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DECOMPOSITIONS OF HOMOMORPHISMS

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Some sufficient conditions for direct decompositions of homomorphisms were investigated for suitable algebras in [5] and [6] and for lattices and the so called *weakly associative lattices* in [7]. The aim of this paper is to derive new conditions for direct decompositions of homomorphisms on universal algebras by using the well-known conditions for direct decompositions of congruences derived by G. A. FRASER and A. HORN in [2] and the results on the Unique Factorization Property derived by G. BIRKHOFF (see [1], [3], [4]).

Let $\prod_{i=1}^n A_i$ be the direct product of given algebras A_1, \dots, A_n of the same type and let pr_i be the projection of $\prod_{i=1}^n A_i$ onto the i -th factor A_i . If σ is a permutation of the subscript set $\{1, \dots, n\}$, denote by $\varkappa(\sigma)$ the isomorphism of $\prod_{i=1}^n A_i$ onto $\prod_{i=1}^n A_{\sigma(i)}$ given by the rule

$$\varkappa(\sigma)(a_1, \dots, a_n) = (a_{\sigma(1)}, \dots, a_{\sigma(n)}),$$

i.e., $\varkappa(\sigma)$ permutes only the direct factors, which is not essential for the direct product representation.

Let A_i, B_i (for $i = 1, \dots, n$) be algebras of the same type and let h_i be a homomorphism of A_i into B_i . The mapping $h = \prod_{i=1}^n h_i$ of $A = \prod_{i=1}^n A_i$ into $B = \prod_{i=1}^n B_i$ introduced by

$$pr_i h(a) = h_i(pr_i a)$$

for every $a \in A$ and each $i = 1, \dots, n$ is called the *direct product of homomorphisms* h_i (see e.g. G. Grätzer, *Universal Algebra*, p. 127).

With respect to the above remark, let us introduce conversely: a homomorphism h of A into B is said to be *direct by decomposable*, provided $h \cdot \varkappa(\sigma) = \prod_{i=1}^n h_i$ for

a suitable permutation σ of $\{1, \dots, n\}$, where h_i is a homomorphism of A_i into $B_{\sigma(i)}$ for all $i = 1, \dots, n$, and $\varkappa(\sigma)$ is an isomorphism of $\prod_{i=1}^n B_i$ onto $\prod_{i=1}^n B_{\sigma(i)}$.

Lemma. Let B and A_1, \dots, A_n be algebras of the same type and let h be a homomorphism of $A = \prod_{i=1}^n A_i$ onto B inducing a congruence Θ_h on A . If $\Theta_h = \prod_{i=1}^n \Theta_i$, where Θ_i is a congruence on A_i , then there exist algebras B_1^*, \dots, B_n^* and homomorphisms h_i of A_i onto B_i^* such that $B \cong \prod_{i=1}^n B_i^*$ (by the isomorphism f) and $h \cdot f = \prod_{i=1}^n h_i$.

Proof. $B = h(\prod_{i=1}^n A_i) \cong \prod_{i=1}^n A_i / \Theta_h = \prod_{i=1}^n A_i / \prod_{i=1}^n \Theta_i = \prod_{i=1}^n (A_i / \Theta_i)$ (see e.g. [1], p. 140). Denote $B_i^* = A_i / \Theta_i$, hence $B \cong \prod_{i=1}^n B_i^*$ and clearly $h \cdot f = \prod_{i=1}^n h_i$, where $h_i : A_i \rightarrow A_i / \Theta_i$ is a canonical homomorphism. Q.E.D.

An algebra is said to have the *Unique Factorization Property over the class \mathcal{M}* (briefly \mathcal{M} -UFP), provided A is isomorphic with a direct product of algebras from \mathcal{M} and this representation is unique up to isomorphism, i.e. provided the condition

$$A \cong \prod_{i \in I} A_i \cong \prod_{j \in J} B_j, \quad I = \{1, \dots, n\}, \quad J = \{1, \dots, m\}$$

(where $\text{card } A_i > 1$, $\text{card } B_j > 1$ for every subscript i, j) with $A_i, B_j \in \mathcal{M}$ for $i \in I$, $j \in J$ always implies the existence of a bijection σ of I onto J with $A_i \cong B_{\sigma(i)}$ for all $i \in I$. If \mathcal{M} is the class of directly irreducible algebras (of the same type as A), A is said briefly to have the *Unique Factorization Property*. (For this definition, see e.g. [3] and [8]).

The concepts involved can be relativized by the following

Definition. Let \mathcal{C} be a class of algebras of the same type and \mathcal{M} its subclass. We say that:

(i) \mathcal{C} has \mathcal{M} -directly decomposable congruences provided for every congruence Θ on each $A \in \mathcal{C}$ with $A = \prod_{i=1}^n A_i$ for $A_1, \dots, A_n \in \mathcal{M}$ there exists Θ_i on A_i such that

$$\Theta = \prod_{i=1}^n \Theta_i.$$

(ii) \mathcal{C} has \mathcal{M} -directly decomposable homomorphisms provided for every $A, B \in \mathcal{C}$, $A = \prod_{i=1}^n A_i$, $B = \prod_{i=1}^n B_i$ with $A_i, B_i \in \mathcal{M}$, every homomorphism h of A onto B is directly decomposable.

