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THE ARGUMENT DELAY AND OSCILLATORY PROPERTIES OF DIFFERENTIAL EQUATION OF n-th ORDER

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In this paper we shall investigate the n-th order nonlinear delay-differential equation

$$u^{(n)}(t) + p(t)\, u^{\alpha}(\tau(t)) = 0\,, \quad n > 1\,, \quad 0 < \alpha < 1 \tag{1}$$

on $[t_0, \infty)$, where

(i) $0 \leqq p(t) \in C_{[t_0,\infty)}$; $p(t)$ is not identically zero in any neighborhood $\mathcal{O}(\infty)$,

(ii) $\tau(t) \in C_{[t_0,\infty)}$, $\tau(t) \leqq t$, $\lim\limits_{t\to\infty} \tau(t) = \infty$,

(iii) $\alpha = r/s$, where r and s are odd natural numbers.

Without mentioning them again, we shall assume the validity of conditions (i), (ii) and (iii) throughout the paper.

Our purpose in this work is to give an analogue of Theorem 1 of [3], and some results concerning the influence of the argument delay on oscillatory properties of solutions of the equation (1).

The basic initial-value problem for the equation (1) can be described as follows: Let $\phi(t)$ be a continuous function on an initial set E_{t_0}. Suppose that $u_0^{(j)}$, $j = 1, 2, \ldots$ $\ldots, n - 1$ are arbitrary real numbers. Find a function $u(t)$ defined on $E_{t_0} \cup [t_0, T)$ $(T \leqq \infty)$ which satisfies the initial conditions

$$u(t_0) = \phi(t_0)\,, \quad u^{(j)}(t_0 + 0) = u_0^{(j)} \quad \text{for} \quad j = 1, 2, \ldots, n - 1\,,$$

$$u(\tau(t)) = \phi(\tau(t)) \quad \text{for} \quad \tau(t) < t_0 \quad (t \in [t_0, T))$$

and for $t \in [t_0, T)$ satisfies the equation (1). This function $u(t)$ is called *the solution of the equation* (1).

Suppose that there exist solutions of the equation (1) on an interval of the form $[b, \infty)$ where $b \geqq t_0$. In the sequel we shall use the term "solution" only to denote a solution which exists on $[b, \infty)$ where $b \geqq t_0$. Moreover, we shall exclude from our considerations solutions of the equation of the type (1) with the property that $u(t) \equiv 0$ for $t \geqq T_1$ where $t_0 \leqq T_1 < \infty$.

We can now define a function that we shall need later.

Definition 1. Let $\gamma(t) = \sup \{s \geqq t_0 \mid \tau(s) \leqq t\}$ for $t \geqq t_0$.

From this definition we see that $t \leqq \gamma(t)$ and $\tau(\gamma(t)) = t$. Another property of the function $\gamma(t)$ is contained in the following lemma.

Lemma 1. *For every t such that $t_0 \leqq t < \infty$, the value $\gamma(t)$ is finite.*

Proof. Assume that for some $t_1 \in [t_0, \infty)$ the assertion of the lemma is false. Then for every natural number k, there exists a point $s_k > k$ such that $\tau(s_k) \leqq t_1$, which yields a contradiction with (ii) and completes the proof of the lemma.

Definition 2. A solution $u(t)$ of (1) is oscillatory for $t \geqq t_0$ if there exists an infinite sequence of points $\{t_i\}_{i=1}^{\infty}$ such that $u(t_i) = 0$ and $t_i \to \infty$ for $i \to \infty$. A solution $u(t)$ of (1) is nonoscillatory if there exists a number T_2 such that $t_0 \leqq T_2 < \infty$ and $u(t) \neq 0$ for $t \geqq T_2$.

I.

I. Ličko and M. Švec [3] investigated the differential equation

$$u^{(n)}(t) + p(t)\, u^{\alpha}(t) = 0\,, \tag{2}$$

which can be obtained from (1) by putting $\tau(t) \equiv t$. For the equation (2) they proved the following theorem (the theorem 1 in [3]):

Theorem A. Let $p(t)$ be a positive continuous function on $[t_0, \infty)$. Let $n > 1$, $0 < \alpha < 1$.

a) Let n be even. Then a necessary and sufficient condition for all solutions of (2) to be oscillatory is

$$\int_t^{\infty} x^{\alpha(n-1)}\, p(x)\, \mathrm{d}x = \infty\,, \quad t > t_0\,. \tag{3}$$

b) Let n be odd. Then condition (3) is necessary and sufficient for every solution of (2) to be either oscillatory or tending monotonically to zero together with its first $n - 1$ derivatives as $t \to \infty$.

For the proof of the theorem that will be an analogue of Theorem A, we shall need the following lemma.

Lemma 2. *Let $u(t) \in C^n_{[T,\infty)}$ and let either*

$$u(t) > 0\,, \quad u^{(n)}(t) \leqq 0 \quad \text{for} \quad t \geqq t_1 \geqq T\,, \tag{4}$$

or

$$u(t) < 0\,, \quad u^{(n)}(t) \geqq 0 \quad \text{for} \quad t \geqq t_1 \geqq T\,. \tag{5}$$

Let $u^{(n)}(t)$ *be not identically zero in any neighborhood* $\mathcal{O}(\infty)$. *Then*

1) *there exists a number* $t_2 \geqq t_1$ *such that the functions* $u^{(j)}(t), j = 1, 2, \ldots, n - 1$ *are eventually of constant sign on* $[t_2, \infty)$;

2) *there exists a number* $k \in \{1, 3, 5, \ldots, n - 1\}$ *or* $k \in \{0, 2, 4, \ldots, n - 1\}$ *if* n *is even or odd, respectively, such that*

$$(6) \qquad u(t)\, u^{(j)}(t) > 0 \quad \textit{for} \quad j = 0, 1, 2, \ldots, k \quad \textit{and} \quad t \geqq t_2\,,$$

$$(-1)^{k+j}\, u(t)\, u^{(j)}(t) > 0 \quad \textit{for} \quad j = k + 1, k + 2, \ldots, n - 1 \quad \textit{and} \quad t \geqq t_2\,;$$

