Czechoslovak Mathematical Journal

Hilbert Levitz

The Cartesian product of sets and the Hessenberg natural product of ordinals

Czechoslovak Mathematical Journal, Vol. 29 (1979), No. 3, 353–358

Persistent URL: http://dml.cz/dmlcz/101618

Czechoslovak Mathematical Journal, 29 (104) 1979, Praha

THE CARTESIAN PRODUCT OF SETS AND THE HESSENBERG NATURAL PRODUCT OF ORDINALS

HILBERT LEVITZ, Tallahassee

(Received July 23, 1976)

Dedicated to LEON J. LEVITZ

1. INTRODUCTION

Let $\prod_{i=1}^{n} A_i$ be the cartesian product of well ordered sets A_i with order type α_i. The *coordinatewise partial ordering* is defined by $(x_1, x_2, \ldots, x_n) \leqq (y_1, y_2, \ldots, y_n)$ if and only if $x_i \leqq y_i$ for each $1 \leqq i \leqq n$. Our result is that for any subset of $\prod_{i=1}^{n} A_i$, all extensions of this partial ordering to a well ordering have order types which do not exceed $\pi_{i=1}^{n} \alpha_i$, product here being understood as the Hessenberg "natural product" of ordinals [1]. A definition of this product will be given later in this article. Our result can be used to obtain ordinal bounds for certain classes of functions well ordered by the majorization relation. We shall show such an application at the end. We came to this result in trying to solve a problem proposed separately by SKOLEM and TARSKI, namely to calculate the ordinal of the full family of functions described in [2]. As yet this problem is still unresolved.

2. CONVENTIONS

N will denote the set of positive integers. n, m range over N. k ranges over $N \cup \{0\}$. i, j range over the set $\{1, 2, \ldots, n\}$. $\alpha, \beta, \gamma, \delta$ will be used in denoting ordinals. If a relation $\leqq$ on Y induces a well ordering on $X \subseteq Y$, then $|X : \leqq|$ denotes the order type of X under this induced ordering. When the ordering is understood from the context, we simply write $|X|$. If $s \in X$ we will denote the initial segment $\{x \in X \mid x < s\}$ by X_s. If $\delta < |X|$ then X^δ will denote the initial segment of X determined by the member of X corresponding to δ when X is mapped by its ordering function onto an initial segment of the ordinals. If δ is of the form $\omega^\gamma . (k + 1)$ then $\bar{A}^\delta$ denotes

$\{x \mid x \in A^{\omega^\gamma(k+1)} \wedge x \notin A^{\omega^\gamma k}\}$. $\alpha \# \beta$ and $\alpha \circ \beta$ will denote the Hessenberg natural sum and product respectively of the ordinals α, β. $\overset{n}{\underset{i=1}{\sigma}}$ and $\overset{n}{\underset{i=1}{\pi}}$ are operators which take the Hessenberg sum and product respectively of $\alpha_1, \alpha_2, \ldots, \alpha_n$. $(\prod_j A_j)[B]_i$ will denote the cartesian product obtained from $\prod_j A_j$ by replacing the factor A_i by B. $(\prod_j \alpha_j)[\beta]_i$ has a similar meaning for Hessenberg product. For $n > 1$ $(x_1, \ldots, \hat{x}_i, \ldots, x_n)$ is the $n - 1$ tuple obtained from $(x_1, x_2, \ldots, x_n)$ by deleting the ith coordinate. $(\prod_{j=1}^{n} A_j)_i$ and $(\overset{n}{\underset{j=1}{\pi}} \alpha_j)_i$ are the products obtained from $\prod_{j=1}^{n} A_j$ and $\overset{n}{\underset{j=1}{\pi}} \alpha_j$ respectively by deleting the ith factor. If $\mathscr{A} \subseteq \prod_j A_j$ then $\mathscr{A}_i^\delta$ denotes $\mathscr{A} \cap ((\prod_j A_j)[(A_i)^\delta]_i)$. If δ is of the form $\omega^\gamma . (k+1)$, $\bar{\mathscr{A}}_i^\delta$ denotes $\mathscr{A} \cap ((\prod_j A_j)[(\bar{A}_i)^\delta]_i)$.

