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A MODIFICATION AND COMPARISON OF FILIPPOV AND VIKTOROVSKIJ GENERALIZED SOLUTIONS

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This paper is an immediate continuation of Chapter III of the first part [3]. We present here the main result, which consists in such a modification of Viktorovskij's definition that the equivalence with Filippov's definition [1] in terms of differential inclusions can be established.

Theorem 7. ($MV \Rightarrow CF$). *If an absolutely continuous function $x(t)$ is an MV-solution of the equation $\dot{x} = f(t, x)$ from Remark 5 on an interval $T = \langle t_1, t_2 \rangle$, then the condition CF from Definition 7 holds for $x(t)$ on T.*

Proof. Let an absolutely continuous function $x(t)$ be given on the interval T and let $x(t)$ be an MV-solution of $\dot{x} = f(t, x)$ on T. Hence for every $\varepsilon > 0$ and every $N \subset G$, $\mu(N) = 0$, there exists a function ψ on T which satisfies (6)–(10) with the norm $\|x\| = \{\max |x_i| : i = 1, \ldots, n\}$. The condition CF can be written in the form

$$\forall(B_j)\, \forall(i)\, \exists(T_1 \subset T : \mu(T_1) = \mu(T))\, \forall(t \in T_1)\, \{\alpha \vee \beta\} \quad \text{(cf. Remark 6)}\,.$$

The negation of this condition has the form:

$$\exists(B_j)\, \exists(i)\, \forall(T_1 \subset T : \mu(T_1) = \mu(T))\, \exists(t \in T_1) \quad \{\text{non}\,(\alpha \vee \beta)\}\,.$$

This is equivalent to the condition

$$\exists(B_j)\, \exists(i)\, \exists(T' \subset T : \mu^*(T') > 0)\, \forall(t \in T') \quad \{\text{non}\,(\alpha \vee \beta)\}\,,$$

where μ^* is the outer measure.

The remaining part of the proof is identical with the proof of Theorem 6, where we insert (6)–(10) instead of (1)–(5). The contradiction obtained proves the theorem.

Remark 7. For brevity, let us introduce $K^U(f, t, x) = \bigcap_{\delta > 0} \bigcap_{N, \mu(N) = 0} \overline{f(t, U(x, \delta) - N)}$ for an arbitrary $(t, x) \in G$ analogously to $K^F(f, t, x)$ in Remark 2.

Lemma 9. *Let us suppose that $x(t)$ is a continuous function on the interval $T = \langle t_1, t_2 \rangle$ and $(t, x(t)) \in G$ holds for every $t \in T$. Then there exists a subset $T_1 \subset T$, $\mu(T_1) = \mu(T)$ such that $K^U(f, t, x(t))$ is compact and nonempty for every $t \in T_1$.*

Proof. Let us choose $\delta_0 > 0$ small enough so that the compact set $\overline{\bigcup_{t \in T} (t, U(x(t), \delta_0))} \subset G$. For this set there exists a subset $T'_1 \subset T$, $\mu(T'_1) = \mu(T)$ and a function $m(t)$ defined on T'_1 with the properties from Remark 5.

Consequently, $K^U(f, t, x(t))$ is compact for every $t \in T'_1$. Further, there exists a subset $T'_2 \subset T$, $\mu(T'_2) = \mu(T)$ such that $K^U(f, t, x(t)) \neq \emptyset$ on T'_2 because we can prove a lemma analogous to Lemma 6 for closures. Now, we choose $T_1 = T'_1 \cap T'_2$ and the proof is complete.

Corollary 2. *Lemma 9 holds also for the sets $K^F(f, t, x(t))$.*

Remark 8. Let a function $z(t)$ be defined and measurable on T and let $z(t) \in K^F(f, t, x(t))$ a. e. on T for a given continuous function $x(t)$ on T. Then the function $z(t)$ is integrable on T. This assertion follows from Remark 5.