3) *either*

$$(7) \qquad u(s) \,.\, \lim_{t\to\infty} u^{(j)}(t) > 0 \quad \textit{for} \quad j = 0, 1, 2, \ldots, q \quad \textit{and} \quad s \geqq t_2\,,$$

$$\lim_{t\to\infty} u^{(j)}(t) = 0 \quad \textit{for} \quad j = q + 1, q + 2, \ldots, n - 1\,,$$

where $q = k$ *if* $u(s) \,.\, \lim_{t\to\infty} u^{(k)}(t) > 0$, *and* $q = k - 1$ *if* $k > 0$ *and* $\lim_{t\to\infty} u^{(k)}(t) = 0$, *or*

$$(8) \qquad \lim_{t\to\infty} u^{(j)}(t) = 0 \quad \textit{for} \quad j = 0, 1, 2, \ldots, n - 1$$

if $k = 0$ *and* $\lim_{t\to\infty} u^{(k)}(t) = \lim_{t\to\infty} u(t) = 0$.

Proof. We give a proof for the case when (4) holds. The case (5) is treated similarly.

Now suppose (4) holds. Then $u^{(n-1)}(t)$ is a nonincreasing function for $t \geqq t_1$ and is not constant in any neighborhood $\mathcal{O}(\infty)$. This implies that exactly one of the following is true:

$$\text{a}_1) \qquad u^{(n-1)}(t) > 0 \quad \text{for} \quad t \geqq t_1\,,$$

$$\text{b}_1) \qquad u^{(n-1)}(t) < 0 \quad \text{for} \quad t \geqq T_1^{(n-1)} \geqq t_1\,.$$

From b_1) and (4) it follows that there exists a number $T_1^{(n-2)} \geqq T_1^{(n-1)}$ such that $u^{(n-2)}(t) < 0$ for $t \geqq T_1^{(n-2)}$. Likewise, we have $u^{(n-3)}(t) < 0$ for $t \geqq T_1^{(n-3)} \geqq \geqq T_1^{(n-2)}, \ldots, u(t) < 0$ for $t \geqq T_1^{(0)} \geqq T_1^{(1)}$, which is a contradiction, since $u(t) > 0$ for $t \geqq t_1$. Thus a_1) holds.

Now we know that $u^{(n-2)}(t)$ is increasing and concave for $t \geqq t_1$. Therefore exactly one of the following possibilities holds true:

$$\text{a}_2) \qquad u^{(n-2)}(t) > 0 \quad \text{for} \quad t \geqq T_2^{(n-2)} \geqq t_1\,,$$

$$\text{b}_2) \qquad u^{(n-2)}(t) < 0 \quad \text{for} \quad t \geqq t_1\,.$$

From a_2) and a_1), we obtain that $u^{(n-3)}(t) > 0$ for $t \geqq T_2^{(n-3)} \geqq T_2^{(n-2)}$. Analogously we get $u^{(n-4)}(t) > 0$ for $t \geqq T_2^{(n-4)} \geqq T_2^{(n-3)}, \ldots, u(t) > 0$ for $t \geqq T_2^{(0)} \geqq T_2^{(1)}$. Thus the functions $u^{(j)}(t)$ $(j = 1, 2, \ldots, n - 1)$ are of constant sign for t sufficiently large.

If b_2) holds, then $u^{(n-3)}$ is decreasing and convex for $t \geqq t_1$. Then exactly one of the following is true:

a_3) $$u^{(n-3)}(t) > 0 \quad \text{for} \quad t \geqq t_1 ,$$

b_3) $$u^{(n-3)}(t) < 0 \quad \text{for} \quad t \geqq T_3^{(n-3)} \geqq t_1 .$$

Here we can repeat the whole argument and show in this way that the functions $u^{(j)}(t)$ $(j = 1, 2, \ldots, n-1)$ are of constant sign for t sufficiently large (e.g. for $t \geqq t_2$), and the first part of Lemma 2 is proved.

The second part of Lemma 2 follows from the proof of the first part.

Now, we prove the third part of Lemma 2. By (6) we have $\lim_{t\to\infty} u^{(k)}(t) \geqq 0$ and if $k > 0$, then $\lim_{t\to\infty} u^{(j)}(t) > 0$ for $j = 0, 1, 2, \ldots, k-1$. Moreover, (6) implies $\lim_{t\to\infty} u^{(k+1)}(t) \leqq 0$. Suppose $\lim_{t\to\infty} u^{(k+1)}(t) = -a^2$ $(a \neq 0)$. Then $u^{(k)}(t) < 0$ for $t(\geqq t_2)$ sufficiently large, which is a contradiction, since $u^{(k)}(t) > 0$ if $t \geqq t_2$. So $\lim_{t\to\infty} u^{(k+1)}(t) = 0$. Further, (6) implies $\lim_{t\to\infty} u^{(k+2)}(t) \geqq 0$. If we assume $\lim_{t\to\infty} u^{(k+2)}(t) = a^2$ $(a \neq 0)$, we obtain $u^{(k+1)}(t) > 0$ for t $(\geqq t_2)$ sufficiently large, which is a contradiction, since $u^{(k+1)}(t) < 0$ if $t \geqq t_2$. Thus $\lim_{t\to\infty} u^{(k+2)}(t) = 0$. Analogously we get

$$\lim_{t\to\infty} u^{(k+3)}(t) = \lim_{t\to\infty} u^{(k+4)}(t) = \ldots = \lim_{t\to\infty} u^{(n-1)}(t) = 0 .$$

This completes the proof.

