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Czechoslovak Mathematical Journal, Vol. 29 (1979), No. 2, 232–245

Persistent URL: http://dml.cz/dmlcz/101600

Czechoslovak Mathematical Journal, 29 (104) 1979, Praha

EMBEDDING OF SEMILATTICES INTO DISTRIBUTIVE LATTICES*)

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(Received May 12, 1977)

Part I of the present paper contains the definition of the r-hull of a meet-semilattice $\mathcal{S}$ (it is an r-distributive lattice, free generated by $\mathcal{S}$ and having some natural properties with respect to $\mathcal{S}$) and some elementary consequences of this definition. Part II contains a construction of the r-hull. Part III contains an other construction of the r-hull (which is similar to the McNeille completization).

This purely algebraic paper is motivated by measure theory: the theory developed so far enables an abstract characterization of semi-rings of sets [2]. On such abstract semi-rings "additive" functions are considered with values from suitable algebraic structures and their additive extensions are investigated.

The authors wish to thank to Dr. JIŘÍ ADÁMEK for helpful discussions and suggestion which made possible to simplificate some of the constructions in part II. They wish to thank also to Dr. JANA RYŠLINKOVÁ for her translation and useful remarks.

I. THE DEFINITION OF THE r-HULL AND SOME CONSEQUENCES

1. Introductory remarks. a. Conventions. k, r, s are infinite cardinals; we shall suppose that k is irregular. r^* will be the smallest of all regular cardinals s such that $r \leqq s$ (i.e. $r^* = r$ if r is regular and $r^* = r^+$ if r is irregular). The support of a structure $\mathcal{A}$ will be denoted by A. Our terminology is that one of [1]. If $\mathcal{P}$ is a poset, then we put, for $X,\ Y \subseteq P,\ z \in P$,

$$X \vee Y =_{\mathrm{Df}} \{p \in P \mid (\exists x \in X)(\exists y \in Y)\, p = x \vee y\}\,, \quad z \vee X =_{\mathrm{Df}} \{z\} \vee X\,,$$

if all joins on the right sides of the defining equations exist in $\mathcal{P}$; $X \wedge Y$, $z \wedge Y$ will be defined dually.

*) This paper has originated at the seminar Algebraic Foundations of Quantum Theories directed by prof. JIŘÍ FÁBERA.

Throughout this paper we shall suppose that $\mathscr{S} = (S; \leqq)$ is a meet-semilattice If $X \subseteq S$, put

$$(X] =_{\mathrm{Df}} \bigcup_{x \in X} \{y \in S \mid y \leqq x\} .$$

b. Definition. Let $\mathscr{K}$, $\mathscr{L}$ be two lattices and let $f : K \to L$. A lattice $\mathscr{K}$ is called *join r-complete*, if for every $X \subseteq K$, $0 < |X| < r$, there exists $\bigvee X$. A map f is called an *r-complete homomorphism* from $\mathscr{K}$ to $\mathscr{L}$, if $\mathscr{K}$, $\mathscr{L}$ are join r-complete lattices, f is a lattice homomorphism from $\mathscr{K}$ to $\mathscr{L}$ and if for every $X \subseteq K$, $0 < |X| < r$, there is $f(\bigvee_{\mathscr{K}} X) = \bigvee_{\mathscr{L}} f(X)$. A join r-complete lattice $\mathscr{K}$ is called *r-distributive*, if for every $X \subseteq K$, $0 < |X| < r$ and for every $x \in K$, there is $x \wedge \bigvee X = \bigvee(x \wedge X)$.

c. Lemma. *Let $\mathscr{K}$, $\mathscr{L}$ be two lattices and let $f : K \to L$. Then the following holds:*

α) *$\mathscr{K}$ is join k-complete iff it is join k^+-complete.*

β) *$\mathscr{K}$ is k-distributive iff it is k^+-distributive.*

γ) *f is a join k-complete homomorphism from $\mathscr{K}$ to $\mathscr{L}$ iff it is a join k^+-complete homomorphism from $\mathscr{K}$ to $\mathscr{L}$.*

The proofs are based on the following consideration: Let X be a set of an irregular cardinality k. Then there exists a system $(X_i)_{i \in I}$ such that $|I| < k$, $(\forall i \in I)\, |X_i| < k$ and $X = \bigcup\{X_i \mid i \in I\}$. We shall prove the statement α, for example. Let $\mathscr{K}$ be join k-complete. Let $X \subseteq K$ with $0 < |X| < k^+$. If $|X| < k$, then $\bigvee X$ exists by assumption. If $|X| = k$, then

$$\bigvee_{i \in I} (\bigvee X_i) = \bigvee(\bigcup_{i \in I} X_i) = \bigvee X ,$$

and $\bigvee_{i \in I} (\bigvee X_i)$ exists, since $\mathscr{K}$ is k-complete.

If $\mathscr{K}$ is join k^+-complete, then it is join r-complete for every $r \leqq k^+$; especially, it is join k-complete.

d. Definition. A subset X of S is called *distributive* (in $\mathscr{S}$), if the following conditions hold:

α) There exists $\bigvee X$.

β) For every $x \in S$, there is $\bigvee(x \wedge X) = x \wedge \bigvee X$.

2. Definition. An ordered pair $(\mathscr{K}, f)$ is called the *r-hull* of a semilattice $\mathscr{S}$, if it satisfies the following conditions:

a) $\mathscr{K}$ is an r-distributive lattice.

b) The map $f : S \to K$ is injective and satisfies the following conditions:

α) If $X \subseteq S$ and if there exists $\bigwedge_{\mathscr{S}} X$, then $f(\bigwedge_{\mathscr{S}} X) = \bigwedge_{\mathscr{K}} f(X)$.

β) Let X be a distributive set in $\mathscr{S}$ with $0 < |X| < r^*$. Then $f(\bigvee_{\mathscr{S}} X) = \bigvee_{\mathscr{K}} f(X)$.