Lemma 10. *For every $(t, x) \in G$ we have the following equivalence: $y \in K^U(f, t, x)$ if and only if*

$$\forall(\varepsilon > 0,\ \delta > 0)\ \mu\{z \in U(x, \delta) : \|y - f(t, z)\| < \varepsilon\} > 0\,.$$

Proof. Let $\forall(\varepsilon > 0, \delta > 0)\ \mu\{z \in U(x, \delta) : \|y - f(t, z)\| < \varepsilon\} > 0$ be satisfied. Let us fix $\delta > 0$; then the preceding condition yields $U(y, \varepsilon) \cap f(t, U(x, \delta) - N_\delta) \neq \emptyset$ for every $\varepsilon > 0$, where the set N_δ of measure zero has the same meaning as the set N_0 in Lemma 5. Consequently, $y \in K^U(f, t, x)$ holds because $y \in \overline{f(t, U(x, \delta) - N_\delta)}$ for an arbitrary $\delta > 0$. Now let us suppose $y \in K^U(f, t, x)$. This yields that $y \in \overline{f(t, U(x, \delta) - N_\delta)}$ for an arbitrary $\delta > 0$. Let us choose a neighbourhood $U(y, \varepsilon)$ for a certain $\varepsilon > 0$ and let us choose a certain $\delta > 0$. This neighbourhood contains at least one point $\bar{y} \in f(t, U(x, \delta) - N_\delta)$. Then there exists a point $\bar{x} \in U(x, \delta) - N_\delta$ such that $\bar{y} = f(t, \bar{x})$ and the function $f(t, z)$ is weakly asymptotically continuous (cf. Definition 1) at the point $\bar{x}$ with respect to the variable z (cf. Lemma 4). Then it holds:

$$\forall(\varepsilon' > 0)\ \forall(\delta' > 0)\ \exists(0 < \delta_0 \leqq \delta')\ \exists(N' : \mu(N') < \mu(U(\bar{x}, \delta_0)))$$
$$\{\|z - \bar{x}\| < \delta_0,\ z \notin N' \Rightarrow \|f(t, z) - f(t, \bar{x})\| < \varepsilon'\}\,.$$

Let us choose $\delta' > 0$ and $\varepsilon' > 0$ such that $U(\bar{x}, \delta') \subset U(x, \delta)$ and $U(\bar{y}, \varepsilon') \subset U(y, \varepsilon)$. Then it can be proved that $\mu\{z \in U(x, \delta) : \|y - f(t, z)\| < \varepsilon\} > 0$.

Lemma 11. *If a set $\Delta \subset E_n$ is open, then the set $\{t \in T : K^U(f, t, x(t)) \cap \Delta \neq \emptyset\}$ is measurable for any measurable function $x(t)$ defined on the interval T, where $(t, x(t)) \in G$ for every $t \in T$.*

Proof. An open set Δ can be written in the form $\Delta = \bigcup_{m=1}^{\infty} Q_{m-1}$, where Q_{m-1} are closed sets fulfilling $Q_0^0 \subset Q_0 \subset \ldots \subset Q_m^0 \subset Q_m \subset Q_{m+1}^0 \subset \ldots$ where Q_m^0 is the interior of Q_m. Let us denote $A_m = \{t \in T : K^U(f, t, x(t)) \cap Q_{m-1} \neq \emptyset\}$ and $A = \{t \in T : K^U(f, t, x(t)) \cap \Delta \neq \emptyset\}$. Then $A = \bigcup_{m=1}^{\infty} A_m$.

Now we must prove that the set A is measurable. Let us choose a fixed index m. The sets $\{x \in U(x(t), \delta) : f(t, x) \in Q_m\}$ are measurable for almost all $t \in T$. First of all we shall show that the sets $T_m^\delta = \{t \in T : \mu\{x \in U(x(t), \delta) : f(t, x) \in Q_m\} > 0\}$ are measurable for an arbitrary $\delta > 0$. The set $T \times E_n$ is measurable in the space E_{n+1} and $M = \{(t, x) \in T \times E_n : x \in U(x(t), \delta), f(t, x) \in Q_m\}$ is a measurable set in E_{n+1} as well. $M(t)$ is the projection of a section of the set M into E_n with a fixed t.