Theorem 1. *Let* $0 < \alpha < 1$, $n > 1$ *being a natural number.*

a) *Let n be even. Then a necessary and sufficient condition for all solutions of* (1) *to be oscillatory is*

(9) $$\int^{\infty} [\tau(t)]^{\alpha(n-1)} \cdot p(t)\, dt = \infty .$$

b) *Let n be odd. Then condition* (9) *is necessary and sufficient for every solution of* (1) *to be either oscillatory of tending monotonically to zero together with its first $n-1$ derivatives as $t \to \infty$.*

Proof. Sufficiency. — Let $u(t)$ be any nonoscillatory solution of (1), i.e. there exists $T \geqq t_0$ such that $u(t) \neq 0$ for $t \geqq T$. Without loss of generality we may suppose further that $u(t)$ is positive on $[T, \infty)$. Then (see (ii)) there exists $t_1 \geqq T \geqq t_0$ such that $u(\tau(t)) > 0$ for $t \geqq t_1$. Now, by (1), we have $u^{(n)}(t) \leqq 0$ for $t \geqq t_1$, but $u^{(n)}(t)$ is not identically zero in any neighborhood $\mathcal{O}(\infty)$ (see (i)). It is clear that this function $u(t)$ satisfies the conditions of Lemma 2. So we can use the assertions of this lemma.

Now suppose that (7) holds. Integrating (1) successively $n - q - 1$ times from t ($\geqq t_2$) to ∞ we have

$$(10) \qquad u^{(n-1)}(t) = \int_t^\infty p(x)\, u^\alpha(\tau(x))\, dx\,,$$

$$-u^{(n-2)}(t) = \int_t^\infty (x - t)\, p(x)\, u^\alpha(\tau(x))\, dx\,,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$(11) \qquad (-1)^{n-q} u^{(q+1)}(t) = \frac{1}{(n-q-2)!} \int_t^\infty (x - t)^{n-q-2}\, p(x)\, u^\alpha(\tau(x))\, dx\,.$$

Integrating (11) from v_1 to $v_2 (t_2 < v_1 < v_2)$ we obtain

$$(12) \qquad (-1)^{n-q} u^{(q)}(v_2) - (-1)^{n-q} u^{(q)}(v_1) =$$

$$= \frac{1}{(n-q-1)!} \int_{v_1}^{v_2} (x - v_1)^{n-q-1}\, p(x)\, u^\alpha(\tau(x))\, dx +$$

$$+ \frac{1}{(n-q-1)!} \int_{v_2}^\infty [(x - v_1)^{n-q-1} - (x - v_2)^{n-q-1}]\, p(x)\, u^\alpha(\tau(x))\, dx\,.$$

It is easy to verify by induction that

$$(13) \qquad (v_2 - v_1)^{n-q-1} \leqq (x - v_1)^{n-q-1} - (x - v_2)^{n-q-1}$$

for $v_1 < v_2 \leqq x$.

Therefore (12), together with (13) implies

$$(14) \qquad (-1)^{n-q} u^{(q)}(v_2) - (-1)^{n-q} u^{(q)}(v_1) \geqq$$

$$\geqq \frac{1}{(n-q-1)!} \int_{v_1}^{v_2} (x - v_1)^{n-q-1}\, p(x)\, u^\alpha(\tau(x))\, dx +$$

$$+ \frac{1}{(n-q-1)!} (v_2 - v_1)^{n-q-1} \int_{v_2}^\infty p(x)\, u^\alpha(\tau(x))\, dx\,.$$

Let $n - q$ be an even natural number. Then we see from (6) that k is odd if n is even and k is even if n is odd. For this reason $n - k$ is always odd. Thus $q \neq k$. But then $q = k - 1$ (by Lemma 2). Now (6) yields

$$u^{(q+1)}(t) = u^{(k)}(t) > 0 \quad \text{for} \quad t \geqq t_2$$

and

$$u^{(q)}(t) = u^{(k-1)}(t) > 0 \quad \text{for} \quad t \geqq t_2\,.$$

Hence $u^{(q)}(t)$ is an increasing and positive function if $t \geqq t_2$. It is clear that the first integral on the right-hand side of (14) is positive. Therefore we have from (14)

$$(15) \qquad u^{(q)}(v_2) > \frac{1}{(n-q-1)!}(v_2 - v_1)^{n-q-1} \int_{v_2}^{\infty} p(x)\, u^{\alpha}(\tau(x))\, dx\,.$$

Let $q > 0$. Integrating (15) from v_1 to $t (t_2 \leqq v_1 \leqq t)$ we get

$$u^{(q-1)}(t) - u^{(q-1)}(v_1) > \frac{1}{(n-q)!}\left[(t - v_1)^{n-q} \int_t^{\infty} p(x)\, u^{\alpha}(\tau(x))\, dx + \right.$$
$$\left. + \int_{v_1}^{t} (v_2 - v_1)^{n-q}\, p(v_2)\, u^{\alpha}(\tau(v_2))\, dv_2 \right].$$

The second integral on the right-hand side of this inequality is nonnegative and $u^{(q-1)}(v_1) > 0$ (by (6)). Thus we can write

$$(16) \qquad u^{(q-1)}(t) > \frac{1}{(n-q)!}(t - v_1)^{n-q} \int_t^{\infty} p(x)\, u^{\alpha}(\tau(x))\, dx\,.$$

Analogously, an integration of (16) $q - 1$ times from v_1 to t yields

$$(17) \qquad u^{(q-2)}(t) > \frac{1}{(n-q+1)!}(t - v_1)^{n-q+1} \int_t^{\infty} p(x)\, u^{\alpha}(\tau(x))\, dx$$

. . . .

$$u(t) > \frac{1}{(n-1)!}(t - v_1)^{n-1} \int_t^{\infty} p(x)\, u^{\alpha}(\tau(x))\, dx\,.$$

Note that for $q = 0$, the inequality (17) reduces to the inequality (15) (just replace v_2 by t). Therefore, for $n - q$ even, we can consider the case $q > 0$ together with the case $q = 0$.