γ) For every $x \in K$, there exists $X \subseteq S$ with $0 < |X| < r^*$ and such that $x = \bigvee_{\mathscr{K}} f(X)$.

c) Let $\mathscr{L}$ be an r-distributive lattice and let φ be a meet-homomorphism from $\mathscr{S}$ to $\mathscr{L}$ such that for every distributive subset X of $\mathscr{S}$ with $0 < |X| < r^*$ we have $\varphi(\bigvee_{\mathscr{S}} X) = \bigvee_{\mathscr{L}} \varphi(X)$. Then there exists at least one join r-complete homomorphism

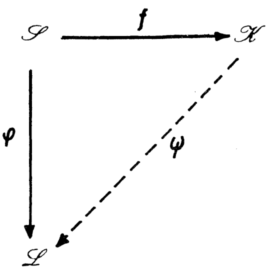

Fig. 1.

$\psi : \mathscr{K} \to \mathscr{L}$ such that $\varphi = \psi f$. (See the commutative diagram of Fig. 1.)

3. Theorem. *Let $(\mathscr{K}, f)$ be an r-hull of $\mathscr{S}$. Then the following statements hold.*

a) *The map f is an isotone monomorphism*) from $\mathscr{S}$ into $\mathscr{K}$.*

b) *Let $X \subseteq S$, $|X| < r^*$ and let there exists $a \in S$ such that $f(a) = \bigvee_{\mathscr{K}} f(X)$. Then X is a distributive set and $a = \bigvee_{\mathscr{S}} X$.* (See also Section 23.)

c) *The join r-complete homomorphism ψ of Section 2.c is unique.*

d) *If a is the greatest or the smallest element of $\mathscr{S}$, then $f(a)$ is the greatest or the smallest element of $\mathscr{K}$ respectively.*

Proof. a) Take $x, y \in S$. Then

$$x \leqq y \Leftrightarrow x = x \wedge y \Leftrightarrow f(x) = f(x \wedge y) = f(x) \wedge f(y) \Leftrightarrow f(x) \lesseqqgtr f(y)\,.$$

(The second equivalence holds, since f is injective.)

b) If $X = \emptyset$, then the statement holds by d). Suppose then, that $0 < |X| < r^*$. Lattice $\mathscr{K}$ is join r-complete, therefore $\bigvee f(X)$ exists (see also 1. c. α). Let us suppose that for some $a \in S$ we have $f(a) = \bigvee f(X)$. Then for every $x \in X$, $x \leqq a$ by a). If y is an upper bound of X in $\mathscr{S}$, then by a), $f(y)$ is an upper bound of $f(X)$ in $\mathscr{K}$; from this fact it follows that $f(a) = \bigvee f(X) \lesseqqgtr f(y)$ and therefore, $a \leqq y$. This proves that $a = \bigvee_{\mathscr{S}} X$.

Take a $z \in S$. It is obvious that $z \wedge a$ is an upper bound of $z \wedge X$ in $\mathscr{S}$. By the r-distributivity of $\mathscr{K}$ and the assumption $0 < |X| < r^*$,

$$\bigvee_{x \in X} f(z \wedge x) = \bigvee_{x \in X} (f(z) \wedge f(x)) = f(z) \wedge \bigvee f(X) = f(z) \wedge f(a) = f(z \wedge a)$$

(in the case of irregular cardinal r it suffices to consider 1. c. β).

*) i.e. if we consider $\mathscr{K}$ as poset $(K; \lesseqqgtr)$, then $(\forall x, y \in S)\; x \leqq y \Leftrightarrow f(x) \lesseqqgtr f(y)$.

Especially: there exists $\bigvee f(z \wedge X)$. Let y be an upper bound of $z \wedge X$ in $\mathcal{S}$. Then $\bigvee f(z \wedge X) \lesssim f(y)$, i.e. $f(z \wedge a) \lesssim f(y)$. This implies, together with the statement a), that $a \wedge z \leqq y$, hence $\bigvee(z \wedge X) = z \wedge a = z \wedge \bigvee X$ for every $z \in S$.

c) For every $x \in K$, there exists $X \subseteq S$ with $0 < |X| < r^*$ and such that $x = \bigvee_{\mathcal{K}} f(X)$ (see 2.b.γ). Then

$$\psi(x) = \psi(\bigvee_{\mathcal{K}} f(X)) = \bigvee_{\mathcal{L}} \psi f(X) = \bigvee_{\mathcal{L}} \varphi(X)$$

since the homomorphism ψ is r-complete (in the case of an irregular cardinal r, we can use 1.c.γ). Hence, such a ψ is unique.

d) This statement follows immediately from 2.b.α, or, from 2.b.α, γ, respectively.

4. Definition. We put $X \leqq' Y$ for $X, Y \subseteq S$, if for every $x \in (X]$ it holds $x = \bigvee_{\mathcal{S}}(x \wedge Y)$.

(The relation $\leqq'$ plays a key role in part II of this paper.)

5. Lemma. *Let $X, Y \subseteq S$, $X \leqq' Y$. Then $x \wedge Y$ is a distributive subset of $\mathcal{S}$ for every $x \in (X]$.*

Proof. Take $z \in S$ and $x \in (X]$. Then $x \wedge z \in (X]$ and since $X \leqq' Y$, then $x \wedge z = \bigvee((x \wedge z) \wedge Y)$ and $x = \bigvee(x \wedge Y)$. Hence,

$$\bigvee(z \wedge (x \wedge Y)) = \bigvee((z \wedge x) \wedge Y) = z \wedge x = z \wedge (\bigvee(x \wedge Y)) .$$

6. Theorem. *Let $(\mathcal{K}, f)$ be an r-hull of $\mathcal{S}$ and let $\varphi : S \to L$ satisfying the requirements of Section 2.c), be injective. Then the homomorphism ψ (the existence of which is ensured in 2.c)) is injective as well.*

Proof. Let $\mathcal{K} = (K; \lesssim)$ and $\mathcal{L} = (L; \lesssim)$. Let $x, y \in K$ and let $\psi(x) \lesssim \psi(y)$; we shall show that $x \lesssim y$ as well (proving the injectivity of ψ).

