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THE CHARACTERIZATION OF $\mathfrak{m}$-COMPACT ELEMENTS IN SOME LATTICES*)

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The present paper is a continuation of the papers [1], [4] and especially of [5–6]. It characterizes the $\mathfrak{m}$-compact elements in the lattice of all congruences on a poset (let us denote it by $(A; \leqq)$) and in the lattice of all convex equivalences on $(A; \leqq)$. Under a congruence on $(A; \leqq)$ we understand a kernel of an arbitrary isotonic mapping with the domain $(A; \leqq)$; under a convex equivalence we understand such an equivalence on A, each equivalence class of which is a convex subset of $(A; \leqq)$.

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1. INTRODUCTORY REMARKS

Throughout this paper the symbol $(A; \leqq)$ shall be reserved for an ordered set with the support A. The set of all non-negative integers will be denoted by N, i.e. $N = \{0, 1, 2, \ldots\}$.

1.1. Notation. (i) By the symbol $[a, b]$, where $a, b \in A$, we denote the set $\{x \in A; a \leqq x \leqq b\} \cup \{x \in A; b \leqq x \leqq a\} \cup \{a, b\}$.

(ii) By the symbol $E(A)$ we denote the set of all equivalences on A.

(iii) By the symbol $G(A; \leqq)$ we denote the set of all congruences on $(A; \leqq)$ in the usual category **Ord** of all ordered sets in the sense of [1], Section 1.1. b**).

*) This paper has originated at the seminar Algebraic Foundations of Quantum Theories, directed by Professor JIŘÍ FÁBERA.

**) I.e. $G(A; \leqq)$ is the set of all kernels of isotonic mappings with the domain $(A; \leqq)$ (see [4], Sections 18 and 49).

(iv) The set of all convex equivalences on $(A; \leqq)$ *) will be denoted by $C(A; \leqq)$.

In the whole paper, the symbol $B(A; \leqq)$ will mean one of the sets $G(A; \leqq)$, $C(A; \leqq)$, fixed for the respective consideration.

1.2. Theorem. (See [5], Section 30 and [6]). *The set $E(A)$, as well as the set $B(A; \leqq)$, forms with respect to the inclusion an algebraic lattice.*

1.3. Definition. We define a relation $\lessapprox$ on the set $\exp A = \{X; X \subseteq A\}$ as follows: if $X, Y \in \exp A$, then

$$X \lessapprox Y \Leftrightarrow_{\mathrm{df}} X = Y = \emptyset \text{ vel } (\exists x \in X)(\exists y \in Y)(x \leqq y).$$

1.4. Remark. The set $(G(A; \leqq); \subseteq)$ is an algebraic closure system in $(E(A); \subseteq)$. The algebraic closure $\bar{\sigma}$ in $(G(A; \leqq); \subseteq)$ (see [4], Section 22') of the equivalence σ can be characterized in terms of the relation $\lessapprox$ like that: if $a, b \in A$, then

$$(a, b) \in \bar{\sigma} \Leftrightarrow (\exists X_0, \ldots, X_n \in A/\sigma)(a \in X_0 \text{ et } b \in X_i \text{ for some}$$
$$0 \leqq i \leqq n \text{ et } X_0 \lessapprox X_1 \lessapprox \ldots \lessapprox X_n \lessapprox X_0).$$

(For the proof see [4], Sections 17, 19.)

1.5. Definition. By induction, for every $n \in N$ we define functions $\psi_n : E(A) \to \exp A^2$ in this manner:

$$\psi_0(\sigma) =_{\mathrm{df}} \sigma - \mathrm{id}_A .$$

Having defined $\psi_n(\sigma)$, we put

$$\psi_{n+1}(\sigma) =_{\mathrm{df}} \bigcup\{[x, y]^2 \circ (\mathrm{id}_A \cup \psi_n(\sigma)); (x, y) \in \psi_n(\sigma)\} .$$

1.6. Remark. In terms of these functions, the closure operator $u : E(A) \to C(A; \leqq)$ induced by the algebraic closure system $C(A; \leqq)$ can be characterized as follows:

$$u(\sigma) = \mathrm{id}_A \cup \bigcup_{n=0}^{\infty} \psi_n(\sigma) .$$

For the proof see [6], Remark 8a.

1.7. Lemma. *Let $C \subseteq B(A; \leqq)$ and $a, b \in A$. If $(a, b) \in \sup_{B(A;\leqq)} C$, then there exists a finite subset $C' \subseteq C$ such that $(a, b) \in \sup_{B(A;\leqq)} C'$.*

Proof. For $B(A; \leqq) = G(A; \leqq)$ the assertion is proved in [5], Corollary 26.