Hence we can write $T_m^\delta = \{t \in T : \mu(M(t)) > 0\}$ and this implies that T_m^δ is measurable set because M is a measurable set in E_{n+1}. There exists a limit $T_m = \lim_{\delta \to 0+} T_m^\delta$, it is measurable and $T_m = \bigcap_{\delta > 0} T_m^\delta$ holds. Let $t \in A_m$, then for this t there exists $y \in K^U(f, t, x(t)) \cap Q_{m-1}$ and from Lemma 10 we obtain that $t \in T_m^\delta$ for an arbitrary $\delta > 0$ and also $t \in T_m$ so that $A_m \subset T_m$.

Now, on the contrary, let $t \in T_m$. This means that $\forall(\delta > 0)\, \mu\{x \in U(x(t), \delta) : f(t, x) \in Q_m\} > 0$ and that $\overline{f(t, U(x(t), \delta) - N_\delta)} \cap Q_m \neq \emptyset$ for an arbitrary $\delta > 0$ and also $K^U(f, t, x(t)) \cap Q_m \neq \emptyset$. This implies $t \in A_{m+1}$ and we obtain $T_m \subset A_{m+1}$. This yields that $A = \bigcup_{m=1}^{\infty} A_m = \bigcup_{m=1}^{\infty} T_m$ is measurable.

Lemma 12. *Let a measurable function $z(t)$ be defined a. e. on T so that $z(t) \in K^F(f, t, x(t))$ a. e. on T, where $x(t)$ is a continuous function on T and $(t, x(t)) \in G$ for every $t \in T$. Then there exist p functions $y_i(t)$, $i = 1, \ldots, p \leqq n + 1$, defined a. e. on T, measurable and locally integrable, with these properties: $y_i(t) \in K^U(f, t, x(t))$ holds a. e. on T for each index i, $z(t) = \sum_{i=1}^{p} \alpha_i(t)\, y_i(t)$ a. e. on T, where $\alpha_i(t)$ are measurable real functions satisfying $0 \leqq \alpha_i(t) \leqq 1$ and $\sum_{i=1}^{p} \alpha_i(t) = 1$ a. e. on T.*

Proof. Let $z(t) \in K^F(f, t, x(t))$ and let $K^F(f, t, x(t))$ be a compact set for every $t \in T_0 \subset T$, where $\mu(T_0) = \mu(T)$ (cf. Corollary 2). We shall find measurable functions $y_i(t)$ on this set T_0 with the properties of this lemma.

It is sufficient to find measurable functions, then the integrability follows from Remark 8. The function $z(t)$ is integrable on T as well. There exist p points $z_i(t)$, $i = 1, \ldots, p$ in $K^U(f, t, x(t))$ for every $t \in T_0$ such that $z(t) = \sum_{i=1}^{p} \beta_i(t)\, z_i(t)$, where $\beta_i(t)$ are real numbers satisfying $\sum_{i=1}^{p} \beta_i(t) = 1$ and $0 \leqq \beta_i(t) \leqq 1$.

Let us introduce the following sets. Let H_1 be the set of all rational points from E_n and H_2 the set of all p-tuples $\alpha_1, \ldots, \alpha_p$ of rational numbers. Now we introduce the cartesian product $H_1^p \times H_2$, where the points of that product have the form $(r_1, \ldots, r_p, \alpha_1, \ldots, \alpha_p)$ and r_i, $i = 1, \ldots, p$ are points from H_1. We define a subset $C \subset H_1^p \times H_2$ by

$$C = \{(r_1, \ldots, r_p, \alpha_1, \ldots, \alpha_p) \in H_1^p \times H_2 : \sum_{i=1}^{p} \alpha_i = 1,\ 0 \leqq \alpha_i \leqq 1,\ i = 1, \ldots, p\} .$$