There exist, by 2.b.γ, X, $Y \subseteq S$ with $0 < |X| < r^*$, $0 < |Y| < r^*$ and such that $x = \bigvee f(X)$ and $y = \bigvee f(Y)$. Since ψ is a join r-complete homomorphism and since the diagram of Fig. 1 commutes, the following holds:

$$\psi(x) = \psi(\bigvee f(X)) = \bigvee \psi f(X) = \bigvee \varphi(X) , \tag{1}$$
$$\psi(y) = \psi(\bigvee f(Y)) = \bigvee \psi f(Y) = \bigvee \varphi(Y) .$$

(If r is an irregular cardinal, we can consider, as usually, Section 1.c).) Let us show that $X \leqq' Y$. Take an arbitrary element $a \in (X]$; then a is an upper bound of $a \wedge Y$ in $\mathcal{S}$. Let b be an arbitrary upper bound of $a \wedge Y$ in $\mathcal{S}$. Then for every $v \in Y$, $a \wedge v \leqq b$, hence $\varphi(a \wedge v) \lesssim \varphi(b)$; therefore $\bigvee \varphi(a \wedge Y) \lesssim \varphi(b)$. By the assumption, there is $\psi(x) \lesssim \psi(y)$, thus $\bigvee \varphi(X) \lesssim \bigvee \varphi(Y)$ by (1). Hence we get

$$\varphi(a) \wedge \bigvee \varphi(X) \lesssim \varphi(a) \wedge \bigvee \varphi(Y) .$$

Since $\mathscr{L}$ is r-distributive, then we also have

$$\bigvee_{u \in X} (\varphi(a) \wedge \varphi(u)) \lesssim \bigvee_{v \in Y} (\varphi(a) \wedge \varphi(v)) .$$

From the properties of φ (see Section 2.c)) it follows that

$$\bigvee \varphi(a \wedge X) \lesssim \bigvee \varphi(a \wedge Y) .$$

Further, $\bigvee \varphi(a \wedge Y) \lesssim \varphi(b)$, hence $\bigvee \varphi(a \wedge X) \lesssim \varphi(b)$. Since $a \in (X]$, then a is the greatest element of the set $a \wedge X$; this implies that $\varphi(a) = \bigvee \varphi(a \wedge X)$, proving the inequality $\varphi(a) \lesssim \varphi(b)$. The injectivity and some other properties of φ (see Section 2.c)) yields

$$\varphi(a) = \varphi(a) \wedge \varphi(b) = \varphi(a \wedge b) \Rightarrow a = a \wedge b \Rightarrow a \leqq b .$$

Thus, for every $a \in (X]$, there is $a = \bigvee (a \wedge Y)$, i.e. $X \leqq' Y$. Then for every $u \in (X]$, $u = \bigvee (u \wedge Y)$; on the other hand, the set $u \wedge Y$ is distributive in $\mathscr{S}$ by Lemma 5. There is $0 < |u \wedge Y| < r^*$, hence it holds

$$x = \bigvee_{u \in X} f(u) = \bigvee_{u \in X} f(\bigvee_{v \in Y} (u \wedge v)) = \bigvee_{u \in X} \bigvee_{v \in Y} (f(u) \wedge f(v)) =$$
$$= (\bigvee f(X)) \wedge (\bigvee f(Y)) = x \wedge y$$

by Section 2.b) (the last but one equality is a consequence of the r-distributivity of $\mathscr{K}$); thus $x \lesseqqgtr y$.

7. Corollary. *Let $r \leqq s$, let $(\mathscr{K}_r, f_r)$ be an r-hull and let $(\mathscr{K}'_s, f'_s)$ be an s-hull of $\mathscr{S}$. Then the map $\psi : K_r \to K'_s$ the existence of which is given by* 2.c)*) *is injective.*

Proof. This statement follows immediately from Section 6.

8. Theorem. *Let $(\mathscr{K}_1, f_1)$, $(\mathscr{K}_2, f_2)$ be r-hulls of $\mathscr{S}$. Then there exist two mutually inverse homomorphisms ψ_1 from $\mathscr{K}_1$ onto $\mathscr{K}_2$ and ψ_2 from $\mathscr{K}_2$ onto $\mathscr{K}_1$ such that the diagram of Fig.* 2 *commutes. (Especially, $\mathscr{K}_1$, $\mathscr{K}_2$ are isomorphic.)*

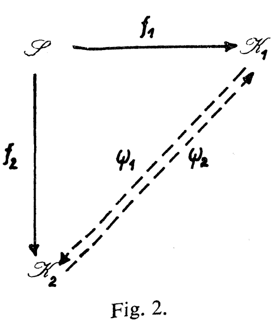

Fig. 2.

*) In this case in Section 2c), we put $\mathscr{K} = \mathscr{K}_r$, $\mathscr{L} = \mathscr{K}_s$, $f = f_r$ and $\varphi = f_s$.

Proof. By Sections 3.c) and 7, there exists exactly one join r-complete monomorphism ψ_1 from $\mathscr{K}_1$ to $\mathscr{K}_2$ and exactly one join r-complete monomorphism ψ_2 from $\mathscr{K}_2$ to $\mathscr{K}_1$ such that the diagram of Fig. 2 is commutative. It remains to prove that $\psi_1 : K_1 \to K_2$ and $\psi_2 : K_2 \to K_1$ are mutually inverse $1 - 1$ mappings. Take an $x \in K_1$. (By Section 2.b.γ), there exists $X \subseteq S$ with $0 < |X| < r^*$ and such that $x = \bigvee_{\mathscr{K}_1} f_1(X)$. Then

$$\psi_2 \psi_1(x) = \psi_2 \psi_1(\bigvee_{\mathscr{K}_1} f_1(X)) = \psi_2(\bigvee_{\mathscr{K}_2} \psi_1 f_1(X)) =$$
$$= \psi_2(\bigvee_{\mathscr{K}_2} f_2(X)) = \bigvee_{\mathscr{K}_1} \psi_2 f_2(X) = \bigvee_{\mathscr{K}_1} f_1(X) = x$$

(if r is an irregular cardinal, then we have to consider Section 1.c)), i.e. $\psi_2 \psi_1 : K_1 \to \to K_1$ is the identity map on K_1.