3. HESSENBERG NATURAL PRODUCT

In this section we review some basic properties of ordinals [1]. Every ordinal $\alpha \neq 0$ has a unique Cantor normal form

$$\alpha = \omega^{\alpha_1} . s_1 + \omega^{\alpha_2} . s_2 + \ldots + \omega^{\alpha_n} . s_n$$

where $\alpha_1 > \alpha_2 > \ldots > \alpha_n$ and $s_i \in N$. It is convenient to think of this as an infinite sum $\sum_v \omega^{\alpha_v} . n_v$ where all but finitely many of the coefficients are zero. The Hessenberg natural sum $\alpha \# \beta$ is defined by $\alpha \# \beta = \sum_v \omega^{\alpha_v} . (s_v + t_v)$ where α, β have the normal forms $\sum_v \omega^{\alpha_v} . s_v$ and $\sum_v \omega^{\alpha_v} . t_v$. In case α or β is zero the natural sum is defined to be zero. Natural sum is commutative, associative, and it is a strictly increasing function of each of its arguments. Ordinals of the form ω^γ are called *main* ordinals. They determine initial segments of the ordinals closed under both ordinary and natural sum. Every ordinal $\alpha \neq 0$ has a normal form $\overset{n}{\underset{i=1}{\sigma}} \omega^{\alpha_i} . s_i = \omega^{\alpha_1} . s_1 \# \omega^{\alpha_2} . s_2 \# \ldots \ldots \# \omega^{\alpha_n} . s_n$ where the exponents no longer need to be decreasing but merely distinct. This representation is unique except for the order of the terms. We can also write a *weak* normal form $\overset{m}{\underset{v=1}{\sigma}} \omega^{\alpha_v}$ where the exponents no longer need to be distinct and the coefficients are ones. This is unique except for the order of the terms.

The Hessenberg natural product $\alpha \circ \beta$ is defined by $\alpha \circ 0 = 0$, $0 \circ \beta = 0$, and for $\alpha = \underset{v}{\sigma} \omega^{\alpha_v} . s_v$, $\beta = \underset{u}{\sigma} \omega^{\beta_u} . t_u$, $\alpha \circ \beta = \underset{u,v}{\sigma} \omega^{\alpha_v \# \beta_u} . s_v . t_u$. It is easy to see that if α, β have weak normal forms $\alpha = \underset{v}{\sigma} \omega^{\alpha_v}$, $\beta = \underset{u}{\sigma} \omega^{\alpha_u}$ then $\alpha \circ \beta = \underset{u,v}{\sigma} \omega^{\alpha_u \# \beta_v}$. Natural product is commutative, associative, and distributes over natural sum; it is a strictly increasing function of each of its arguments when the other argument is not zero.

Lemma 1. *Suppose that $A \cup B$ is a well ordered set and that under the induced ordering $|A| = \alpha$ and $|B| = \beta$, then $|A \cup B| \leqq \alpha \# \beta$.*

Proof. By transfinite induction on $\alpha \# \beta$ over the usual ordering of the ordinals. Let A, B, α, β be given as stated above. Our induction hypothesis is that for any well ordered set $C \cup D$ where $|C| = \gamma$, $|D| = \delta$ and $\gamma \# \delta < \alpha \# \beta$, it is the case that $|C \cup D| \leqq \gamma \# \delta$. If $A \cup B = \emptyset$ the desired result is trivial. Assume $A \cup B \neq \emptyset$. We shall show that $|A \cup B| \leqq \alpha \# \beta$ by showing that $|(A \cup B)_s| < \alpha \# \beta$ for each initial segment $(A \cup B)_s$. Let $s \in (A \cup B)$ be given.

Case 1. $s \in A$; then $(A \cup B)_s = A_s \cup D$ where $D \subseteq B$. Thus

$$(*) \qquad |(A \cup B)_s| = |A_s \cup D| .$$

Now $|A_s| < \alpha$ and $|D| \leqq \beta$, so $|A_s| \# |D| < \alpha \# \beta$. This gives us the right to invoke the induction hypothesis to get

$$|A_s \cup D| \leqq |A_s| \# |D| < \alpha \# \beta .$$

This with $(*)$ gives $|(A \cup B)_s| < \alpha \# \beta$ as desired.

Case 2. $s \in B$; then owing to the fact that set union and natural sum of ordinals are commutative, we can argue the same as in case 1 but with the roles of A and B interchanged.