The set C is countable. Hence we can arrange its elements into a sequence, say $C = \{(r_{1j}, \ldots, r_{pj}, \alpha_{1j}, \ldots, \alpha_{pj})\}_{j=1}^{\infty}$. Let us choose any fixed positive integer k. Now we define the following sets for each positive integer j.

$$\hat{T}_{0j}^k = \left\{t \in T_0 : \left\|z(t) - \sum_{i=1}^{p} \alpha_{ij} r_{ij}\right\| < \frac{1}{k}\right\},$$

$$\hat{T}_{mj}^k = \left\{t \in T_0 : U\left(r_{mj}, \frac{1}{k}\right) \cap K^U(f, t, x(t)) \neq \emptyset\right\},$$

where $m = 1, \ldots, p$. The sets $\hat{T}_{0j}^k$ are measurable because the function $z(t)$ is measurable on T_0. According to Lemma 11 the sets $\hat{T}_{mj}^k$ are measurable. We introduce sets $\hat{T}_j^k = \bigcap_{m=0}^{p} \hat{T}_{mj}^k$ for each j and we prove that $T_0 = \bigcup_{j=1}^{\infty} \hat{T}_j^k$. We choose any $t \in T_0$. To that t there exist points $z_i(t)$, $i = 1, \ldots, p$ from $K^U(f, t, x(t))$ so that $z(t) = \sum_{i=1}^{p} \beta_i(t)\, z_i(t)$, where $\beta_i(t)$, $i = 1, \ldots, p$ satisfy $\sum_{i=1}^{p} \beta_i(t) = 1$ and $0 \leqq \beta_i(t) \leqq 1$. Moreover, this t satisfies the inequality

$$\left\|z(t) - \sum_{i=1}^{p} \alpha_{ij} r_{ij}\right\| = \left\|\sum_{i=1}^{p} \beta_i(t)\, z_i(t) - \sum_{i=1}^{p} \alpha_{ij} r_{ij}\right\| \leqq \leqq \sum_{i=1}^{p} |\beta_i(t) - \alpha_{ij}|\, \|z_i(t)\| + \sum_{i=1}^{p} |\alpha_{ij}|\, \|z_i(t) - r_{ij}\| . \tag{23}$$

Now we can choose such an index j that the element $(r_{ij}, \ldots, r_{pj}, \alpha_{1j}, \ldots, \alpha_{pj})$ from the set C satisfies the inequalities

$$\left\|z(t) - \sum_{i=1}^{p} \alpha_{ij} r_{ij}\right\| \leqq \sum_{i=1}^{p} |\beta_i(t) - \alpha_{ij}|\, \|z_i(t)\| + \sum_{i=1}^{p} |\alpha_{ij}|\, \|z_i(t) - r_{ij}\| < \frac{1}{k}$$

and

$$\|z_i(t) - r_{ij}\| < \frac{1}{k},$$

$i = 1, \ldots, p$. For this index j it holds $t \in \hat{T}_j^k$. Hence $T_0 = \bigcup_{j=1}^{\infty} \hat{T}_j^k$ holds. Let us set successively $T_1^k = \hat{T}_1^k$, $T_2^k = \hat{T}_2^k - T_1^k, \ldots, T_j^k = \hat{T}_j^k - \bigcup_{i=1}^{j-1} T_i^k$. Then $T_0 = \bigcup_{j=1}^{\infty} T_j^k$ is a disjoint covering of the set T_0 by measurable sets. The following formula defines measurable functions $z_i^k(t), \alpha_i^k(t)$ on T_0, $i = 1, \ldots, p$: $z_i^k(t) = r_{ij}$, $\alpha_i^k(t) = \alpha_{ij}$ for $t \in T_j^k$. These functions fulfil $\|z(t) - \sum_{i=1}^{p} \alpha_i^k(t)\, z_i^k(t)\| < 1/k$, $\sum_{i=1}^{p} \alpha_i^k(t) = 1$, $0 \leqq \alpha_i^k(t) \leqq 1$, $i = 1, \ldots, p$ and $z_i^k(t) \in U(K^U(f, t, x(t)), 1/k)$ on T_0. We have found measurable functions $z_i^k(t)$, $\alpha_i^k(t)$ which form a sequence $\{(z_1^k(t), \ldots, z_p^k(t), \alpha_1^k(t), \ldots, \alpha_p^k(t))\}_{k=1}^{\infty}$. Let us denote $y_k(t) = (z_1^k(t), \ldots, z_p^k(t), \alpha_1^k(t), \ldots, \alpha_p^k(t))$, where $y_k(t) \in E_n^p \times E_n$.