9. Theorem. *$(\mathscr{K}, f)$ is a k-hull iff it is a k^+-hull of $\mathscr{S}$*).*

Proof. It follows immediately from Definition 2, considering Lemma 1.c) and the fact that $|X| < k^*$ iff $|X| < (k^+)^*$ for any set X.

II. A CONSTRUCTION OF THE r-HULL

10. Lemma. *Relation $\leqq'$ is a quasiordering on* $\exp S$.

Proof. Let $X \in \exp S$ and let $x \in (X]$. Then x is the greatest element of $x \wedge X$, thus $x = \bigvee(x \wedge X)$, i.e. $X \leqq' X$.

Let us prove the transitivity of $\leqq'$. Let $X, Y, Z \in \exp S$ with $X \leqq' Y \leqq' Z$ and let $u \in (X]$. Then there exists $x \in X$ such that $u \leqq x$. Hence we have

$$u = \bigvee_{y \in Y} (u \wedge y) = \bigvee_{y \in Y} (\bigvee_{z \in Z} ((u \wedge y) \wedge z)) =$$
$$= \bigvee_{z \in Z} (\bigvee_{y \in Y} ((u \wedge z) \wedge y)) = \bigvee_{z \in Z} (u \wedge z)$$

(the second equality follows from the fact that $u \wedge y \leqq y \in Y$ and that $Y \leqq' Z$, the fourth one from $u \wedge z \leqq u \leqq x \in X$ and $X \leqq' Y$). Therefore, we have $X \leqq' Z$ as well.

11. Convention. Throughout the following, we shall suppose the infinite cardinal r to be regular.

12. Construction. We shall use the following notation:

$$S_r =_{\mathrm{Df}} \{X \subseteq S \mid 0 < |X| < r\},$$
$$S_r^\circ =_{\mathrm{Df}} S_r/((\leqq' \cap (\leqq')^{-1}) \cap (S_r \times S_r)).$$

*) The cardinal k is supposed to be an infinite irregular cardinal — see Section 1a).

Put $\xi \leqq_r \eta$ for $\xi, \eta \in S_r^\circ$ if there exist $X \in \xi$ and $Y \in \eta$ such that $X \leqq' Y$. The well-known properties of quasiordered sets (see [1], pp. 20–21) imply that

$$\mathscr{S}_r^\circ =_{\mathrm{Df}} (S_r^\circ; \leqq_r)$$

is a poset, where $\xi \leqq_r \eta$ $(\xi, \eta \in S_r^\circ)$ iff $X \leqq' Y$ for every $X \in \xi$ and $Y \in \eta$. The map $g_r : S_r \to S_r^0$ is the useful canonical surjection, i.e. if $X \in S_r$, then $X \in g_r(X) \in S_r^\circ$. Put $h_r(x) =_{\mathrm{Df}} g_r(\{x\})$ for every $x \in S$; then $h_r : S \to S_r^\circ$. In the following proofs, we shall often omit the index r of the symbols S_r°, $\mathscr{S}_r^\circ$, g_r, h_r.

In Section 21 it will be proved that $(\mathscr{S}_r^\circ, h_r)$ is an r-hull of $\mathscr{S}$.

13. Lemma. *There is $X \wedge Y \in S_r$ whenever $X, Y \in S_r$. If $(X_i)_{i \in I}$ is a system of elements of S_r with $0 < |I| < r$, then $\bigcup_{i \in I} X_i \in S_r$.*

Proof. The statement is obvious and therefore it will be used hereafter without exact reference.

14. Lemma. *If $X, Y \in S_r$, then*

$$g_r(X \wedge Y) = \inf_{\mathscr{S}^\circ_r}\{g_r(X), g_r(Y)\}\,.$$

Proof. If $z \in (X \wedge Y]$, then there exist $x \in X$, $y \in Y$ with $z \leqq x \wedge y$. Since $z \leqq x \wedge y \leqq x$, then z is the greatest element of $z \wedge X$, i.e. $z = \bigvee_{\mathscr{S}}(z \wedge X)$; hence $X \wedge Y \leqq' X$, i.e. $g(X \wedge Y) \leqq_r g(X)$. Similarly can be proved that $g(X \wedge Y) \leqq_r g(Y)$.

Let $\xi \in S^\circ$ be such that $\xi \leqq_r g(X)$ and $\xi \leqq_r g(Y)$. Then $Z \leqq' X$, $Z \leqq' Y$ for any $Z \in \xi$, and for any $z \in (Z]$ it holds

$$z = \bigvee_{x \in X}(z \wedge x) = \bigvee_{x \in X}\bigvee_{y \in Y}((z \wedge x) \wedge y) = \bigvee(z \wedge (X \wedge Y))$$

(the second equality follows from the relations $z \wedge x \leqq z \in (Z]$ and $Z \leqq' Y$). Thus $Z \leqq' X \wedge Y$ which implies $\xi = g(Z) \leqq_r g(X \wedge Y)$.

15. Lemma. *Let $(X_i)_{i \in I}$ be a system of elements of S_r with $0 < |I| < r$. Then*

$$g_r(\bigcup_{i \in I} X_i) = \sup_{\mathscr{S}^\circ_r}\{g_r(X_i) \mid i \in I\}\,.$$

Proof. Denoting by Y the set $\bigcup\{X_i \mid i \in I\}$, we get

$$(Y] = \bigcup_{i \in I}(X_i]\,. \tag{2}$$

Then $x \in (Y]$ whenever $x \in (X_j]$ and $j \in I$; further, $\leqq'$ is reflexive, thus $x = \bigvee(x \wedge Y)$. Hence $X_j \leqq' Y$ for any $j \in I$, i.e. $g(Y)$ is an upper bound of $\{g(X_i) \mid i \in I\}$ in $\mathscr{S}^\circ$.