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*) An equivalence $\sigma \in E(A)$ is called *convex on* $(A; \leqq)$, if and only if every $X \in A/\sigma$ is a convex subset of $(A; \leqq)$, i.e. $X = \bigcup\{[x, y]; x, y \in X\}$.

Let us give the proof for $B(A; \leqq) = C(A; \leqq)$:
Given $\tau \in E(A)$ with

$$(1) \qquad \tau = \sup_{E(A)} C \quad \text{and} \quad u(\tau) = \bar{\tau} = \sup_{C(A;\leqq)} C\,,$$

we have – by Remark 1.6 –

$$(2) \qquad \bar{\tau} = \bigcup_{n=0}^{\infty} \psi_n(\tau) \cup \mathrm{id}_A\,.$$

Hence it holds for any $a, b \in A$, that

$$(a, b) \in \bar{\tau} \Leftrightarrow (\exists n \in N)\,((a, b) \in \psi_n(\tau)) \text{ vel } a = b\,.$$

If $a = b$ we take $C' = \emptyset$. If $a \neq b$, we continue by induction over n: let $(a, b) \in \psi_0(\tau) = \tau - \mathrm{id}_A$. By the well-known properties of the lattice $(E(A); \subseteq)$, there exists a finite set $C' \subseteq C$ such that

$$(3) \qquad (a, b) \in \sup_{E(A)} C' \subseteq \sup_{C(A;\leqq)} C'\,.$$

Assume, for all $k \leqq n$, $k \in N$, that the condition

$$(4) \qquad (\forall (x, y))\,((x, y) \in \psi_k(\tau) \Rightarrow (\exists C'(x, y) \text{ finite})\,(C'(x, y) \subseteq C \text{ et } (x, y) \in \sup_{C(A;\leqq)} C'(x, y))$$

is valid and let $(a, b) \in \psi_{n+1}(\tau)$.

By the definition of ψ_{n+1} we have: there exist $x, y \in A$ so that

$$(x, y) \in \psi_n(\tau)$$

and there exists also $c \in A$ such that

$$(a, c) \in [x, y]^2 \quad \text{et} \quad (c, b) \in \mathrm{id}_A \cup \psi_n(\tau)\,.$$

From (4) it follows that there exists a finite set $C'(x, y) \subseteq C$ so that

$$(5) \qquad (x, y) \in \sup_{C(A;\leqq)} C'(x, y) \subseteq \bar{\tau}\,.$$

Since $\sup_{C(A;\leqq)} C'(x, y)$ is convex, we have also

$$(a, c) \in \sup_{C(A;\leqq)} C'(x, y)\,.$$

This completes the proof for $c = b$. So, let $c \neq b$. Then (4) implies again that there exists a finite set $C'(c, b) \subseteq C$ such that

$$(7) \qquad (c, b) \in \sup_{C(A;\leqq)} C'(c, b)\,.$$

Then (6) and (7) yield

$$(8) \qquad (a, b) \in (\sup_{C(A;\leqq)} C'(x, y)) \circ (\sup_{C(A;\leqq)} C'(c, b)) .$$

It is

$$(9) \qquad (\sup_{C(A;\leqq)} C'(x, y)) \circ (\sup_{C(A;\leqq)} C'(c, b)) \subseteq$$

$$\subseteq (\sup_{C(A;\leqq)} (C'(x, y) \cup C'(c, b))) \circ (\sup_{C(A;\leqq)} (C'(x, y) \cup C'(c, b))) =$$

$$= \sup_{C(A;\leqq)} (C'(x, y) \cup C'(c, b)) .$$

We put $C' = C'(x, y) \cup C'(c, b)$.

Then by (8) and (9) we have

$$(a, b) \in \sup_{C(A;\leqq)} C' ,$$

where C' is a finite set.

1.8. Remark. The assertion of Lemma 1.7 for $B(A; \leqq) = C(A; \leqq)$ is a direct consequence of Theorem 37 in [5]. However, the axiom of choice is essentially used in the proof of this theorem. This is why the proof of this special case in which the axiom of choice does not appear is introduced here.

1.9. Lemma. *Let $\sigma \in B(A; \leqq)$ and let $\tau \subseteq \sigma$ be any equivalence on A. Then $\tau \in B(A; \leqq)$ if and only if for every $X \in A/\sigma$ the partial relation $\tau \cap X^2$ is an element of $B(X; \leqq)$.*

Proof. For $B(A; \leqq) = G(A; \leqq)$ the assertion is proved in [5], Corollary 40. If $B(A, \leqq) = C(A, \leqq)$, it is trivial.

