Jan Troják; Jiří Vanžura

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A CHARACTERIZATION OF HYPERSPHERES IN THE QUATERNIONIC SPACE

JAN TROJÁK, Praha, JIŘÍ VANŽURA, Olomouc

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Let us consider the 4-dimensional euclidean space R^4, which we shall identify in the natural way with the division algebra H of quaternions. The left multiplication by the quaternionic units i, j, k induces on R^4 three tensor fields I_1, I_2, I_3 of type $(1,1)$ satisfying

$$I_i I_j + I_j I_i = -2\delta_{ij} I ,$$

i.e. a quaternionic structure. At any point $x \in R^4$ the tensors I_1, I_2, I_3 are orthogonal automorphisms of the tangent space $T_x(R^4)$.

We shall investigate the structure induced on a 3-dimensional submanifold $M \subset R^4$ by the quaternionic structure on R^4. Let us suppose that M is orientable and let us denote by N the field of positive unit normals on M. From the above mentioned properties of the tensors I_1, I_2, I_3 it follows easily that

$$\langle I_i N, N \rangle = 0 , \quad \langle I_i N, I_j N \rangle = \delta_{ij}$$

for any $i, j = 1, 2, 3$.

It enables us to define three orthonormal tangent vector fields $V_1 = I_1 N$, $V_2 = I_2 N$, $V_3 = I_3 N$ on M obtaining thus on M a complete parallelism. We write

$$[V_1, V_2] = a_1 V_1 + a_2 V_2 + a_3 V_3 ,$$
$$[V_2, V_3] = b_1 V_1 + b_2 V_2 + b_3 V_3 ,$$
$$[V_3, V_1] = c_1 V_1 + c_2 V_2 + c_3 V_3$$

with a_i, b_i, c_i; $i = 1, 2, 3$ being functions on M.

Taking for M a hypersphere of radius r we get

$$[V_1, V_2] = -\frac{2}{r} V_3 , \quad [V_2, V_3] = -\frac{2}{r} V_1 , \quad [V_3, V_1] = -\frac{2}{r} V_2 .$$

The goal of the present note is to prove the following

Theorem. *Let M be a connected oriented 3-dimensional submanifold of R^4 on which the complete parallelism V_1, V_2, V_3 satisfies*

$$[V_1, V_2] = -\frac{2}{r} V_3\,, \quad [V_2, V_3] = -\frac{2}{r} V_1\,, \quad [V_3, V_1] = -\frac{2}{r} V_2\,.$$

Then M is part of a hypersphere with radius r.

For the proof we shall need two lemmas.

Lemma 1. *Let ∇ denote the Levi-Civita connection on M and let us write $\nabla_{I_iN}(I_jN) = \Gamma^k_{ij} I_k N$. Then*

$$\Gamma^k_{ij} = -\frac{1}{r} \operatorname{sgn} \begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix} \text{ if } \begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix} \text{ is a permutation and } \Gamma^k_{ij} = 0 \text{ otherwise}.$$

Proof. Using the basic properties of the Levi-Civita connection we can write the identities

$$(1) \qquad 2\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle + \\ + \langle Z, [X, Y]\rangle + \langle Y, [Z, X]\rangle - \langle X, [Y, Z]\rangle\,,$$

$$(2) \qquad \langle \nabla_X Y, Z\rangle - \langle Z, [X, Y]\rangle = \langle \nabla_Y X, Z\rangle$$

with any $X, Y, Z \in T(M)$. The identity (1) enables us to evaluate

$$\nabla_{I_1N}(I_2N), I_3N\rangle = -\frac{1}{r}\{\langle I_3N, I_3N\rangle + \langle I_2N, I_2N\rangle - \langle I_1N, I_1N\rangle\}\,,$$

i.e. $\langle \Gamma^k_{12} I_k N, I_3 N\rangle = -1/r$. It follows that $\Gamma^3_{12} = -1/r$ and due to (2) we have $\Gamma^3_{21} = 1/r$. Similarly it can be shown that $\Gamma^2_{13} = \Gamma^1_{32} = 1/r$, $\Gamma^1_{23} = \Gamma^2_{31} = -1/r$. Furthermore,

$$\nabla_{I_1N}(I_2N), I_1N\rangle = \tfrac{1}{2}(I_2N)\langle I_1N, I_1N\rangle = 0$$

implies $\Gamma^2_{11} = 0$ and using the same argument we get $\Gamma^k_{ij} = 0$ whenever at least two of the indices i, j, k are equal.

Lemma 2. *Let b_{ij} denote the components of the second fundamental form of M with respect to the basis I_1N, I_2N, I_2N. Then*

$$b_{ij} = -\frac{1}{r}\delta_{ij}\,.$$

Proof. We denote by $\hat{\nabla}$ the canonical connection in R^4. Using Lemma 1 and the Gauss formula

$$\hat{\nabla}_{I_iN}(I_jN) = \nabla_{I_iN}(I_jN) + b_{ij}N$$

we can evaluate

$$\hat{\nabla}_{I_1N}N = -\hat{\nabla}_{I_1N}(I_1^2N) = -I_1\hat{\nabla}_{I_1N}(I_1N) = -I_1(\nabla_{I_1N}(I_1N) + b_{11}N) = -b_{11}I_1N\,,$$

$$\hat{\nabla}_{I_1N}N = -\hat{\nabla}_{I_1N}(I_2^2N) = -I_2\,\hat{\nabla}_{I_1N}(I_2N) = -I_2(\nabla_{I_1N}(I_2N) + b_{12}N) = \\ = \frac{1}{r}I_1N - b_{12}I_2N\,,$$

$$\hat{\nabla}_{I_1N}N = -\hat{\nabla}_{I_1N}(I_3^2N) = -I_3\,\hat{\nabla}_{I_1N}(I_3N) = -I_3(\nabla_{I_1N}(I_3N) + b_{13}N) = \\ = \frac{1}{r}I_1N - b_{13}I_3N\,.$$

Comparing the right hand sides of the above equations we get $b_{11} = -1/r$, $b_{12} = b_{13} = 0$. Proceeding along the same lines we find easily $b_{22} = b_{33} = -1/r$, $b_{23} = 0$.

The proof of our theorem follows now easily from Lemma 2, which in fact says that every point of M is umbilical. See e.g. Theorem 5.1 in Chap. VII of [1].

Reference

[1] *Kobayashi S., Nomizu K.:* Foundations of differential geometry, Vol. II.

Authors' addresses: J. Troják, 186 00 Praha 8, Sokolovská 83, ČSSR (Matematicko-fyzikální fakulta UK); J. Vanžura, 771 46 Olomouc, Leninova 26, ČSSR (Přírodovědecká fakulta UP).