Let ζ be an upper bound of $\{g(X_i) \mid i \in I\}$ in $\mathscr{S}^\circ$. Then $X_j \leqq Z$ for each $Z \in \zeta$ and each $j \in I$; therefore, $x = \bigvee_{\mathscr{S}}(x \wedge Z)$ for any $x \in (Y]$ by (2). Hence $Y \leqq' Z$, i.e. $g(Y) \leqq_r g(Z) = \zeta$.

16. Lemma. *Let $X \subseteq S$ and let there exists $\inf_{\mathscr{S}} X$. Then there exists $\inf_{\mathscr{S}^\circ_r} h_r(X)$ as well and it holds*

$$h_r(\inf_{\mathscr{S}} X) = \inf_{\mathscr{S}^\circ_r} h_r(X) .$$

Proof. Denoting by $a = \bigwedge_{\mathscr{S}} X$, there is $a \leqq x$ for every $x \in X$, hence $\{a\} \leqq' \{x\}$. Thus $h(a) = g(\{a\}) \leqq_r g(\{x\}) = h(x)$ for every $x \in X$, i.e. $h(a)$ is a lower bound of $h(X)$ in $\mathscr{S}^\circ$.

Let η be a lower bound of $h(X)$ in $\mathscr{S}^\circ$. Then $Y \leqq' \{x\}$ whenever $Y \in \eta$ and $x \in X$, thus $y = y \wedge x$, i.e. $y \leqq x$ for every $y \in (Y]$ and every $x \in X$. Hence $y \leqq a$ for every $y \in (Y]$, i.e. $Y \leqq' \{a\}$ as well, which implies that

$$\eta = g(Y) \leqq_r g(\{a\}) = h(a) .$$

17. Lemma. *If $X \in S_r$ is a distributive subset of $\mathscr{S}$, then $g_r(X) = h_r(\bigvee_{\mathscr{S}} X)$.*

Proof. With respect to the assumption of the Lemma, we have to prove that $g(X) = h(\bigvee X)$, i.e. that $X \leqq' \{\bigvee X\} \leqq' X$. The relation $X \leqq' \{\bigvee X\}$ follows immediately from the definition of $\leqq'$. Let $z \in (\{\bigvee X\}]$, i.e. let $z \in S$ be such that $z \leqq \bigvee X$. Following the distributivity of X, there is $\bigvee(z \wedge X) = z \wedge \bigvee X$, thus $\bigvee(z \wedge X) = z$. Therefore, $\{\bigvee X\} \leqq' X$.

18. Corollary. *If $X \in S_r$ is distributive, then*

$$h_r(\bigvee_{\mathscr{S}} X) = \bigvee_{\mathscr{S}^\circ_r} h_r(X) .$$

Proof. There is, by Lemma 15

$$\bigvee_{\mathscr{S}^\circ} h(X) = \bigvee_{\mathscr{S}^\circ}\{g(\{x\}) \mid x \in X\} = g(X) , \tag{3}$$

hence if we suppose $X \in S_r$ then there exists $\bigvee_{\mathscr{S}^\circ} h(X)$. The assertion of this Section follows then from (3) and Section 17.

19. Lemma. *Let $(X_i)_{i \in I}$ be a system of elements of S_r with $0 < |I| < r$. Then $(Y \wedge \bigcup_{i \in I} X_i) \in S_r$ for any $Y \in S_r$* where

$$Y \wedge \bigcup_{i \in I} X_i = \bigcup_{i \in I} (Y \wedge X_i) .$$

The proof is easy. (See also Section 13.)

20. Lemma. *The following statements hold:*

a) *$\mathscr{S}^\circ_r$ is an r-distributive lattice.*

b) *h_r is an isotonic monomorphism of $\mathscr{S}$ to $\mathscr{S}^\circ_r$. (Especially: $h_r : S \to S^\circ_r$ is injective.)*

c) *For every $\xi \in S^\circ_r$, there exists $X \in S_r$ such that $\xi = \bigvee_{\mathscr{S}^\circ_r} h(X)$.*

Proof. a) Let $\xi, \eta \in S^\circ$; take $X \in \xi$ and $Y \in \eta$. Then $\xi \wedge \eta = g(X \wedge Y)$ in $\mathscr{S}^\circ$ by Section 14. Let $\Gamma \subseteq S^\circ$, $0 < |\Gamma| < r$. Taking a representative $v(\gamma)$ of each $\gamma \in \Gamma$, it holds in $\mathscr{S}^\circ$ (by Lemma 15)

$$\bigvee \Gamma = \bigvee\{\gamma \mid \gamma \in \Gamma\} = g(\bigcup_{\gamma \in \Gamma} v(\gamma))\,;$$

especially, $\mathscr{S}^\circ$ is a join r-complete lattice. Let $\eta \in S^\circ$. Take an arbitrary $Y \in \eta$. Then, by Sections 14, 15 and 19, the following holds in $\mathscr{S}^\circ$:

$$\bigvee(\eta \wedge \Gamma) = \bigvee\{g(Y \wedge v(\gamma)) \mid \gamma \in \Gamma\} = g(\bigcup_{\gamma \in \Gamma}(Y \wedge v(\gamma))) =$$

$$= g(Y \wedge \bigcup_{\gamma \in \Gamma} v(\gamma)) = g(Y) \wedge g(\bigcup_{\gamma \in \Gamma} v(\gamma)) = \eta \wedge \bigvee \Gamma\,.$$

b) If $x, y \in S$, then $x \leqq y$ iff $\{x\} \leqq' \{y\}$, i.e. iff $h(x) \leqq_r h(y)$.

c) Take a set $X \in \xi$. Then $X \in S_r$ and

$$\xi = g(X) = g(\bigcup_{x \in X}\{x\}) = \bigvee\{g(\{x\}) \mid x \in X\} = \bigvee h(X)$$

holds in $\mathscr{S}^\circ$ by Section 15.