1.10. Definition. Let $\sigma \in E(A)$ and let $X \in A/\sigma$ be such that $|X| \geqq 2$; we denote by σ_X the equivalence

$$\sigma_X =_{\mathrm{df}} X^2 \cup \mathrm{id}_A$$

and by E the set

$$E =_{\mathrm{df}} \{\sigma_X;\ X \in A/\sigma \text{ et } |X| \geqq 2\} .$$

1.11. Lemma. *Let $\sigma \in B(A; \leqq)$ and $X \in A/\sigma$; then*

(i) $E \subseteq B(A; \leqq)$,

(ii) *for every C such that $\emptyset \neq C \subseteq E$ we have*

$$\sup_{B(A;\leqq)} C = \bigcup C ,$$

(iii) $\sigma = \bigcup E$.

Proof. It follows immediately from Lemma 1.9 because $\bigcup C$ is an equivalence on A.

1.12. Definition. Let $\sigma \in E(A)$ and $X \in A/\sigma$. Suppose that Y and Z are arbitrary non-empty subsets of X. For every $y \in Y$ and $z \in Z$ define

$$\sigma_{yz} =_{\mathrm{df}} [y, z]^2 \cup \mathrm{id}_X \cup (\sigma \cap (A - X)^2) .$$

The set of all such equivalences will be denoted by $D(Y, Z)$ (or shortly by D), i.e.

$$D = D(Y, Z) = \{\sigma_{yz};\ y \in Y,\ z \in Z\} .$$

1.13. Lemma. *Suppose $\sigma \in B(A; \leqq)$ and $X \in A/\sigma$. Then for every non-empty subsets $Y \subseteq X$ and $Z \subseteq X$ the following condition holds:*

(i) $D(Y, Z) \subseteq B(A, \leqq)$.

If, moreover, Y is confinal with X and Z coninitial) with X, then*

(ii) $\sigma = \sup_{B(A;\leqq)} D(Y, Z)$.

Proof. The assertion (i) follows immediately from Lemma 1.9 because $[y, z]^2 \cup \cup\, \mathrm{id}_X \in B(X; \leqq)$.

To prove the assertion (ii), it is sufficient to establish the following implication:

$$\text{if} \quad a, b \in X , \quad \text{then} \quad (a, b) \in \sup_{B(A;\leqq)} D .$$

To this end take a, $b \in X$. The set Y being confinal and Z coninitial with X, there are $y_1, y_2 \in Y$ and $z_1, z_2 \in Z$ such that

$$a \in [z_1, y_1] \quad \text{and} \quad b \in [z_2, y_2] .$$

Then

$$(a, b) \in \sigma_{y_1 z_1} \circ \sigma_{y_1 z_2} \circ \sigma_{y_2 z_2} \subseteq \sup_{E(A)} D \subseteq \sup_{B(A;\leqq)} D .$$

The assertion (ii) is now easily seen.

1.14. Lemma. *Let $\emptyset \neq X \subseteq A$ and $V \subseteq X$ be such that* $\mathrm{cf}\, V < \mathrm{cf}\, X$ **).
Then there exists $a_0 \in X$ such that no $v \in V$ satisfies $a_0 \leqq v$.

Proof. Let us suppose to the contrary that for every $a \in X$ there exists $v \in V$ such that $a \leqq v$. Then V is confinal with X which contradicts the assumption $\mathrm{cf}\, V < \mathrm{cf}\, X$.

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*) The set Z is called *coninitial* with X (more exactly with $(X; \leqq)$, but in the whole paper the ordering "$\leqq$" is given) if $(\forall x \in X)(\exists z \in Z)(z \leqq x)$. The term "coinitial", is also used in literature, however, it is perhaps less suitable owing to the strictly limited sense of the prefix "co" in the theory of categories.

**) The symbol $\mathrm{cf}\, X$ (or more exactly $\mathrm{cf}\,(X; \leqq)$) means the least cardinal number $\varkappa$ such that for some $Y \subseteq X$ confinal with $(X; \leqq)$ it is $|Y| = \varkappa$.

1.15. Definition. Let $\sigma \in E(A)$, $X \in A/\sigma$ and let Y, Z be any non-empty subsets of X. For every $C \subseteq D(Y, Z)$ we define

$$V_C =_{\mathrm{df}} \{u \in X;\ (\exists v \in Y \cup Z)\,(\sigma_{uv} \in C)\}\,.$$

1.16. Lemma. *Let $\sigma \in C(A; \leqq)$, let $\mathfrak{m}$ be an infinite cardinal and $X \in A/\sigma$ with $\operatorname{cf} X \geqq \mathfrak{m}$. Suppose that $Y \subseteq X$ is confinal with X and $Z \subseteq X$ is a non-empty subset. Then for every $C \subseteq D(Y, Z)$ with $|C| < \mathfrak{m}$ there exists $a_0 \in X$ such that*

$$\{a_0\} \in A/\sup_{E(A)} C$$

and no $v \in V_C$ satisfies $a_0 \leqq v$.