(iii) \mathcal{C} has \mathcal{M} -UFP provided every $A \in \mathcal{C}$ has \mathcal{M} -UFP.

Theorem 1. Let \mathcal{C} be a class of algebras of the same type and \mathcal{M} its arbitrary subclass closed under homomorphic images. Then the implication (a) \Rightarrow (b) is satisfied, where:

- (a) \mathcal{C} has \mathcal{M} -directly decomposable congruences and \mathcal{M} -UFP;
- (b) \mathcal{C} has \mathcal{M} -directly decomposable homomorphisms.

Proof. Suppose $A, B \in \mathcal{C}$, $A = \prod_{i=1}^n A_i$, $B = \prod_{i=1}^n B_i$ with $A_i, B_i \in \mathcal{M}$ and let h be a homomorphism of A onto B . Let Θ_h be a congruence induced by h on A . By (a), $\Theta_h = \prod_{i=1}^n \Theta_i$, where Θ_i is a congruence on A_i . By Lemma, there exist $B_i^* \in \mathcal{C}$ ($i = 1, \dots, n$) and homomorphisms h_i^* of A_i onto B_i^* with $B \cong \prod_{i=1}^n B_i^*$ (an isomorphism f) and $h \cdot f = \prod_{i=1}^n h_i$. Since \mathcal{M} is closed under homomorphic images and $B_i^* = h_i^*(A_i)$, also $B_i^* \in \mathcal{M}$ for $i = 1, \dots, n$. As \mathcal{C} has \mathcal{M} -UFP, there exists a permutation σ of $\{1, \dots, n\}$ such that $B_i \cong B_{\sigma(i)}^*$. Denote by g_i this isomorphism $B_{\sigma(i)}^* \cong B_i$ and put $h_i = h_i^* \cdot g_i$. Then clearly

$$h = h \cdot f \cdot \prod_{i=1}^n g_i = \prod_{i=1}^n h_i^* \cdot \prod_{i=1}^n g_i = \prod_{i=1}^n (h_i^* \cdot g_i) = \prod_{i=1}^n h_i.$$

Hence \mathcal{C} has \mathcal{M} -directly decomposable homomorphisms. Q.E.D.

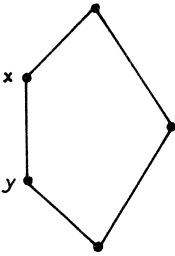


Fig. 1.

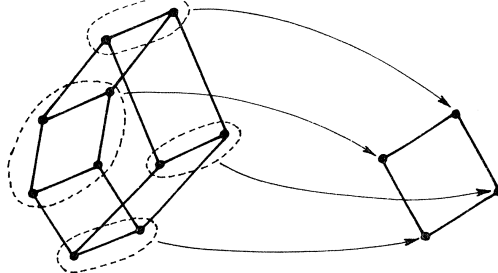


Fig. 2.

Remark. If a subclass \mathcal{M} of \mathcal{C} is not closed under homomorphic images, the implication (a) \Rightarrow (b) of Theorem 1 need not be satisfied, even if \mathcal{M} is a subclass of directly irreducible algebras. It can be demonstrated by the following

Example. Let $\mathcal{M} = \{N_5, I_2\}$, where N_5 is the non-modular pentagon (Fig. 1) and I_2 the two-element lattice, and let \mathcal{C} be a class formed by five lattices: a one-element lattice E and $N_5, I_2, A = N_5 \times I_2, B = I_2 \times I_2$. Let h be a homomorphism of A onto B visualized by arrows in Fig. 2. Clearly \mathcal{C} has \mathcal{M} -UFP and \mathcal{M} -directly

decomposable congruences as follows directly from Corollary 1 in [2] (namely the congruence Θ induced by h satisfies $\Theta = \Theta_1 \times \Theta_2$, where Θ_1 on N_5 collapses onto x, y – see Fig. 1, and $\Theta_2 = I_2 \times I_2$), however h is not directly decomposable because h_1 induced by Θ_1 maps N_5 onto the whole B contrary to h_2 induced by Θ_2 which collapses I_2 onto the one-element lattice E .

Theorem 2. *Let \mathcal{C} be a class of algebras of the same type containing a one-element algebra E , let \mathcal{M} be an arbitrary subclass of \mathcal{C} closed under homomorphic images such that $E \in \mathcal{M}$. If \mathcal{C} has \mathcal{M} -directly decomposable congruences, the following conditions are equivalent:*

- (1) \mathcal{C} has \mathcal{M} -directly decomposable homomorphisms,
- (2) \mathcal{C} has \mathcal{M} -UFP.