Now we shall introduce the sets $M_s(t) = \overline{\{y_k(t)\}_{k=s}^{\infty}}$ and $Q(t) = \bigcap_{s=1}^{\infty} M_s(t) = \bigcap_{s=1}^{\infty} \overline{(\bigcup_{k=s}^{\infty} \{y_k(t)\})}$ on T_0. The sets $Q(t)$ are nonempty for every $t \in T_0$ because the sequence $\{y_k(t)\}_{k=1}^{\infty}$ is bounded for every $t \in T_0$. Further, $M_s(t)$ are compact sets for every $t \in T_0$. This implies that the sets $Q(t)$ are compact as well. If $y(t) = (z_1(t), \ldots, z_p(t), \alpha_1(t), \ldots, \alpha_p(t)) \in Q(t)$, then $z_i(t) \in K^U(f, t, x(t))$ for $i = 1, \ldots, p$ and $0 \leqq \alpha_i(t) \leqq 1$ for $i = 1, \ldots, p$ and $\sum_{i=1}^{p} \alpha_i(t) = 1$. It holds $z(t) = \sum_{i=1}^{p} \alpha_i(t)\, z_i(t)$ on T_0 as well. We shall prove that the set function $Q(t)$ is measurable on T_0. It suffices to show that the set $B = \{t \in T_0 : Q(t) \cap F \neq \emptyset\}$ is measurable for every closed set F in the space $E_n^p \times E_n$. We introduce the auxiliary set

$$A = \bigcap_{n=1}^{\infty} \bigcap_{i=1}^{\infty} \bigcap_{j=i}^{\infty} \left\{t \in T_0 : y_j(t) \in U\left(F, \frac{1}{n}\right)\right\},$$

which is measurable. Now we shall prove that $A = B$. First, let $t \in A$, then $\forall(n)$ $\forall(i)\, \exists(j \geqq i)$ such that $y_j(t) \in U(F, 1/n)$ and $y_j(t) \in M_i(t)$. Hence for each index n there exists such an index j_n that $y_{j_n}(t) \in U(F, 1/n)$ and hence $Q(t) \cap F \neq \emptyset$. Consequently, it is $t \in B$ which proves $A \subset B$.

On the other hand, let $t \in B$. It means that $Q(t) \cap F \neq \emptyset$. This implies that there exists $y \in Q(t) \cap F$. With respect to the definition of $Q(t)$ there exists a subsequence $\{y_{k(s)}(t)\}_{s=1}^{\infty}$ whose limit is y. For each n and i there exists an index $k(s) \geqq i$ such that $y_{k(s)}(t) \in U(F, 1/n)$, and this yields $t \in A$. Thus we have proved that $A = B$. This is sufficient for the measurability of the set function $Q(t)$ on T_0. Now we shall find a measurable function $\psi(t) \in Q(t)$ on T_0 and the proof will be complete. The set $Q(t)$ is compact and nonempty for every $t \in T_0$ and $Q(t) \subset E_m = E_n^p \times E_n$ of the dimension $m = np + n$. Let us write the points y of the space E_m in the form $y = (y^1, \ldots, y^m)$. We introduce the function $\varphi'(t) = \sup\{y^1(t) : (y^1(t), \ldots, y^m(t)) \in Q(t)\}$ on T_0. We show that the function $\varphi^1(t)$ is measurable on T_0. This immediately follows from the measurability of the set

$$\{t : \varphi^1(t) \geqq \lambda\} = \{t : Q(t) \cap \{(y^1, \ldots, y^m) : y^1 \geqq \lambda\} \neq \emptyset\}$$

for every real value λ. Further, we define the set function

$$Z_1(t) = \{(y^1, \ldots, y^m) : y^1 \geqq \varphi^1(t),\ \|y\| \leqq c(t)\} .$$