Proof. According to the assumption we have

$$|Y| \geqq \mathfrak{m} \tag{10}$$

and so

$$|D| \geqq \mathfrak{m}\,. \tag{11}$$

Let C be any subset of D with $|C| < \mathfrak{m}$. It is $|V_C| \leqq |C|^2 < \mathfrak{m}^2 = \mathfrak{m}$. Especially, we have $|V_C| < \operatorname{cf} X$ and hence

$$\operatorname{cf} V_C < \operatorname{cf} X\,. \tag{12}$$

According to Lemma 1.14 there exists $a_0 \in X$ such that no $v \in V_C$ satisfies $a_0 \leqq v$. Hence for every $y \in Y$ and every $z \in Z$ we have $a_0 \notin [y, z]$, i.e.

$$\{a_0\} \in A/\sup_{E(A)} C\,. \tag{13}$$

2. CHARACTERIZATION OF $\mathfrak{m}$-COMPACT ELEMENTS IN $(G(A; \leqq); \subseteq)$ AND $(C(A; \leqq); \subseteq)$ FOR $\mathfrak{m}$ REGULAR

2.1. Theorem. *Let $\mathfrak{m}$ be an infinite cardinal and let $\sigma \in B(A; \leqq)$ be $\mathfrak{m}$-compact in $(B(A; \leqq); \subseteq)$. Then the cardinality of the set*

$$\mathcal{M} = \{X \in A/\sigma;\ |X| \geqq 2\}$$

is smaller than $\mathfrak{m}$.

Proof. Let $\sigma \in B(A; \leqq)$ be $\mathfrak{m}$-compact in $(B(A; \leqq); \subseteq)$ and suppose

$$|\mathcal{M}| \geqq \mathfrak{m}\,. \tag{14}$$

By Lemma 1.11, the set

$$E = \{\sigma_X;\ X \in \mathcal{M}\}$$

(see Definition 1.10) forms a covering of σ in $(B(A; \leqq); \subseteq)$ and (14) yields

$$|E| \geqq \mathfrak{m} .$$

Let C by any non-empty subset of E with

$$|C| < \mathfrak{m} \tag{15}$$

and let τ denote

$$\tau = \sup_{B(A;\leqq)} C .$$

According to Lemma 1.11 we have

$$\tau = \bigcup C \quad \text{and} \quad \sigma = \bigcup E . \tag{16}$$

Due to (15) there exists $X_0 \in \mathscr{M}$ such that

$$\sigma_{X_0} \notin C ,$$

hence

$$\bigcup C \subset \bigcup E , \quad \bigcup C \neq \bigcup E ,$$

i.e. (with respect to (16))

$$\tau \subset \sigma , \quad \tau \neq \sigma .$$

This contradicts the assumption that σ is $\mathfrak{m}$-compact. The theorem is proved.

2.2. Theorem. *Let $\mathfrak{m}$ be an infinite cardinal and $\sigma \in G(A; \leqq)$ an $\mathfrak{m}$-compact element of $(G(A; \leqq); \subseteq)$. Then for every $X \in A/\sigma$ it holds*

(i) $\operatorname{cf} X < \mathfrak{m}$,*)
(ii) $\operatorname{ci} X < \mathfrak{m}$ **).

Proof. (i) Suppose that there exists $X \in A/\sigma$ such that

$$\operatorname{cf} X \geqq \mathfrak{m} . \tag{17}$$

Let $Y \subseteq X$ be confinal with X. We have

$$|Y| \geqq \mathfrak{m} .$$

Let $Z \subseteq X$ be coinitial with X. According to Lemma 1.13 the set

$$D = \{\sigma_{yz};\ y \in Y,\ z \in Z\}$$

*) See the footnote on the page 256.

**) The coninitial of $(X; \leqq)$ denoted by ci X or more exactly ci $(X; \leqq)$ is the confinal of the dually ordered set X.

covers the congruence σ. Given $C \subseteq D$ with $|C| < \mathfrak{m}$ denote

$$\tau = \sup_{E(A)} C .$$

According to Lemma 1.16 there exists $a_0 \in X$ such that

$$\{a_0\} \in A/\tau \tag{18}$$

and no $v \in V_C$ (see Definition 1.15) satisfies $a_0 \leqq v$ (by [4], Section 36 we have $G(A; \leqq) \subseteq C(A; \leqq)$; hence $\sigma \in C(A; \leqq)$ and we can apply Lemma 1.16). It remains to prove that $\sigma \nsubseteq \bar{\tau}$, where $\bar{\tau}$ is the algebraic closure of τ in $G(A; \leqq)$. Since $\sigma \in G(A; \leqq)$, we can – with respect to Sections 1.4, 1.9 – restrict our proof to X: we have $\tau \subseteq \sigma$ and, therefore, $\bar{\tau} \subseteq \sigma$ as well.