It is easy to show that the lemma fails for unions of infinitely many sets.

Lemma 2. *For every finite sequence $\{A_i\}_{i=1}^n$ of well ordered sets such that $|A_i|$ is a main ordinal ω^{α_i} each i, for every $\mathscr{A} \subseteq \prod_i A_i$, and for every extension $\leqq_{\mathscr{A}}$ of the coordinatewise ordering on $\mathscr{A}$ to a well ordering of A, $|A : \leqq_{\mathscr{A}}| \leqq \pi_i \omega^{\alpha_i}$.*

Proof. We argue by transfinite induction over the class of ordered pairs of ordinals $(n, \pi_i \omega^{\alpha_i})$ ordered lexicographically from the left. Let $\{A_i\}_{i=1}^n$, $\mathscr{A}$, $\leqq_{\mathscr{A}}$ be given as stated. Our induction assumption is that the lemma holds for all sequences $\{B_i\}_{i=1}^m$ where $|B_i|$ is a main ordinal ω^{β_i} provided that the ordered pair $(m, \pi_i \omega^{\beta_i})$ precedes the ordered pair $(n, \pi_i \omega^{\alpha_i})$ in the lexicographical ordering from the left. If $n = 1$, the result is trivial. Let $n > 1$ be given.

Case 1. $\alpha_i = 0$ for at least one i. Let such an i be given. In this case A_i consists of a single element. We can define a one to one mapping ψ from $\prod_j A_j$ into $(\prod_j A_j)_i$ by

$$\psi(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, \hat{x}_i, \ldots, x_n) .$$

This induces a one to one mapping ψ^* from $\mathscr{A}$ into $(\prod_j A_j)_i$. Let $\mathscr{B}$ denote the range of ψ^*. Order $\mathscr{B}$ by an ordering $\leqq_{\mathscr{B}}$ defined by the condition that ψ^* be order preserving. From the very definition of $\mathscr{B}$ and $\leqq_{\mathscr{B}}$ we have that $|\mathscr{A} : \leqq_{\mathscr{A}}| = |\mathscr{B} : \leqq_{\mathscr{B}}|$. Moreover, it is easy to see that $\leqq_{\mathscr{B}}$ is an extension of the coordinatewise ordering. On the other hand since $\mathscr{B} \subseteq (\prod_j A_j)_i$ which is a product of $n - 1$ sets we can invoke the induction hypothesis to get $|\mathscr{B} : \leqq_{\mathscr{B}}| \leqq (\prod_j \omega^{\alpha_j})_i = \prod_j \omega^{\alpha_j}$, the latter equality holding because $\omega^{\alpha_i} = 1$. This with the equality a few lines back gives us the desired conclusion for this case.

Case 2. $\alpha_i > 0$ all i. Throughout this case all the orderings in question will be induced by $\leqq_{\mathscr{A}}$, so we shall suppress the symbol $\leqq_{\mathscr{A}}$. We shall now show that for each $\delta < \omega^{\alpha_i}$

$$|\mathscr{A}_i^\delta| < \pi_j \omega^{\alpha_j} . \tag{1}$$

Let i, δ be given

Case 2.1. α_i is a limit ordinal. Then there exists ω^γ such that $\delta < \omega^\gamma < \omega^{\alpha_i}$. The following assertions are easy to verify:

a) $\mathscr{A}_i^\delta \subseteq \mathscr{A}_i^{\omega^\gamma}$

b) $|\mathscr{A}_i^\delta| \leqq |\mathscr{A}_i^{\omega^\gamma}|$

c) $\mathscr{A}_i^{\omega^\gamma} \subseteq (\prod_j A_j) [A_i^{\omega^\gamma}]_i$

d) $(\pi_j \omega^{\alpha_j}) [\omega^\gamma]_i < \pi_j \omega^{\alpha_j}$

From c) and d) we earn the right to invoke the induction hypothesis to get $|\mathscr{A}_i^{\omega^\gamma}| \leqq$ $\leqq (\pi_j \omega^{\alpha_j}) [\omega^\gamma]_i < \pi_j \omega^{\alpha_j}$, this with b) gives the desired conclusion (1).