This set function is measurable and $Z_1(t)$ is nonempty for $c(t) = \max(m(t), 1)$ with the norm $\|y\| = \max\{|y^i| : i = 1, \ldots, m\}$. The sets $Z_1(t)$ and $Q(t)$ have a nonempty intersection for every $t \in T_0$. Hence the set function $Q_1(t) = Q(t) \cap Z_1(t)$ is measurable on T_0 since both $Q(t)$ and $Z_1(t)$ are measurable set functions. Let us introduce analogously the function $\varphi^2(t) = \sup\{y^2(t) : (y^1(t), \ldots, y^m(t)) \in Q_1(t)\}$ on T_0. The function $\varphi^2(t)$ is measurable on T_0 as well as the function $\varphi^1(t)$. Further, we define

$$Z_2(t) = \{(y^1, \ldots, y^m) : y^2 \geqq \varphi^2(t),\ \|y\| \leqq c(t)\}$$

and $Q_2(t) = Q_1(t) \cap Z_2(t)$. In this way we can obtain functions $\varphi^i(t)$ on T_0 for each index $i = 1, \ldots, m$ in the form $\varphi^i(t) = \sup\{y^i(t) : (y^1(t), \ldots, y^m(t)) \in Q_{i-1}(t)\}$ and measurable set functions

$$Z_i(t) = \{(y^1, \ldots, y^m) : y^i \geqq \varphi^i(t),\ \|y\| \leqq c(t)\}$$

and $Q_i(t) = Q_{i-1}(t) \cap Z_i(t)$ with $Q_0(t) = Q(t)$. The set functions $Q_i(t)$ are measurable on T_0. This construction implies that the function $\varphi(t) = (\varphi^1(t), \ldots, \varphi^m(t)) \in Q(t)$ for every $t \in T_0$ is the desired measurable function.

Theorem 8. $(F \Rightarrow MV)$. *Let a function $x(t)$ be defined and absolutely continuous on $T = \langle t_1, t_2 \rangle$, let it map the interval T into E_n and let $(t, x(t)) \in G$ for every $t \in T$, where $G \subset E_{n+1}$ is an open connected set. If the function $x(t)$ is an F-solution of the equation $\dot{x} = f(t, x)$ from Remark 5 on T, then $x(t)$ is an MV-solution on T.*

Proof. Let us choose $\varepsilon > 0$ small enough so that the compact set $\overline{\bigcup_{t \in T} (t, U(x(t), \varepsilon))}$ is a subset of G and let us choose an arbitrary set $N \subset G$, $\mu(N) = 0$. We shall find a function $\psi(t)$ on T with respect to ε and N such that the function ψ satisfies the following properties:

$$(t, \psi(t)) \in G \quad \text{on} \quad T, \tag{24}$$

$$f(t, \psi(t)) \quad \text{is integrable on} \quad T, \tag{25}$$

$$\|x(t) - \psi(t)\| < \varepsilon \quad \text{on} \quad T, \tag{26}$$

$$\left\| x(t) - \left(x(t_1) + \int_{t_1}^{t} f(\tau, \psi(\tau))\, d\tau\right) \right\| < \varepsilon \quad \text{on} \quad T, \tag{27}$$