Denote

$$\bar{\pi} = \bar{\tau} \cap X^2 , \quad \pi = \tau \cap X^2 .$$

Let there exist $v_0 \in V_C$ with

$$(v_0, a_0) \in \bar{\pi} . \tag{19}$$

Then by Remark 1.4 there exist subsets $X_0, \ldots, X_n$ $(n \in N)$ such that $X_k \in X/\pi$ for each $k = 0, \ldots, n$ and

$$X_0 \lessapprox X_1 \lessapprox \ldots \lessapprox X_n = X_0 . \tag{20}$$

Moreover,

$$v_0 \in X_0 \quad \text{et} \quad (\exists j)\,(1 \leqq j \leqq n-1 \quad \text{and} \quad a_0 \in X_j) . \tag{21}$$

By (18), we have $X_j = \{a_0\}$. Let k be the smallest of the natural numbers $j+1, \ldots, n$ such that

$$|X_k| \geqq 2 . \tag{22}$$

Such k must exist since, assuming the contrary, one obtains

$$a_0 \leqq x_{j+1} \leqq \ldots \leqq x_n = v_0 \quad (\{x_{j+1}\} = X_{j+1}, \ldots, \{x_n\} = X_n)$$

– a contradiction. Denote by x_k the element of X_k with

$$a_0 \leqq x_k .$$

According to (22), there exists $v \in V_C$ such that $x_k \leqq v$, hence $a_0 \leqq v$ – a contradiction again.

Thus

$$\{a_0\} \in X/\bar{\pi} ,$$

i.e., $\bar{\pi} \neq X^2$ and, therefore, also $\bar{\tau} \subset \sigma$, $\bar{\tau} \neq \sigma$.

The inequality (ii) can be proved dually.

2.3. Definition. Let $\sigma \in E(A)$. The function

$$m_\sigma : A \to N \cup \{-1\}$$

is defined in this way: Let $x \in A$. If there exist $n \in N$ and $y \in A$ such that

$$(x, y) \in \psi_n(\sigma) \cup (\psi_n(\sigma))^{-1}, \tag{23}$$

then $m_\sigma(x)$ is by the definition equal to the smallest of n's satisfying (23). If there is no such n, we set $m_\sigma(x) =_{\mathrm{df}} -1$.

2.4. Lemma. *Let $\sigma \in E(A)$ and $x \in A$ such that*

$$m_\sigma(x) \geqq 1 .$$

Then there exist $x_1, x_2 \in A$ such that $(x_1, x_2) \in \psi_{m_\sigma(x)-1}(\sigma)$ and $x_1 < x < x_2$.

Proof See [6], Section 8b.

2.5. Theorem. *Let $\mathfrak{m}$ be an infinite cardinal and $\sigma \in C(A; \leqq)$ an $\mathfrak{m}$-compact element of $(C(A; \leqq); \subseteq)$. Then for every $X \in A/\sigma$ it holds*

(i) $\operatorname{cf} X < \mathfrak{m}$,
(ii) $\operatorname{ci} X < \mathfrak{m}$.

Proof. Let, to the contrary, there be $X \in A/\sigma$ such that $\operatorname{cf} X \geqq \mathfrak{m}$. Let $Y \subseteq X$ be confinal with X; then

$$|Y| \geqq \mathfrak{m} . \tag{24}$$

Let $Z \subseteq X$ be coninitial with X. According to Lemma 1.13, the set

$$D = \{\sigma_{yz};\ y \in Y,\ z \in Z\}$$

covers the equivalence σ in $(C(A; \leqq); \subseteq)$ and in virtue of (24) we have

$$|D| \geqq \mathfrak{m} .$$

Let $C \subseteq D$ be such that

$$|C| < \mathfrak{m} \tag{25}$$

and let us denote

$$\tau = \sup_{E(A)} C .$$

By Lemma 1.16 there exists $a_0 \in X$ such that

$$\{a_0\} \in A/\tau \tag{26}$$

and no $v \in V_C$ satisfies $a_0 \leqq v$.

It remains to prove that

$$\{a_0\} \in A/\bar{\tau}\,,$$

where $\bar{\tau}$ is the convex closure of τ. By Remark 1.6 we have

$$\bar{\tau} = \bigcup_{n=0}^{\infty} \psi_n(\tau) \cup \mathrm{id}_A\,.$$

If $(a_0, v_0) \in \bar{\tau}$ for some $v_0 \in V_C$, then there must exist $n \in N$ such that

$$(a_0, v_0) \in \psi_n(\tau)$$

and $n \geqq 1$, because (26) yields $(a_0, v_0) \notin \tau$. Hence

$$m_\tau(a_0) \geqq 1\,.$$

By induction we shall prove this assertion: If $x \in X$ with $m_\tau(x) \geqq 1$, then there exists $v \in V_C$ such that $x \leqq v$.