21. Theorem. *$(\mathscr{S}^\circ_r, h_r)$ is an r-hull of $\mathscr{S}$.*

Proof. Requirement 2.a) is satisfied following Section 20.a), the map $h : S \to S^\circ$ is injective by Section 20.b), requirements 2.b.α), 2.b.β) and 2.b.γ) are satisfied following Section 16, Section 17 and Section 20.c), respectively (by assumption, r is regular, hence $r^* = r$).

Suppose the assumptions of Section 2.c) concerning $\mathscr{L}$ and φ to be true. Let $\xi \in S^\circ$. First of all, we shall prove that $\bigvee_{\mathscr{L}} \varphi(X) = \bigvee_{\mathscr{L}} \varphi(Y)$ for any two sets $X, Y \in \xi$ (the joins $\bigvee_{\mathscr{L}} \varphi(X)$ and $\bigvee_{\mathscr{L}} \varphi(Y)$ exist since $X, Y \in S_r$). Then we shall show that the map $\psi : S^\circ \to L$ defined by

$$(4) \qquad \psi(\xi) =_{\mathrm{Df}} \bigvee \varphi(X) \quad (X \in \xi \in S^\circ)$$

is a join r-complete homomorphism of $\mathscr{S}^\circ$ to $\mathscr{L}$ satisfying the equality $\varphi = \psi h$.

There is $X \leqq' Y \leqq' X$ for any $X, Y \in \xi \in S^\circ$. The sets $x \wedge Y$, $y \wedge X$ are distributive for every $x \in X$, $y \in Y$ following Lemma 5. Further, $x \wedge Y \in S_r$ as well as $y \wedge X \in S_r$; from this fact together with the properties of φ we get

$$\varphi(x) = \varphi(\bigvee_{\mathscr{S}}(x \wedge Y)) = \bigvee_{\mathscr{L}} \varphi(x \wedge Y) = \bigvee_{\mathscr{L}}\{\varphi(x) \wedge \varphi(v) \mid v \in Y\}$$

and similarly

$$\varphi(y) = \bigvee_{\mathscr{L}}\{\varphi(u) \wedge \varphi(y) \mid u \in X\}\,.$$

This implies immediately the following:

$$\bigvee\varphi(X) = \bigvee_{x\in X} \varphi(x) = \bigvee_{x\in X} \bigvee_{v\in Y} (\varphi(x) \wedge \varphi(v)) ,$$

$$\bigvee\varphi(Y) = \bigvee_{y\in Y} \varphi(y) = \bigvee_{y\in Y} \bigvee_{u\in X} (\varphi(y) \wedge \varphi(u)) .$$

Since the lattice $\mathscr{L}$ is join r-complete, then all the above mentioned joins exist. Hence

$$\bigvee\varphi(X) = \bigvee\{\varphi(x) \wedge \varphi(y) \mid x \in X,\ y \in Y\} = \bigvee\varphi(Y) ,$$

showing that (4) is a correct definition of ψ.

Let $\xi, \eta \in S^\circ$, $X \in \xi$, $Y \in \eta$; then $\psi(\xi) = \bigvee\varphi(X)$, $\psi(\eta) = \bigvee\varphi(Y)$. Section 14, the definition of ψ and the properties of φ imply that

$$\psi(\xi \wedge \eta) = \psi(g(X) \wedge g(Y)) = \psi\, g(X \wedge Y) = \bigvee\varphi(X \wedge Y) =$$
$$= \bigvee\{\varphi(x) \wedge \varphi(y) \mid x \in X,\ y \in Y\} .$$

Following the r-distributivity of the lattice $\mathscr{L}$, there is

$$\psi(\xi \wedge \eta) = \bigvee\{\varphi(x) \wedge \varphi(y) \mid x \in X,\ y \in Y\} = \bigvee_{y\in Y} (\bigvee_{x\in X} (\varphi(x) \wedge \varphi(y))) =$$
$$= \bigvee_{y\in Y} (\varphi(y) \wedge \bigvee_{x\in X} \varphi(x)) = (\bigvee_{x\in X} \varphi(x)) \wedge (\bigvee_{y\in Y} \varphi(y)) = \psi(\xi) \wedge \psi(\eta) .$$

Hence, ψ is a meet-homomorphism from $\mathscr{S}^\circ$ to $\mathscr{L}$.

Let $\Gamma \subseteq S^\circ$, $0 < |\Gamma| < r$. Let us take a representative $v(\gamma)$ of γ for each $\gamma \in \Gamma$. Then $v(\gamma) \in S_r$ and, by Section 15, the equality $\bigvee\Gamma = g(\bigcup_{\gamma\in\Gamma} v(\gamma))$ holds in $\mathscr{S}^\circ$. Further, considering that $\mathscr{L}$ is join r-complete, we get

$$\bigvee_{\mathscr{L}} \psi(\Gamma) = \bigvee_{\mathscr{L}}\{\psi(\gamma) \mid \gamma \in \Gamma\} = \bigvee_{\mathscr{L}}\{\bigvee_{\mathscr{L}} \varphi\, v(\gamma) \mid \gamma \in \Gamma\} =$$
$$= \bigvee_{\mathscr{L}}(\bigcup\{\varphi\, v(\gamma) \mid \gamma \in \Gamma\}) = \bigvee_{\mathscr{L}} \varphi(\bigcup_{\gamma\in\Gamma} v(\gamma)) =$$
$$= \psi\, g(\bigcup_{\gamma\in\Gamma} v(\gamma)) = \psi(\bigvee_{\mathscr{S}^\circ}\Gamma) .$$

(The third equality: there is $\varphi(v(\gamma_0)) \subseteq \bigcup\{\varphi(v(\gamma)) \mid \gamma \in \Gamma\}$ for every $\gamma_0 \in \Gamma$; if $\lesssim$ denotes the ordering of the lattice $\mathscr{L}$, then this inclusion implies the inequality

$$\bigvee_{\mathscr{L}}\{\bigvee_{\mathscr{L}} \varphi\, v(\gamma) \mid \gamma \in \Gamma\} \lesssim \bigvee_{\mathscr{L}}(\bigcup\{\varphi(v(\gamma)) \mid \gamma \in \Gamma\}) .$$

The other inequality follows from the fact that $\bigvee_{\mathscr{L}}(\bigvee_{\mathscr{L}} \varphi(v(\gamma)) \mid \gamma \in \Gamma)$ is an upper bound of $\bigcap\{\varphi(v(\gamma)) \mid \gamma \in \Gamma\}$ in $\mathscr{L}$.)