Since (17) holds true, we get for $t \geqq \gamma(v_1)$

$$u(\tau(t)) > \frac{1}{(n-1)!}(\tau(t) - v_1)^{n-1} \int_{\tau(t)}^{\infty} p(x)\, u^{\alpha}(\tau(x))\, dx\,;$$

this implies directly

$$(17') \qquad u(\tau(t)) > \frac{1}{(n-1)!}(\tau(t) - v_1)^{n-1} \int_t^{\infty} p(x)\, u^{\alpha}(\tau(x))\, dx\,.$$

Raising both sides of the inequality (17′) to the power α and multiplying by $p(t)$, we obtain

$$\frac{p(t)\, u^{\alpha}(\tau(t))}{\left[\int_t^{\infty} p(x)\, u^{\alpha}(\tau(x))\, dx\right]^{\alpha}} > \left[\frac{1}{(n-1)!}\right]^{\alpha} (\tau(t) - v_1)^{\alpha(n-1)} \cdot p(t)\,.$$

Integrating the last inequality from $\gamma(v_1)$ to t $(t \geqq \gamma(v_1))$ we conclude

$$(18) \qquad \frac{1}{1-\alpha}\left\{\left[\int_{\gamma(v_1)}^{\infty} p(x)\, u^{\alpha}(\tau(x))\,\mathrm{d}x\right]^{1-\alpha} - \left[\int_{t}^{\infty} p(x)\, u^{\alpha}(\tau(x))\,\mathrm{d}x\right]^{1-\alpha}\right\} >$$

$$> \left[\frac{1}{(n-1)!}\right]^{\alpha} \int_{\gamma(v_1)}^{t} (\tau(x) - v_1)^{\alpha(n-1)} \,.\, p(x)\,\mathrm{d}x\,.$$

We already know that $u^{(n-1)}(t)$ is a positive nonincreasing function. Therefore, (10) implies that

$$0 < \int_t^{\infty} p(x)\, u^{\alpha}(\tau(x))\,\mathrm{d}x < \infty\,.$$

Thus from (18) for $t \to \infty$, we get

$$\frac{1}{1-\alpha}\left[\int_{\gamma(v_1)}^{\infty} p(x)\, u^{\alpha}(\tau(x))\,\mathrm{d}x\right]^{1-\alpha} \geqq \left[\frac{1}{(n-1)!}\right]^{\alpha} \int_{\gamma(v_1)}^{\infty} (\tau(x) - v_1)^{\alpha(n-1)} \,.\, p(x)\,\mathrm{d}x\,.$$

The left-hand side of the last inequality is a positive and finite number. Since the right-hand side is nonnegative, it is also finite, which yields a contradiction with (9). So, if (7) holds and $n - q$ is an even number, we have a contradiction.

Again, let (7) hold. Now let $n - q$ be an odd natural number. Then, by Lemma 2, we obtain $q = k$. Thus from (6) we have

$$u^{(q+1)}(t) = u^{(k+1)}(t) < 0 \quad \text{for} \quad t \geqq t_2$$

and

$$u^{(q)}(t) = u^{(k)}(t) > 0 \quad \text{for} \quad t \geqq t_2\,.$$

Hence $u^{(q)}(t)$ is a positive decreasing function for $t \geqq t_2$. It is easy to see that the second integral on the right-hand side of the inequality (14) is positive. Therefore from (14) we get

$$u^{(q)}(v_1) - u^{(q)}(v_2) > \frac{1}{(n-q-1)!}\int_{v_1}^{v_2} (x - v_1)^{n-q-1}\, p(x)\, u^{\alpha}(\tau(x))\,\mathrm{d}x$$

and for $v_2 \to \infty$

$$(19) \qquad u^{(q)}(v_1) > \frac{1}{(n-q-1)!}\int_{v_1}^{\infty} (x - v_1)^{n-q-1}\, p(x)\, u^{\alpha}(\tau(x))\,\mathrm{d}x\,.$$

Let $q > 0$. Integrating (19) from v_3 to t $(t_2 < v_3 < t)$ yields

$$(20) \qquad u^{(q-1)}(t) - u^{(q-1)}(v_3) > \frac{1}{(n-q)!}\int_{v_3}^{t} (x - v_3)^{n-q}\, p(x)\, u^{\alpha}(\tau(x))\,\mathrm{d}x +$$

$$+ \frac{1}{(n-q)!}\int_{t}^{\infty} [(x - v_3)^{n-q} - (x - t)^{n-q}]\, p(x)\, u^{\alpha}(\tau(x))\,\mathrm{d}x\,.$$

It is easy to see by (13) that $(t - v_3)^{n-q} \leqq (x - v_3)^{n-q} - (x - t)^{n-q}$ for $v_3 < t \lesseqq x$. The first integral on the right-hand side of (20) is positive, and by (6), $u^{(q-1)}(v_3) = u^{(k-1)}(v_3) > 0$. Thus from (20) (replacing v_3 by v_1) we obtain

$$u^{(q-1)}(t) > \frac{1}{(n-q)!} (t - v_1)^{n-q} \int_t^\infty p(x)\, u^\alpha(\tau(x))\, dx \,. \tag{21}$$

The inequality (21) coincides with the inequality (16). So continuing as above we successively get the inequalities (17), (17′) and (18), contradicting again (9). Thus we have a contradiction with (9) for $n - q$ odd, $q > 0$ provided (7) holds.