Let $i = m_\tau(x) \geqq 1$. Then by Lemma 2.4 there exist $x_1, x_2 \in X$ such that

$$(x_1, x_2) \in \psi_{i-1}(\tau) \tag{27}$$

and

$$x_1 < x < x_2\,. \tag{28}$$

Let $m_\tau(x) = 1$. Then $(x_1, x_2) \in \tau - \mathrm{id}_A$, which means there exists $\{u, v\} \subseteq V_C$ such that

$$u \leqq x_2 \leqq v$$

and in virtue of (28) we have

$$x \leqq v$$

as well.

Let $k \in N$, $k \geqq 1$. Suppose that for every $y \in X$ with $m_\tau(y) \leqq k$ there exists $v \in V_C$ such that

$$y \leqq v \tag{29}$$

and let $m_\tau(x) = k + 1$. Then there exist $x_1, x_2 \in X$ such that

$$(x_1, x_2) \in \psi_k(\tau) \tag{27'}$$

and

$$x_1 < x < x_2\,. \tag{28'}$$

By $(27')$ we have $m_\tau(x_2) \leqq k$. According to the assumption of induction there exists $v \in V_C$ such that

$$x_2 \leqq v$$

and so

$$x \leqq v .$$

Particularly, for a_0 there exists $v_0 \in V_C$ such that

$$a_0 \leqq v_0 .$$

This is a contradiction. Thus, necessarily, $m_\tau(a_0) = -1$, i.e.

$$\{a_0\} \in A/\bar{\tau}$$

and therefore $\bar{\tau} \subset \sigma$, $\bar{\tau} \neq \sigma$. We have again obtained a contradiction – σ is $\mathfrak{m}$--compact.

The inequality (ii) can be proved dually.

2.6. Theorem. *Let* $\mathfrak{m}$ *be an infinite regular cardinal and* $\sigma \in B(A; \leqq)$. *Suppose that the following conditions hold*:

(i) *the cardinality of the set*

$$\mathcal{M} = \{X \in A/\sigma;\ |X| \geqq 2\}$$

is smaller than $\mathfrak{m}$,

(ii) $\operatorname{cf} X < \mathfrak{m}$ *for every* $X \in A/\sigma$;

(iii) $\operatorname{ci} X < \mathfrak{m}$ *for every* $X \in A/\sigma$.

Then σ *is* $\mathfrak{m}$-*compact in* $(B(A; \leqq), \subseteq)$.

Proof. Let $\sigma \in B(A; \leqq)$ and $C \subseteq B(A; \leqq)$ be such that

$$\sigma \subseteq \sup_{B(A;\leqq)} C .$$

Let $X \in A/\sigma$ with $|X| \geqq 2$ and let Y, Z be subsets of X such that Y is confinal with X, $|Y| = \operatorname{cf} X$ and Z is coninitial with X, $|Z| = \operatorname{ci} X$. By assumptions we have

$$|Y| < \mathfrak{m} \quad \text{et} \quad |Z| < \mathfrak{m} . \tag{30}$$

According to Lemma 1.7, for every $y \in Y$ and every $z \in Z$ there exists a finite set $C'_X(y, z) \subseteq C$ such that

$$(y, z) \in \sup_{B(A;\leqq)} C'_X(y, z) .$$

By (30) we have $|Y \times Z| < \mathfrak{m}$.

Denoting

$$C'_X = \bigcup\{C'_X(y, z);\ y \in Y,\ z \in Z\}\,,$$

we obtain – taking into account the regularity of $\mathfrak{m}$ –

$$|C'_X| < \mathfrak{m} \tag{31}$$

and for every $(y, z) \in Y \times Z$,

$$(y, z) \in \sup_{B(A;\leqq)} C'_X\,. \tag{32}$$

Further, for every $(a, b) \in X^2$ there exist $y_1, y_2 \in Y$ and $z_1, z_2 \in Z$ such that

$$a \in [y_1, z_1] \quad \text{and} \quad b \in [y_2, z_2]\,.$$

Thus

$$(a, b) \in \sigma_{y_1z_1} \circ \sigma_{y_2z_1} \circ \sigma_{y_2z_2} \subseteq \sup_{E(A)} C\,. \tag{33}$$

Hence, we obtain

$$\sigma \subseteq \sup_{B(A;\leqq)}(\bigcup_{X\in\mathcal{M}} C'_X)\,.$$

Combining (31) with the assumption (i) we obtain the inequality

$$|\bigcup_{X\in\mathcal{M}} C'_X| < \mathfrak{m}\,,$$

since $\mathfrak{m}$ is regular.

This completes the proof.

