Let $R = \{(a, b)\}$ and let T be a compatible tolerance on S generated by the relation R. Then $(a, a) \in T$, $(b, b) \in T$, $(a, b) \in T$, $(b, a) \in T$ and thus $\{a, b\} \times \{a, b\} \subseteq T$ and by Lemma 4 there exists a block B of T such that $a \in B$, $b \in B$. Let B be an arbitrary block of T containing a and b and suppose $c \in B$. Then $(z, c) \in T$ for each $z \in B$, which is a contradiction, because by Lemma 3 we have $(b, c) \notin T$.

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