Case 2.2. α_i is a successor ordinal $\gamma + 1$. In this case there exists an integer $k \geqq 0$ such that $\omega^\gamma . k \leqq \delta < \omega^\gamma . (k + 1)$. From our notational conventions together with the fact that cartesian product and intersection distribute over set union we get

$$\mathscr{A}_i^{\omega^\gamma (k+1)} = \bigcup_{0 \leqq s \leqq k} \bar{\mathscr{A}}_i^{\omega^\gamma (s+1)} . \tag{2}$$

The following are easy to verify:

a′) $\mathscr{A}_i^\delta \subseteq \mathscr{A}_i^{\omega^\gamma (k+1)}$

b′) $|\mathscr{A}_i^\delta| \leqq |\mathscr{A}_i^{\omega^\gamma (k+1)}|$

c′) $|\bar{\mathscr{A}}_i^{\omega^\gamma (s+1)}| \leqq \omega^\gamma$ for each $0 \leqq s \leqq k$

d′) $\bar{\mathscr{A}}_i^{\omega^\gamma (s+1)} \subseteq (\prod_j A_j) [\bar{A}_i^{\omega^\gamma (s+1)}]_i$ for $0 \leqq s \leqq k$

e′) $(\pi_j \omega^{\alpha_j}) [\omega^\gamma]_i < \pi_j \omega^{\alpha_j}$

By virtue of c′), d′) and e′) we have the right to invoke the induction hypothesis to get

$$(3) \qquad \left|\overline{\mathscr{A}}_i^{\omega^\gamma(s+1)}\right| \leqq \left(\pi_j \omega^{\alpha_j}\right)[\omega^\gamma]_i \quad \text{for each} \quad 0 \leqq s \leqq k\,.$$

Using this with (2) and lemma 1 we get

$$(4) \qquad \left|\mathscr{A}_i^{\omega^\gamma(k+1)}\right| = \left|\bigcup_{0 \leqq s \leqq k} \overline{\mathscr{A}}_i^{\omega^\gamma(s+1)}\right| \leqq \left(\left(\pi_j \omega^{\alpha_j}\right)[\omega^\gamma]_i\right) . (k+1)\,.$$

Using the fact that main ordinals determine initial segments closed under addition we get

$$(5) \qquad \left(\left(\pi_j \omega^{\alpha_j}\right)[\omega^\gamma]_i\right) . (k+1) < \left(\pi_j \omega^{\alpha_j}\right)[\omega^{\gamma+1}]_i = \pi_j \omega^{\alpha_j}\,.$$

Putting (4) and (5) together with b′) we get (1) as desired.

We now proceed to show $|\mathscr{A}| \leqq \pi_i \omega^{\alpha_i}$ by showing that for each $\boldsymbol{a} = (a_1, a_2, \ldots, a_n) \in \mathscr{A}$ the initial segment $\mathscr{A}_{\boldsymbol{a}}$ has order type less than $\pi_i \omega^{\alpha_i}$. Let $\boldsymbol{a} = (a_1, a_2, \ldots, a_n) \in \mathscr{A}$ be given. Let $(r_1, r_2, \ldots, r_n) <_{\mathscr{A}} (a_1, a_2, \ldots, a_n)$ be given. Clearly $r_i < a_i$ for at least one i and for such an i, $(r_1, r_2, \ldots, r_n) \in \mathscr{A}_i^{\delta_i}$ where δ_i denotes the order type of the initial segment of A_i determined by a_i. Thus $\mathscr{A}_{\boldsymbol{a}} \subseteq \bigcup_{i=1}^{n} \mathscr{A}_i^{\delta_i}$. From this along with lemma 1 follows

$$(6) \qquad |\mathscr{A}_{\boldsymbol{a}}| \leqq \left|\bigcup_{i=1}^{n} \mathscr{A}_i^{\delta_i}\right| \leqq \sigma_{i=1}^{n} \left(\left|\mathscr{A}_i^{\delta_i}\right|\right).$$

Now by (1) each term in the right hand sum is strictly less than $\pi_j \omega^{\alpha_j}$, and since main ordinals determine initial segments closed under addition,

$$\sigma_{i=1}^{n} \left(\left|\mathscr{A}_i^{\delta_i}\right|\right) < \pi_j \omega^{\alpha_j}\,.$$

This with (6) gives

$$|\mathscr{A}_{\boldsymbol{a}}| < \pi_j \omega^{\alpha_j}\,. \quad \text{(QED)}$$