2.7. Theorem. *Let $\mathfrak{m}$ be an infinite regular cardinal and $\sigma \in B(A; \leqq)$. Then σ is $\mathfrak{m}$-compact in $(B(A; \leqq); \subseteq)$ if and only if the following conditions hold:*

(i) *the cardinality of the set*

$$\mathcal{M} = \{X \in A/\sigma;\ |X| \geqq 2\}$$

is smaller than $\mathfrak{m}$;

(ii) $\operatorname{cf} X < \mathfrak{m}$ *for every* $X \in A/\sigma$;

(iii) $\operatorname{ci} X < \mathfrak{m}$ *for every* $X \in A/\sigma$.

Proof. The assertion of this theorem follows immediately from 2.1, 2.2, 2.5 and 2.6.

2.8. Remark. If $\mathfrak{m}$ is an infinite regular cardinal and $\sigma \in G(A; \leqq)$, then σ is $\mathfrak{m}$-compact in $(G(A; \leqq); \subseteq)$ if and only if σ is $\mathfrak{m}$-compact in $(C(A; \leqq); \subseteq)$. This follows from Theorem 2.7 and from the fact that $G(A; \leqq) \subseteq C(A; \leqq)$. (See [4], Section 36.)

3. THE CHARACTERIZATION OF $\mathfrak{m}$-COMPACT ELEMENTS IN $(G(A; \leqq); \subseteq)$ AND $(C(A; \leqq); \subseteq)$ FOR $\mathfrak{m}$ IRREGULAR

The following example shows that Theorem 2.7 does not generally hold for $\mathfrak{m}$ irregular.

3.1. Example. Let $\mathfrak{m}$ be an infinite irregular cardinal and let $\mathfrak{m}$ denote simultaneously the initial ordinal of the cardinality $\mathfrak{m}$. Let I be a set of indices with $|I| = \operatorname{cf} \mathfrak{m}$.

Then

$$(34) \qquad |I| < \mathfrak{m}\,,$$

$$(35) \qquad (\forall i \in I)\,(\exists \mathfrak{m}_i < \mathfrak{m})\,(\sup_{i \in I} \mathfrak{m}_i = \mathfrak{m} \text{ et } \mathfrak{m}_i \text{ are regular cardinal numbers})\,.$$

(As above, the symbol $\mathfrak{m}_i$ denotes at the same time the initial ordinal of the cardinality $\mathfrak{m}_i$.)

Let $\{(M_i; \leqq_i);\ i \in I\}$ be a disjoint system of posets such that $(M_i, \leqq_i)$ is of the type $\mathfrak{m}_i$ for all $i \in I$. We define a poset $(M; \leqq)$ as the cardinal sum of all $(M_i; \leqq_i)$:

$$M = \bigcup\{M_i;\ i \in I\}$$

and for $a, b \in M$ we set

$$a \leqq b \Leftrightarrow_{\mathrm{df}} (\exists i \in I)\,(a \in M_i \text{ et } b \in M_i \text{ et } a \leqq_i b)$$

(see [2], page 55).

Let $\sigma \in G(M; \leqq)$ be such that

$$M/\sigma = \{M_i;\ i \in I\}\,.$$

Every set M_i can be written in the form

$$M_i = \{a_{i\lambda}\}_{\lambda < \mathfrak{m}_i}$$

so that $a_{i\lambda} \leqq a_{j\mu}$ if and only if $i = j$ and $\lambda \leqq \mu$. Given any $i \in I$ and $\varkappa < \mathfrak{m}_i$ we denote

$$M_{i\varkappa} = \{a_{i\lambda};\ \lambda < \varkappa\} \subseteq M_i\,.$$

On $(M; \leqq)$ we introduce the congruences

$$\alpha_{i\varkappa} = M_{i\varkappa}^2 \cup \mathrm{id}_M$$

for every $i \in I$ and every $\varkappa < \mathfrak{m}_i$.

Let

$$E_i = \{\alpha_{i\varkappa};\ \varkappa < \mathfrak{m}_i\}$$

and

$$E = \bigcup\{E_i;\ i \in I\}.$$

Then

$$\sup_{G(M;\leqq)} E = \sigma \tag{36}$$

and

(37) for every $i \in I$, the set E_i is a covering of $\mathrm{id}_M \cup (\sigma \cap M_i^2)$ and $|E_i| = \mathfrak{m}_i$.

Since $\mathfrak{m}_i$ is regular, it is impossible to choose from E_i a subcovering of a cardinality smaller than $\mathfrak{m}_i$.

If any subcovering $C \subseteq E$ from E were chosen so that

$$|C| < \mathfrak{m},$$

there would exist $i \in I$ such that

$$|C| < \mathfrak{m}_i < \mathfrak{m},$$

and so C could not cover even $\sigma \cap M_i^2$.