Proof. The implication (2) \Rightarrow (1) follows directly from Theorem 1. Prove (1) \Rightarrow (2). Let $A \in \mathcal{C}$ and $A \cong \prod_{i=1}^n A_i \cong \prod_{j=1}^m B_j$, where $A_i, B_j \in \mathcal{M}$ and $\text{card } A_i > 1, \text{card } B_j > 1$. Without loss of generality, suppose $m \leq n$. Put $B_j = E$ for $j = m + 1, \dots, n$. Then $\prod_{i=1}^n A_i \cong \prod_{j=1}^n B_j$; denote this isomorphism by h . As h is \mathcal{M} -directly decomposable, thus, for every $i = 1, \dots, n$, A_i is isomorphic to $B_{\sigma(i)}$ for a suitable permutation σ of $\{1, \dots, n\}$. Hence $\text{card } B_j > 1$ for $j = 1, \dots, n$, i.e. $m = n$ and A has \mathcal{M} -UFP. Q.E.D.

Necessary and sufficient conditions for the direct decomposition of congruences are derived by G. A. Fraser and A. Horn in [2].

Corollary 1. *Let \mathcal{C} be a class of algebras with distributive congruences and \mathcal{M} an arbitrary class of directly irreducible algebras of \mathcal{C} closed under homomorphic images and containing a trivial algebra E of \mathcal{C} . Then*

- (a) \mathcal{C} has \mathcal{M} -directly decomposable homomorphisms if and only if \mathcal{C} has \mathcal{M} -UFP.
- (b) If $A \in \mathcal{M}$ implies that A is finite and a trivial algebra is a subalgebra of A , then \mathcal{C} has \mathcal{M} -directly decomposable homomorphisms.

Proof. If \mathcal{C} has distributive congruences, then by Corollary 1 in [2], \mathcal{C} has \mathcal{M} -directly decomposable congruences and (a) follows immediately from Theorem 2.

If algebras from \mathcal{M} are finite and contain a trivial algebra of \mathcal{C} as a subalgebra, then by (i) in [4] (p. 285), \mathcal{C} has also \mathcal{M} -UFP. By virtue of the just proved (a), also (b) is valid. Q.E.D.

As an application, we can concentrate on the class of lattices. Let \mathcal{C} be an arbitrary non-trivial variety of lattices and \mathcal{M} the class of all chains with least elements. Then:

- (i) \mathcal{C} has distributive congruences (see [1], Theorem VI.9),
- (ii) \mathcal{C} has \mathcal{M} -UFP (see Theorem 2 in [8] for $\oplus = \vee$ and $\omega = \wedge$),
- (iii) $\mathcal{M} \subseteq \mathcal{C}$ (since every non-trivial variety of lattices contains a variety of distributive lattices containing \mathcal{M}),

- (iv) \mathcal{M} is closed under homomorphic images (evident),
- (v) \mathcal{M} contains a one-element lattice (evident),
- (vi) \mathcal{M} contains only directly irreducible lattices (see [1]).

Hence, by Corollary 1 (a), we obtain

Corollary 2. *Let $A_1, \dots, A_n, B_1, \dots, B_n$ be chains with least elements and $A = \prod_{i=1}^n A_i, B = \prod_{i=1}^n B_i$. Then every homomorphism h of the lattice A onto B is directly decomposable.*

An analogous result can be proved if a variety of weakly associative lattices or the class of tournaments is considered instead of \mathcal{C} or \mathcal{M} , respectively. The result obtained is contained in [7].

Let \mathcal{C} be a class of relatively complemented finite modular lattices and \mathcal{M} a subclass of all directly irreducible lattices of \mathcal{C} . Then clearly the previous conditions (i), (iii), (v) and (vi) are satisfied. By Theorem 7 in [1] (p. 89), \mathcal{C} contains only modular geometric lattices (Definition in [1], p. 80) and by Corollary on p. 93 in [1], all lattices in \mathcal{M} are simple. Hence, \mathcal{M} is closed under homomorphic images, thus also (iv) is satisfied. Since lattices of \mathcal{C} are relative by complemented, \mathcal{C} has permutable congruences ([1], p. 163, Ex. 7). As lattices in \mathcal{C} are finite, they have congruence lattices with finite lengths and, by Corollary 1 on p. 169 in [1], \mathcal{C} has \mathcal{M} -UFP. Hence (ii) is also true and by Corollary 2(a), we have

Corollary 3. *Let \mathcal{C} be a class of relatively complemented finite modular lattices, let $A, B \in \mathcal{C}$ and $A = \prod_{i=1}^n A_i, B = \prod_{i=1}^n B_i$, where A_i, B_i are directly irreducible lattices of \mathcal{C} . Then every homomorphism h of A onto B is directly decomposable.*

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