and

$$(t, \psi(t)) \notin N \quad \text{almost everywhere on} \quad T. \tag{28}$$

Let $T' \subset T$, $\mu(T') = \mu(T)$ be a set, where $\dot{x}(t) \in K^F(f, t, x(t))$ and $K^F(f, t, x(t))$ are compact sets. According to Lemma 12 the function $\dot{x}(t)$ can be written on T' in the form $\dot{x}(t) = \sum_{i=1}^{p} \alpha_i(t)\, y_i(t)$, where $0 \leqq \alpha_i(t) \leqq 1$, $\sum_{i=1}^{p} \alpha_i(t) = 1$, $p \leqq n + 1$ and $\alpha_i(t)$ are real measurable functions defined on the interval T while $y_i(t)$ are local integrable on T and $y_i(t) \in K^U(f, t, x(t))$ for every $t \in T'$. First of all we shall find an approximation of the function $\dot{x}(t)$ on T' which has the form $\sum_{i=1}^{p} \bar{\beta}_i(t)\, g_i(t)$, where $\bar{\beta}_i(t)$ are simple measurable functions defined on T with rational values satisfying $0 \leqq \bar{\beta}_i(t) \leqq 1$, $\sum_{i=1}^{p} \bar{\beta}_i(t) = 1$ on T' while the functions $g_i(t)$ are step functions on T.

Now we shall construct the functions $\bar{\beta}_i(t)$ and $g_i(t)$ with these properties so that the inequality

$$\left\| \int_{t_1}^{t} (\dot{x}(\tau) - \sum_{i=1}^{p} \bar{\beta}_i(\tau)\, g_i(\tau))\, d\tau \right\| < \varepsilon_0 \tag{29}$$

is satisfied for every $t \in T$, where $\varepsilon_0 = \varepsilon/3$. The inequality (29) can be expressed in the form

$$\left\| \int_{t_1}^{t} (\dot{x}(\tau) - \sum_{i=1}^{p} \bar{\beta}_i(\tau)\, g_i(\tau))\, d\tau \right\| \leqq$$

$$\leqq \left\| \int_{t_1}^{t} (\sum_{i=1}^{p} \alpha_i(\tau)\, y_i(\tau) - \sum_{i=1}^{p} \alpha_i(\tau)\, f(\tau, \psi^i(\tau)))\, d\tau \right\| +$$

$$+ \left\| \int_{t_1}^{t} (\sum_{i=1}^{p} \alpha_i(\tau)\, f(\tau, \psi^i(\tau)) - \sum_{i=1}^{p} \alpha_i(\tau)\, g_i(\tau))\, d\tau \right\| +$$

$$+ \left\| \int_{t_1}^{t} (\sum_{i=1}^{p} \alpha_i(\tau)\, g_i(\tau) - \sum_{i=1}^{p} \beta_i(\tau)\, g_i(\tau))\, d\tau \right\| +$$

$$+ \left\| \int_{t_1}^{t} \sum_{i=1}^{p} (\beta_i(\tau) - \bar{\beta}_i(\tau))\, g_i(\tau)\, d\tau \right\| < \varepsilon_0 .$$

We must find functions $\psi^i(t)$ on T, $i = 1, \ldots, p$ such that the inequality

$$\left\| \int_{t_1}^{t} (\sum_{i=1}^{p} \alpha_i(\tau)\, y_i(\tau) - \sum_{i=1}^{p} \alpha_i(\tau)\, f(\tau, \psi^i(\tau)))\, d\tau \right\| < \frac{\varepsilon_0}{4} \tag{30}$$

is satisfied for every $t \in T$ and, at the same time, the functions ψ^i satisfy the conditions (24), (25), (26), (28). Let us choose $\delta = \varepsilon$. According to Lemma 10 it holds $\forall(\varepsilon' > 0)\ \mu(M^{y_i(t)}_{\varepsilon',\delta,t}) > 0$ on T' for each $i = 1, \ldots, p$, where

$$M^{y_i(t)}_{\varepsilon',\delta,t} = \{x \in U(x(t), \delta) : \|y_i(t) - f(t, x)\| < \varepsilon'\} .$$