The same counter-example could be constructed in the set of all convex equivalences.

The characterization of $\mathfrak{m}$-compact elements in $B(A;\leqq)$ for $\mathfrak{m}$ irregular is a consequence of the following general theorem.

3.2. Theorem. *Let $\mathfrak{m}$ be a limit cardinal*) and $\mathfrak{n}$ an arbitrary cardinal smaller than $\mathfrak{m}$. If $(M;\leqq)$ is an arbitrary $\mathfrak{n}$-algebraic lattice and $a \in M$ an $\mathfrak{m}$-compact element, then there exists an isolated**) cardinal $\mathfrak{k}$ smaller then $\mathfrak{m}$ so that a is $\mathfrak{k}$-compact.*

Proof. Let $a \in M$ be given. Since the lattice $(M;\leqq)$ is $\mathfrak{n}$-algebraic, there exists $B \subseteq M$, $B = \{b_i;\ i \in I\}$ with

$$a = \sup_M B \tag{38}$$

and such that b_i is $\mathfrak{n}$-compact for every $i \in I$. By the assumption a is $\mathfrak{m}$-compact, i.e. there exists $J \subseteq I$, $|J| < \mathfrak{m}$ such that

$$a = \sup_M \{b_j;\ j \in J\}. \tag{39}$$

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*) A cardinal number is called *limit* if it is uncountable and such that for every cardinal $\mathfrak{n} < \mathfrak{m}$ there exists a cardinal $\mathfrak{k}$ so that $\mathfrak{n} < \mathfrak{k} < \mathfrak{m}$.

**) A cardinal number is called *isolated* if it is infinite but not limit. An isolated cardinal is always regular.

Let $C \subseteq M$ be any covering of a, i.e.

$$a \leqq \sup_M C . \tag{40}$$

Then by (39) we have

$$\sup_M \{b_j; j \in J\} \leqq \sup_M C ,$$

and in particular

$$(\forall j \in J)(b_j \leqq \sup_M C) .$$

Hence for every $j \in J$ there exists $C_j \subseteq C$, $|C_j| < \mathfrak{n}$ such that

$$b_j \leqq \sup_M C_j . \tag{41}$$

Altogether, from (39) and (41) we obtain

$$a = \sup_M \{b_j; j \in J\} \leqq \sup_M \{\sup_M C_j; j \in J\} = \sup_M \big(\bigcup_{j \in J} C_j\big) .$$

Further,

$$\big|\bigcup_{j \in J} C_j\big| = \mathfrak{n} \,.\, |J| < (\mathfrak{n} \,.\, |J|)^+ < \mathfrak{m}$$

since $\mathfrak{m}$ is a limit cardinal. This completes the proof if we set $\mathfrak{k} = (\mathfrak{n} \,.\, |J|)^+$. The following example shows that the requirement of $\mathfrak{n}$-algebraicity of $(M; \leqq)$ in Theorem 3.2 is really essential.

3.3. Example. Let $(M; \leqq)$ be a lattice with the following property: for every $k \in N$ let $(M_k, \leqq_k)$ be a poset of the type ω_k and for any $k_1, k_2 \in N$, $k_1 \neq k_2$ let $M_{k_1} \cap M_{k_2} = \emptyset$. We define a set

$$M =_{\mathrm{df}} \bigcup_{k \in N} M_k \cup \{o, j\} \tag{42}$$

and an ordering $\leqq$ on M in this way: for every $x \in M$ we set $o \leqq x$ and $x \leqq j$; if $x, y \in \bigcup_{k \in N} M_k$, we define

$$x \leqq y \Leftrightarrow_{\mathrm{df}} (\exists k \in N)(x \in M_k \text{ et } y \in M_k \text{ et } x \leqq_k y) .$$

Every element different from σ of this lattice is $\aleph_{\omega 0}$ – compact but not $\mathfrak{k}$-compact for any $\mathfrak{k} < \aleph_{\omega 0}$ since every M_k covers every element of M. Such a lattice obviously cannot be n-algebraic for any $n < \aleph_{\omega 0}$.

3.4. Theorem. *Let $\mathfrak{m}$ be an irregular cardinal and $\sigma \in B(A; \leqq)$. Then σ is $\mathfrak{m}$-compact in $(B(A; \leqq); \subseteq)$ if and only if there exists a regular cardinal $\mathfrak{k}$, $\mathfrak{k} < \mathfrak{m}$ such that σ is $\mathfrak{k}$-compact in $(B(A; \leqq); \subseteq)$.*

Proof. It is well-known that $(G(A; \leqq); \subseteq)$ as well as $(C(A; \leqq); \subseteq)$ are algebraic lattices (see [5], Section 30 and [6]). Thus, Theorem 3.4 is a direct consequence of Theorem 3.2.

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