Now let $q = 0$, $n - q = n$ being odd. Since $q = k$ so the inequality (6) gives $u'(t) < 0$ for each $t \geqq t_2$. This means that $u(t)$ is a positive decreasing function for $t \geqq t_2$. Because the conditions (7) still hold true, so $\lim_{t\to\infty} u(t) = c > 0$. Hence $0 < c < u(t) \leqq u(t_2)$ for each $t \geqq t_2$, and for $t \geqq \gamma(t_2)$ we get

$$0 < c < u(t) \leqq u(\tau(t)) \leqq u(t_2)\,.$$

These inequalities imply that

$$0 < c^\alpha\, p(t) < p(t)\, u^\alpha(\tau(t)) = -u^{(n)}(t)\,.$$

Integrating the last inequality n-times from t to ∞ $(t \geqq \gamma(t_2))$ and using (7), we obtain

$$\frac{c^\alpha}{(n-1)!} \int_t^\infty (x - t)^{n-1}\, p(x)\, dx < u(t) - c\,.$$

Since this inequality holds for $t \geqq \gamma(t_2)$, we get

$$\frac{c^\alpha}{(n-1)!} \int_{\gamma(t_2)}^\infty (x - \gamma(t_2))^{n-1}\, p(x)\, dx < u(\gamma(t_2)) - c$$

and also

$$\frac{c^\alpha}{(n-1)!} \int_{\gamma_2(t_2)}^\infty (\tau(x) - \gamma(t_2))^{n-1}\, p(x)\, dx < u(\gamma(t_2)) - c$$

where $\gamma_2(t_2) = \gamma(\gamma(t_2))$. Since $\alpha(n-1) < n-1$, we conclude that

$$\frac{c^\alpha}{(n-1)!} \int_{\gamma_2(t_2)}^\infty (\tau(x) - \gamma(t_2))^{\alpha(n-1)}\, p(x)\, dx < u(\gamma(t_2)) - c\,. \tag{22}$$

We already know that $u(\gamma(t_2)) - c$ is a positive and finite number. Because the left-hand side of (22) is also positive, it is finite, i.e.

$$\int^\infty (\tau(x) - \gamma(t_2))^{\alpha(n-1)}\, p(x)\, dx < \infty\,,$$

but this is a contradiction with (9). Then we have a contradiction with (9) if $q = 0$, $n - q = n$ is odd, and (7) holds.

Now consider the case when conditions (7) are not satisfied. But then conditions (8) are satisfied. This means that if (9) holds and (1) has a nonoscillatory solution, then this solution satisfies (8). We know (see Lemma 2) that conditions (8) are satisfied only when $k = 0$ and if $\lim_{t \to \infty} u(t) = 0$ and, further, that $k = 0$ only if n is odd. Hence the proof of sufficiency is complete.

Necessity. – Let

$$\int^{\infty} [\tau(t)]^{\alpha(n-1)} p(t) \, \mathrm{d}t < \infty .$$

Then there exists a number $t_1 > t_0$ (we can assume $t_1 \geqq 0$) such that

$$\left[\frac{1}{(n-1)!}\right]^{\alpha} \int_{\gamma(t_1)}^{\infty} [\tau(x)]^{\alpha(n-1)} p(x) \, \mathrm{d}x < 1 . \tag{23}$$

We show that if (23) holds then there exists a nonoscillatory solution of (1) which does not tend monotonically to zero together with its first $n - 1$ derivatives as $t \to \infty$.

Let us consider a solution $u(t)$ of (1) which satisfies the initial conditions

$$\begin{gathered} u(t) = 0 \quad \text{for} \quad t \in E_{\gamma(t_1)} = [t_1, \gamma(t_1)] , \\ u'(\gamma(t_1) + 0) = u''(\gamma(t_1) + 0) = \ldots = u^{(n-2)}(\gamma(t_1) + 0) = 0 , \\ u^{(n-1)}(\gamma(t_1) + 0) = 1 . \end{gathered} \tag{24}$$

Let t_2 be the first zero of $u(t)$ greater than $\gamma(t_1)$. Then $u(t) > 0$, and $u(\tau(t)) \geqq 0$ for $t \in (\gamma(t_1), t_2)$. Looking at (1), we see that $u^{(n)}(t) \leqq 0$ for $t \in (\gamma(t_1), t_2)$, i.e. $u^{(n-1)}(t)$ is nonincreasing on this interval. It is easy to show, using (24), that there exists $\xi \in (\gamma(t_1), t_2)$ such that $u^{(n-1)}(\xi) = 0$ and $u^{(n-1)}(t) > 0$ for $t \in (\gamma(t_1), \xi)$. According to the Taylor theorem, we have by (24)

$$u(t) = \frac{u^{(n-1)}(\gamma(t_1) + 0)}{(n-1)!} (t - \gamma(t_1))^{n-1} + \frac{(t - \gamma(t_1))^n}{n!} u^{(n)}(\eta)$$

for $t \in [\gamma(t_1), t_2]$, where $\eta = \gamma(t_1) + \vartheta(t - \gamma(t_1))$, $0 < \vartheta < 1$. Hence (1) and (24) imply

$$u(t) = \frac{(t - \gamma(t_1))^{n-1}}{(n-1)!} - \frac{(t - \gamma(t_1))^n}{n!} p(\eta) \, u^{\alpha}(\tau(\eta))$$

or

$$u(t) \leqq \frac{(t - \gamma(t_1))^{n-1}}{(n-1)!} \leqq \frac{1}{(n-1)!} t^{n-1} \quad \text{for} \quad t \in [\gamma(t_1), t_2] . \tag{25}$$

We know that $\tau(t) \in [t_1, t_2]$ for $t \in [\gamma(t_1), t_2]$. Because $t_1 \geqq 0$, then also (using (24))

$$u(t) \leqq \frac{1}{(n-1)!} t^{n-1} \quad \text{for} \quad t \in [t_1, \gamma(t_1)] . \tag{26}$$

From (25) and (26) we obtain

$$u(\tau(t)) \leqq \frac{1}{(n-1)!} [\tau(t)]^{n-1} \quad \text{for} \quad t \in [\gamma(t_1), t_2] . \tag{27}$$