We have proved that $\psi : S^\circ \to L$ is a join r-complete homomorphism from $\mathscr{S}^\circ$ to $\mathscr{L}$. Following the definition of ψ, the following holds for every $x \in S$:

$$\varphi(x) = \bigvee\varphi(\{x\}) = \psi(g(\{x\})) = \psi\, h(x) ,$$

hence $\varphi = \psi h$.

This proves the theorem.

22. Corollary. *For every infinite cardinal s, there exists an s-hull of $\mathscr{S}$.*
The proof follows immediately from Theorems 21 and 9.

23. Theorem. *Let $(\mathscr{K}, f)$ be an s-hull of $\mathscr{S}$. Then $f(\bigvee_{\mathscr{S}} X) = \bigvee_{\mathscr{K}} f(X)$ for every distributive subset X of $\mathscr{S}$.*

Proof. Let $\lesssim$ be the ordering of the lattice $\mathscr{K}$. If $0 < |X| < s^*$, then the theorem holds by Section 2.b.β). Let $|X| \geqq s^*$. Then the cardinal $t = |X|^+$ is infinite and regular and such that $|X| < t$. $(\mathscr{S}_t^\circ, h_t)$ is a t-hull of $\mathscr{S}$ by Theorem 21, hence X is distributive iff $\bigvee_{\mathscr{S}^\circ_t} h_t(X) = h_t(\bigvee_{\mathscr{S}} X)$. Following Corollary 7, there exists an injective join s-complete homomorphism ψ from $\mathscr{K}$ to $\mathscr{S}_t^\circ$ such that $h_t = \psi f$.

Let b be an upper bound of $f(X)$ in $\mathscr{K}$. Then $\psi(b)$ is an upper bound of $\psi f(X) = h_t(X)$ in $\mathscr{S}_t^\circ$, hence

$$h_t(\bigvee_{\mathscr{S}} X) = \bigvee_{\mathscr{S}^\circ_t} h_t(X) \leqq_t \psi(b) .$$

Further, $h_t(\bigvee X) = \psi(f(\bigvee X))$; hence, $f(\bigvee X) \lesssim b$, since ψ is injective.

(Would not be $f(\bigvee X) \lesssim b$ satisfied then $f(\bigvee X) \wedge b \prec f(\bigvee X)$. This fact together with the injectivity of the isotonic homomorphism ψ implies

$$\psi(b) \wedge h_t(\bigvee X) = \psi(b) \wedge \psi(f(\bigvee X)) =$$
$$= \psi(b \wedge f(\bigvee X)) <_t \psi(f(\bigvee X)) = h_t(\bigvee X) .$$

A contradiction with the proved relation $h_t(\bigvee X) \leqq_t \psi(b)$.)

Since $f(\bigvee X)$ is an upper bound of $f(X)$ in $\mathscr{K}$ as well, then $f(\bigvee_{\mathscr{S}} X) = \bigvee_{\mathscr{K}} f(X)$.

The statement is obvious for $X = \emptyset : \emptyset$ is distributive iff there exists $\bigwedge_{\mathscr{S}} S$; for the following – see Section 3.d).

24. Example. We shall show that the converse statement to Theorem 23 need not be true in general. Let A be an infinite set, $o, j \notin A$. Put $S = A \cup \{o, j\}$. Let id denote

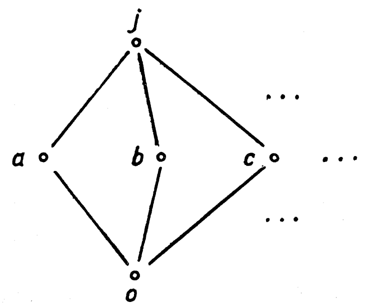

Fig. 3.

the identity relation. Put

$$\mathcal{S} = (\{o\};\ \mathrm{id}) \oplus (A;\ \mathrm{id}) \oplus (\{j\};\ \mathrm{id}),$$

where $\oplus$ denotes the ordinal sum; see also Fig. 3, where $A = \{a, b, c, \ldots\}$. Let $X \subseteq S$. Then X is distributive in $\mathcal{S}$ iff either $j \in X$ or $A \subseteq X$ or $|X \cap A| \leqq 1$. An $\aleph_0$-distributive hull of $\mathcal{S}$ is for example the system

$$\mathcal{A} = \{X \in \exp A \mid X = A \quad \text{or} \quad |X| < \aleph_0\},$$

ordered by inclusion, together with the map $f : S \to A$, defined by

$$f(o) = \emptyset, \quad f(j) = A, \quad f(x) = \{x\} \quad \text{for} \quad x \in A.$$

($(\mathcal{A}, f)$ is an $\aleph_0$-hull following Definition 2 or Theorem 21; see also Section 31.) If $a \in A$, then $A - \{a\}$ is not a distributive subset of $\mathcal{S}$, but

$$f(\bigvee(A - \{a\})) = f(j) = A = \sup_{(\mathcal{A};\subseteq)} f(A - \{a\}).$$

III. AN OTHER CONSTRUCTION OF THE r-HULL

25. Definition. If $X \subseteq S$, then

$$X^- =_{\mathrm{Df}} \{\bigvee_{\mathcal{S}} Y \mid Y \subseteq (X],\ Y \text{ is distributive}\}.$$

26. Lemma. *Let $X \in \exp S$. Then*

$$X^- = \{y \in S \mid \{y\} \leqq' X\}.$$

Proof. Let $y \in S$, $\{y\} \leqq' X$. Then $y \wedge X$ is distributive by Section 5. Further, $y \wedge X \subseteq (X]$, and $\{y\} \leqq' X$, hence $y = \bigvee(y \wedge X)$. This implies that $y \in X^-$.