Theorem. *Let $\{A_i\}_{i=1}^n$ be a finite sequence of non-empty well ordered sets with $|A_i| = \alpha_i$ each i. Then for every $\mathscr{A} \subseteq \prod_i A_i$ and every well ordered extension $\leqq_{\mathscr{A}}$ of the coordinatewise partial ordering of $\mathscr{A}$, $|\mathscr{A} : \leqq_{\mathscr{A}}| \leqq \pi_i \alpha_i$.*

Proof. Let $\{A_i\}$, $\mathscr{A}$, $\leqq_{\mathscr{A}}$ be given as stated. Write $|A_i|$ in weak Cantor normal form $\sigma_{j=1}^{m_i} \omega^{\alpha_{ij}}$. Using this we can see that A_i can be written $\bigcup_{j=1}^{m_i} A_{ij}$ where $|A_{ij}| = \omega^{\alpha_{ij}}$ Let $S = \{(j_1, j_2, \ldots, j_n) \mid 1 \leqq j_i \leqq m_i\}$. For $s = (j_1, j_2, \ldots, j_n) \in S$ let $\mathscr{B}_s$ denote

$\mathscr{A} \cap (\prod_{i=1}^{n} A_{ij_i})$ and let β_s denote $\pi_{i=1}^{n} \omega^{\alpha_{ij_i}}$. By Lemma 2 $|\mathscr{B}_s| \leqq \beta_s$, so by lemma 1 $|\bigcup_{s\in S} \mathscr{B}_s| \leqq \sigma_{s\in S}(\beta_s)$. Now using these inequalities with the facts that cartesian product distributes over union, intersection distributes over union, and Hessenberg product distributes over Hessenberg sum we get,

$$|\mathscr{A}| = |\mathscr{A} \cap \prod_i A_i| = |\bigcup_{s\in S} \mathscr{B}_s| \leqq \sigma_{s\in S}(\beta_s) = \pi_i \alpha_i \, .$$

5. APPLICATION

Consider the family $\mathscr{F}$ of functions from N into N of the form

$$p(x)^x + q(x)^x \tag{7}$$

where $p(x)$, $q(x)$ range over the set $N[x]$ of polynomial functions with coefficients in N. It is known [2] that this family is well ordered by the *majorization* relation $\leqq$ defined by $f \leqq g$ if and only if there exists $n_0 \in N$ such that $f(x) \leqq g(x)$ for all $x \geqq n_0$. We shall show that $\omega^{\omega \cdot 2}$ is a bound for this ordering. For the polynomial functions, the majorization relation coincides with the ordering obtained by ordering the coefficients lexicographically, and on account of this, $|N[x]| = \omega^\omega$. Consider the functional which sends the ordered pair $(p(x), g(x)) \in N[x] \times N[x]$ to the function (7). This functional, of course is many to one. We can get a one to one "inverse" functional ψ from $\mathscr{F}$ back into $N[x] \times N[x]$ by choosing for each $f \in \mathscr{F}$ just one of its inverse images. Let $\mathscr{A}$ denote the range of ψ. Define a well ordering $\leqq_{\mathscr{A}}$ of $\mathscr{A}$ by the condition that ψ be order preserving. It is easy to verify that $\leqq_{\mathscr{A}}$ is an extension of the coordinatewise ordering of $\mathscr{A} \subseteq N[x] \times N[x]$, the ordering in the factor spaces being majorization. By the main theorem the Hessenberg product $\omega^\omega \circ \omega^\omega$ (which in this case is also the usual product $\omega^\omega \omega^\omega = \omega^{\omega \cdot 2}$) is a bound for $|\mathscr{A} : \leqq_{\mathscr{A}}|$ and consequently for $|\mathscr{F} : \leqq|$.

Similar results can be obtained when (7) has more than 2 terms, and for the class of functions of the form $p(x)^{q(x)}$ where $p(x), q(x) \in N[x]$.

References

[1] *Bachmann, H.:* Transfinite Zahlen, Springer Verlag, Berlin—Heidelberg—New York, 1967.

[2] *Ehrenfeucht, A.:* Polynomical functions with exponentiation are well ordered, Algebra Universalis, vol. *3*, (1973) pp. 261—262.

Author's address: The Florida State University, Department of Mathematics, Tallahassee, Florida 32306, U.S.A.