Using (27) in (1) we have

$$-u^{(n)}(t) \leqq \left[\frac{1}{(n-1)!}\right]^{\alpha} p(t) [\tau(t)]^{\alpha(n-1)} \quad \text{for} \quad t \in (\gamma(t_1), t_2] . \tag{28}$$

Integrating (28) from $\gamma(t_1)$ to ξ we have

$$\begin{aligned} & -u^{(n-1)}(\xi) + u^{(n-1)}(\gamma(t_1) + 0) \leqq \\ & \leqq \left[\frac{1}{(n-1)!}\right]^{\alpha} \int_{\gamma(t_1)}^{\xi} [\tau(t)]^{\alpha(n-1)} p(t) \, dt . \end{aligned} \tag{29}$$

Since $u^{(n-1)}(\xi) = 0$, we have from (29), by (23) and (24),

$$\begin{aligned} 1 & \leqq \left[\frac{1}{(n-1)!}\right]^{\alpha} \int_{\gamma(t_1)}^{\xi} [\tau(t)]^{\alpha(n-1)} p(t) \, dt \leqq \\ & \leqq \left[\frac{1}{(n-1)!}\right]^{\alpha} \int_{\gamma(t_1)}^{\infty} [\tau(t)]^{\alpha(n-1)} p(t) \, dt < 1 , \end{aligned}$$

i.e. $1 < 1$ and this contradiction completes the proof.

As we have said, Theorem 1 is an analogue of Theorem A. Moreover, it is a generalization of Theorem 3.6 in [1], of Theorem 2 in [2], and also of those parts of Theorems 1 and 2 in [4] which hold true for $0 < \alpha < 1$.

II.

We start the second part of this paper by two definitions.

Definition 3. We shall say that the equation (j) $(j = 1, 2)$ has the property $\mathcal{O}$, iff at least one of the following conditions is satisfied:

a) All solutions of (j) $(j = 1, 2)$ are oscillatory.

b) Every solution of (j) $(j = 1, 2)$ is either oscillatory or tending monotonically to zero together with its first $n - 1$ derivatives as $t \to \infty$.

Definition 4. Let the equation (2) have the property $\mathcal{O}$. We shall say that an argument delay $\tau(t)$ influences the property $\mathcal{O}$ of the equation (1) if for this $\tau(t)$ the equation (1) has not the property $\mathcal{O}$. On the other hand, if for some argument delay $\tau(t)$ the equation (1) has the property $\mathcal{O}$, we shall say that $\tau(t)$ does not influence the property $\mathcal{O}$ of the equation (1).

It is clear that if $\int^\infty p(t)\,\mathrm{d}t = \infty$, then the conditions (3) and (9) are satisfied, i.e. the equations (2) and (1) have the property $\mathcal{O}$. Thus in this case the argument delay does not influence the property $\mathcal{O}$ of (1). We shall therefore assume in the sequel that $\int^\infty p(t)\,\mathrm{d}t < \infty$.

Theorem 2. *Let $0 < \alpha < 1$ and let an argument delay $\tau(t)$ influence the property $\mathcal{O}$ of the equation* (1). *Then*

$$\liminf_{t\to\infty} \frac{\tau(t)}{t} = 0\,. \tag{30}$$

Proof. Let (30) be false, i.e.

$$\liminf_{t\to\infty} \frac{\tau(t)}{t} = 2\delta > 0 \quad \text{for some} \quad \delta > 0\,.$$

Then there exists an infinite sequence $\{t_k\}_{k=1}^\infty$, $t_k \to \infty$ for $k \to \infty$ such that the sequence

$$\left\{\frac{\tau(t_k)}{t_k}\right\}_{k=1}^\infty$$

tends to 2δ. Therefore there exists $T(\delta)$ such that

$$\frac{\tau(t)}{t} > 2\delta - \delta = \delta \quad \text{for every} \quad t > T(\delta)\,.$$

Hence

$$0 < \delta < \frac{\tau(t)}{t} \leqq 1 \quad \text{for} \quad t > T(\delta)$$

which yields

$$\delta^{\alpha(n-1)} \leqq \left[\frac{\tau(t)}{t}\right]^{\alpha(n-1)} \leqq 1 \quad \text{for} \quad t > T(\delta)\,.$$

Now we have

$$\delta^{\alpha(n-1)} \,.\, \int^\infty t^{\alpha(n-1)} \,.\, p(t)\,\mathrm{d}t \leqq \int^\infty [\tau(t)]^{\alpha(n-1)} \,.\, p(t)\,\mathrm{d}t \leqq \int^\infty t^{\alpha(n-1)} \,.\, p(t)\,\mathrm{d}t$$

and from this we see that $\int^\infty [\tau(t)]^{\alpha(n-1)} \,.\, p(t)\,\mathrm{d}t = \infty$ holds true if and only if $\int^\infty t^{\alpha(n-1)} \,.\, p(t)\,\mathrm{d}t = \infty$. Now, using Theorem A and Theorem 1, one can see that the equation (1) has the property $\mathcal{O}$ if and only if the equation (2) has the property $\mathcal{O}$, i.e. the argument delay does not influence the property $\mathcal{O}$ of (1). This is a contradiction with the assumption of the theorem and our proof is complete.

The following example shows that the condition (30) is not sufficient for the property $\mathcal{O}$ of (1) to be influenced by the argument delay $\tau(t)$.

Example 1. Consider the equation

$$u''(t) + t^{-11/10} \,.\, u^{1/3}(t^{1/2}) = 0 \,. \tag{31}$$

Then $\int^\infty p(t)\,\mathrm{d}t < \infty$, $\lim_{t\to\infty} [\tau(t)/t] = 0$, $\int^\infty \tau^\alpha(t)\, p(t)\,\mathrm{d}t = \infty$. Now we see that (30) holds true, but both (31) and the equation

$$u''(t) + t^{-11/10} \,.\, u^{1/3}(t) = 0$$

have the property $\mathcal{O}$.