Let $y \in X^-$. Then there exists a distributive set Y with $y = \bigvee Y$ and $Y \subseteq (X]$. Let $z \in (\{y\}]$, i.e. let $z \leqq y$. Then

$$z = z \wedge y = z \wedge \bigvee Y = \bigvee(z \wedge Y) \leqq \bigvee(z \wedge X) = z.$$

Hence $z = \bigvee(z \wedge X)$ for every $z \leqq y$, i.e. $\{y\} \leqq' X$.

27. Lemma. $X^- \leqq' X$ *for each* $X \in \exp S$.

Proof. Let $y \in (X^-]$. Then there exists $z \in X^-$ such that $y \leqq z$. Further, there exists a distributive set $Z \subseteq (X]$ with $z = \bigvee Z$. Then it holds

$$y = y \wedge z = y \wedge \bigvee Z = \bigvee(y \wedge Z) \leqq \bigvee(y \wedge X) \leqq y,$$

i.e. $X^- \leqq' X$.

28. Theorem. *The map* $^-$: $\exp S \to \exp S$ *is a closure operator on the complete lattice* $(\exp S; \subseteq)$.

Proof. Let X, $Y \in \exp S$, $X \subseteq Y$. From the definition of X^- it follows immediately that $X \subseteq X^-$ (any one-point set is distributive and $X \subseteq (X]$). Let $x \in X^-$, then $\{x\} \leqq' X$ by Section 26. There is $X \subseteq Y$ and immediately from the definition of the relation $\leqq'$ we get $X \leqq' Y$, in this case. Relation $\leqq'$ is transitive (see Section 10), hence $\{x\} \leqq' Y$. Then $x \in Y^-$ by Lemma 26. Thus $X^- \subseteq Y^-$.

We have $X \subseteq X^-$, hence $X^- \subseteq X^{--}$ as well. Let $x \in X^{--}$. Then $\{x\} \leqq' X^-$ by Section 26; further $X^- \leqq' X$ by Section 27. Hence $\{x\} \leqq' X$ by Lemma 10; following Section 26, $x \in X^-$, proving the inclusion $X^{--} \subseteq X^-$.

29. Remark. For some semilattices $\mathscr{S}$, the closure operator $X \mapsto X^-$ is neither topologic (see [1], p. 116) nor algebraic (see [3], Section 1.b.)).

30. Lemma. *If* X, $Y \in \exp S$, *then* $X \leqq' Y$ *iff* $X^- \subseteq Y^-$.

Proof. Let $X \leqq' Y$. If $x \in X^-$, then $\{x\} \leqq' X$ (Section 26), hence $\{x\} \leqq' Y$ as well ($\leqq'$ is transitive by Section 10). Then $x \in Y^-$ by Lemma 26.

Suppose now $X^- \subseteq Y^-$ and let $x \in (X]$. Since $\{x\}$ is distributive, then $x \in X^-$ as it follows from the definition of X^-. Then $x \in Y^-$ as well and there exists a distributive set Z such that $Z \subseteq (Y]$ and $x = \bigvee Z$. Then

$$x = x \wedge \bigvee Z = \bigvee (x \wedge Z) \leqq \bigvee (x \wedge Y) \leqq x ,$$

thus, $X \leqq' Y$ by the definition of $\leqq'$.

31. Theorem. *Let r be an infinite regular cardinal. Then the system* $\{X^- \mid X \in S_r\}$, *ordered by inclusion together with the map* $x \mapsto \{x\}^- = (\{x\}]$ (*for* $x \in S$) *is an r-hull of* $\mathscr{S}$.

Proof. It follows immediately from the construction of $\mathscr{S}^\circ$ (see Section 12) and from Lemma 30: if ξ $\eta \in S^\circ$, then $\xi \leqq_r \eta$ iff for some (hence for all) representatives X of ξ, Y of η there is $X^- \subseteq Y^-$. The remaining follows from the fact that $X^- = \{x\}^-$ for all $X \in h(x)$; the equality $\{x\}^- = (\{x\}]$ is obvious.

32. Remark. Let us define categories $\boldsymbol{SL_s}$ and $\boldsymbol{DL_s}$ in the following way. Objects of $\boldsymbol{SL_s}$ are all meet semilattices. If $\mathscr{A}$, $\mathscr{B}$ are $\boldsymbol{SL_s}$-objects then $\varphi : A \to B$ is an $\boldsymbol{SL_s}$-morphism if it satisfies the following conditions:

(5) φ is a meet-homomorphism from $\mathscr{A}$ to $\mathscr{B}$.

(6) If X is a distributive subset of $\mathscr{A}$ with $0 < (X) < s^*$, then $\varphi(X)$ is distributive in $\mathscr{B}$ and $\varphi(\bigvee_{\mathscr{A}} X) = \bigvee_{\mathscr{B}} \varphi(X)$.

Objects of category $\boldsymbol{DL}_s$ are all s-distributive lattices, $\boldsymbol{DL}_s$-morphisms are all join s-complete homomorphisms between $\boldsymbol{DL}_s$-objects. It can be easily seen that $\boldsymbol{DL}_s$ is a full subcategory of $\boldsymbol{SL}_s$. The following statement holds by Sections 2 and 22:

$\boldsymbol{DL}_s$ is a full reflexive subcategory of $\boldsymbol{SL}_s$.

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