Theorem 3. *Let $0 < \alpha < 1$ and $\int^\infty p(t)\,\mathrm{d}t < \infty$. Let $\int^\infty t^\beta\, p(t)\,\mathrm{d}t < \infty$ for some $\beta \in (0, \alpha(n-1))$ and let (3) hold true. Then every argument delay $\tau(t)$ which satisfies (for t sufficiently large) the inequality*

$$\tau(t) \leqq K \,.\, t^{\beta/\alpha(n-1)}\,, \quad K > 0 \tag{32}$$

influences the property $\mathcal{O}$ of the equation (1).

Proof. From (32), by (i), we have

$$p(t)\,[\tau(t)]^{\alpha(n-1)} \leqq K^{\alpha(n-1)} \,.\, t^\beta\, p(t) \tag{33}$$

for t sufficiently large. Integrating (33) from T (sufficiently large) to ∞ gives

$$0 < \int_T^\infty [\tau(t)]^{\alpha(n-1)} \,.\, p(t)\,\mathrm{d}t \leqq K^{\alpha(n-1)} \,.\, \int_T^\infty t^\beta\, p(t)\,\mathrm{d}t < \infty \,.$$

Thus $\int^\infty [\tau(t)]^{\alpha(n-1)} \,.\, p(t)\,\mathrm{d}t < \infty$, which means by Theorem 1 that the equation (1) has not the property $\mathcal{O}$. Since (3) holds true, so the equation (2) has the property $\mathcal{O}$ and the theorem is proved.

Before stating further results we establish the following lemmas.

Lemma 3. *Let $f(t) \in C_{[t_0,\infty)}$. Then*

$$\liminf_{t\to\infty} \frac{\ln f(t)}{\ln t} > 0$$

if and only if there exists a number $\varepsilon > 0$ such that $t^\varepsilon \leqq f(t)$ for t sufficiently large.

Proof. Suppose that $t^\varepsilon \leqq f(t)$ for some $\varepsilon > 0$ and t sufficiently large. Then $\varepsilon \ln t \leqq \ln f(t)$ or

$$\frac{\ln f(t)}{\ln t} \geqq \varepsilon$$

for t sufficiently large. Therefore also

$$\liminf_{t\to\infty} \frac{\ln f(t)}{\ln t} \geqq \varepsilon > 0 .$$

Conversely, if

$$\liminf_{t\to\infty} \frac{\ln f(t)}{\ln t} = c > 0 ,$$

then for every $\varepsilon_1 > 0$ ($\varepsilon_1 < c$) there exists $T(\varepsilon_1)$ such that

$$\frac{\ln f(t)}{\ln t} > c - \varepsilon_1 \quad \text{or} \quad f(t) \geqq t^{c-\varepsilon_1} \quad \text{for} \quad t > T(\varepsilon_1) .$$

If we put $\varepsilon = c - \varepsilon_1 > 0$, we have $t^\varepsilon \leqq f(t)$ for $t > T(\varepsilon_1)$. The lemma is thus completely proved.

Lemma 4. *If $f(t) \in C_{[t_0,\infty)}$, $f(t) \leqq t$ and $\lim_{t\to\infty} f(t) = \infty$, then*

$$\liminf_{t\to\infty} \frac{f(t)}{t} \leqq \liminf_{t\to\infty} \frac{\ln f(t)}{\ln t} .$$

Proof. The function $F(x) = \ln x/x$ is decreasing for $x > e$ because

$$F'(x) = \frac{1 - \ln x}{x^2} < 0 \quad \text{for} \quad x > e .$$

Now we choose T so large that $f(t) > e$ for $t > T$ (this is always possible since $\lim_{t\to\infty} f(t) = \infty$). Since $f(t) \leqq t$, we have $F(f(t)) \geqq F(t)$ for $t > T$, i.e.

$$\frac{\ln f(t)}{f(t)} \geqq \frac{\ln t}{t} \quad \text{or} \quad \frac{f(t)}{t} \leqq \frac{\ln f(t)}{\ln t}$$

For $t > T$. From this we obtain

$$\liminf_{t\to\infty} \frac{f(t)}{t} \leqq \liminf_{t\to\infty} \frac{\ln f(t)}{\ln t}$$

and the lemma is proved.

Lemma 5. *Let $f(t) \in C_{[t_0,\infty)}$ and $f(t) > 0$ on some neighborhood $\mathcal{O}_1(\infty)$. Then*

$$\limsup_{t\to\infty} \frac{\ln f(t)}{\ln t} < 1$$

if and only if there exists a number $\varepsilon > 0$ such that $f(t) \leqq t^{1-\varepsilon}$ for t sufficiently large.

Proof. Suppose that $f(t) \leqq t^{1-\varepsilon}$ for some $\varepsilon > 0$ and t sufficiently large. Then $\ln f(t) \leqq (1 - \varepsilon) \ln t$ or equivalently

$$\frac{\ln f(t)}{\ln t} \leqq 1 - \varepsilon$$

for t sufficiently large. This implies that

$$\limsup_{t \to \infty} \frac{\ln f(t)}{\ln t} \leqq 1 - \varepsilon < 1 .$$

Conversely, if

$$\limsup_{t \to \infty} \frac{\ln f(t)}{\ln t} = d < 1 ,$$

then for every $\varepsilon_1 > 0$ such that $d + \varepsilon_1 < 1$ there exists a real $T(\varepsilon_1)$ (we can assume $T(\varepsilon_1) > 1$) such that

$$\frac{\ln f(t)}{\ln t} < d + \varepsilon_1 \quad \text{or} \quad f(t) \leqq t^{d+\varepsilon_1}$$

for $t > T(\varepsilon_1)$. This completes the proof since the number $d + \varepsilon_1$ may be written in the form $1 - \varepsilon$ where $\varepsilon > 0$.

Theorem 4. *Let* $0 < \alpha < 1$ *and* $\int^\infty p(t)\,dt < \infty$. *If* $\int^\infty t^\varepsilon p(t)\,dt = \infty$ *for every* $\varepsilon > 0$, *then an argument delay* $\tau(t)$ *which satisfies the condition*

$$\liminf_{t \to \infty} \frac{\ln \tau(t)}{\ln t} > 0 \tag{34}$$

does not influence the property $\mathcal{O}$ *of the equation* (1).

Proof. Since $\int^\infty t^\varepsilon p(t)\,dt = \infty$ for every $\varepsilon > 0$, the equation (2) has the property $\mathcal{O}$. We know by Lemma 4 that both the conditions (30) and (34) can be satisfied simultaneously. It means that the condition (34) alone does not ensure the validity of the assertion of Theorem 4 if no further assumptions are made.

If (34) holds true, then by Lemma 3 there exists a positive real ε_1 such that $t^{\varepsilon_1} \leqq \tau(t)$ for each sufficiently large t. From this we have

$$t^{\varepsilon_1 \alpha(n-1)} \, . \, p(t) \leqq [\tau(t)]^{\alpha(n-1)} \, . \, p(t)$$

and since $\int^\infty t^{\varepsilon_1 \alpha(n-1)} \, . \, p(t)\,dt = \infty$ (because $\int^\infty t^\varepsilon p(t)\,dt = \infty$ for every $\varepsilon > 0$), we get

$$\int^\infty [\tau(t)]^{\alpha(n-1)} \, . \, p(t)\,dt = \infty ,$$

i.e. the equation (1) has the property $\mathcal{O}$. It means that the argument delay $\tau(t)$ does not influence the property $\mathcal{O}$ of the equation (1) and our proof is complete.

Theorem 5. *Let* $0 < \alpha < 1$. *Let* $\int^\infty t^{\alpha(n-1)} . p(t)\, dt = \infty$, *and*

$$\int^\infty t^{\alpha(n-1)-\varepsilon} . p(t)\, dt < \infty \tag{35}$$

for every $\varepsilon > 0$. *Then every argument delay* $\tau(t)$ *which satisfies the condition*

$$\limsup_{t\to\infty} \frac{\ln \tau(t)}{\ln t} < 1 \tag{36}$$

influences the property $\mathcal{O}$ *of the equation* (1).

Proof. Since $\int^\infty t^{\alpha(n-1)} . p(t)\, dt = \infty$ the equation (2) has the property $\mathcal{O}$. Now, in order to prove the theorem we must show that the equation (1) has not the property $\mathcal{O}$ when (35) and (36) hold true.

If (36) holds true then by Lemma 5 there exists a positive real ε_1 such that $\tau(t) \leqq \leqq t^{1-\varepsilon_1}$ for sufficiently large t.

Hence

$$[\tau(t)]^{\alpha(n-1)} . p(t) \leqq t^{(1-\varepsilon_1)\alpha(n-1)} . p(t) .$$

Since (35) holds true we have

$$\int^\infty t^{(1-\varepsilon_1)\alpha(n-1)} . p(t)\, dt < \infty$$

and also

$$\int^\infty [\tau(t)]^{\alpha(n-1)} . p(t)\, dt < \infty .$$

From this inequality we conclude that the equation (1) has not the property $\mathcal{O}$ and the theorem is proved.

Theorem 6. *Let* $0 < \alpha < 1$ *and* $\int^\infty t^{\alpha(n-1)} . p(t)\, dt = \infty$. *Let*

$$\int^\infty t^c\, p(t)\, dt < \infty \tag{37}$$

for some $c \in (0, \alpha(n-1))$. *Then every argument delay* $\tau(t)$ *which satisfies the condition*

$$\limsup_{t\to\infty} \frac{\ln \tau(t)}{\ln t} < \frac{c}{\alpha(n-1)} \tag{38}$$

influences the property $\mathcal{O}$ *of the equation* (1).

Proof. We know that the equation (2) has the property $\mathcal{O}$ because $\int^{\infty} t^{\alpha(n-1)} \, . \, p(t) \, \mathrm{d}t = \infty$. As above we must show that the equation (1) has not the property $\mathcal{O}$ when (37) and (38) hold true and then the theorem will be proved.

From (38) we have

$$\limsup_{t\to\infty} \frac{\ln \tau(t)}{\ln t} = d < \frac{c}{\alpha(n-1)}.$$

Then for every $\varepsilon > 0$ there exists $T(\varepsilon)$ (we may assume $T(\varepsilon) > 1$) such that

$$\frac{\ln \tau(t)}{\ln t} < d + \varepsilon \quad \text{for} \quad t > T(\varepsilon)$$

and

$$\tau(t) \leqq t^{d+\varepsilon} \quad \text{for} \quad t > T(\varepsilon).$$

From this we conclude that

$$[\tau(t)]^{\alpha(n-1)} . \, p(t) \leqq t^{(d+\varepsilon)\alpha(n-1)} . \, p(t) \tag{39}$$

for $t > T(\varepsilon) > 1$. Now if we choose ε such that

$$0 < \varepsilon \leqq \frac{c}{\alpha(n-1)} - d,$$

we obtain from (39) that

$$[\tau(t)]^{\alpha(n-1)} . \, p(t) \leqq t^c \, p(t) \quad \text{for} \quad t > T(\varepsilon) > 1.$$

This implies by (37) that

$$\int^{\infty} [\tau(t)]^{\alpha(n-1)} . \, p(t) \, \mathrm{d}t < \infty.$$

It means that the equation (1) has not the property $\mathcal{O}$. This completes the proof.

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