Let $\varepsilon' > 0$ be such that $\mu(T)\, \varepsilon' < \varepsilon_0/4p$. Lemma 8 implies the existence of functions $\psi^i(t)$ on T which for each $i = 1, \ldots, p$ fulfil the following condition: $f(t, \psi^i(t))$ is integrable on T, $(t, \psi^i(t)) \notin N$ a. e. on T, $\psi^i(t) \in M^{y_i(t)}_{\varepsilon',\delta,t}$ on T' and $\psi^i(t) \in U(x(t), \delta)$ on $T - T'$. Since $\delta = \varepsilon$, the functions $\psi^i(t)$ satisfy (24) and (26). Hence the functions $\psi^i(t)$ satisfy the conditions (24), (25), (26), (28). We can write

$$\left\| \int_{t_1}^{t} \sum_{i=1}^{p} \alpha_i(\tau) \left(y_i(\tau) - f(\tau, \psi^i(\tau))\right) \mathrm{d}\tau \right\| \leqq \tag{31}$$

$$\leqq \int_{t_1}^{t} \sum_{i=1}^{p} \|\alpha_i(\tau)\| \, \|y_i(\tau) - f(\tau, \psi^i(\tau))\| \, \mathrm{d}\tau =$$

$$= \sum_{i=1}^{p} \int_{t_1}^{t} \|\alpha_i(\tau)\| \, \|y_i(\tau) - f(\tau, \psi^i(\tau))\| \, \mathrm{d}\tau \leqq \sum_{i=1}^{p} \mu(T) \, \varepsilon' < \frac{\varepsilon_0}{4}.$$

We have proved that the inequality (30) holds for every $t \in T$. Further, we can find an approximation of $f(t, \psi^i(t))$ by step functions $g_i(t)$ on T such that

$$\int_{t_1}^{t} \|f(\tau, \psi^i(\tau)) - g_i(\tau)\| \, \mathrm{d}\tau < \frac{\varepsilon_0}{4p}$$

holds for every $t \in T$. Hence

$$\left\| \int_{t_1}^{t} \sum_{i=1}^{p} \alpha_i(\tau) \left(f(\tau, \psi^i(\tau)) - g_i(\tau)\right) \mathrm{d}\tau \right\| \leqq \tag{32}$$

$$\leqq \sum_{i=1}^{p} \int_{t_1}^{t} \|\alpha_i(\tau)\| \, \|f(\tau, \psi^i(\tau)) - g_i(\tau)\| \, \mathrm{d}\tau \leqq$$

$$\leqq \sum_{i=1}^{p} \int_{t_1}^{t} \|f(\tau, \psi^i(\tau)) - g_i(\tau)\| \, \mathrm{d}\tau < \sum_{i=1}^{p} \frac{\varepsilon_0}{4p} = \frac{\varepsilon_0}{4}.$$

To each function $\alpha_i(t)$ there exists a sequence of simple measurable functions $\{\alpha_i^j(t)\}_{j=1}^{\infty}$ defined on T which converges uniformly to $\alpha_i(t)$ on T. It is sufficient to introduce a function $\beta(t)$ equal to a certain member of the sequence $\{\alpha_i^j(t)\}_{j=1}^{\infty}$ so that the inequality

$$|\alpha_i(t) - \beta_i(t)| < \frac{1}{2} \cdot \frac{\varepsilon_0}{4pk \, \mu(T)} \tag{33}$$

holds for every $t \in T$ where $k = \max \{1, \|g_i(t)\| : i = 1, \ldots, p, \; t \in T\}$. Hence

$$\left\| \int_{t_1}^{t} \left(\sum_{i=1}^{p} (\alpha_i(\tau) - \beta_i(\tau)) \, g_i(\tau) \right) \mathrm{d}\tau \right\| \leqq \sum_{i=1}^{p} k \int_{t_1}^{t} |\alpha_i(\tau) - \beta_i(\tau)| \, \mathrm{d}\tau < \frac{\varepsilon_0}{4}. \tag{34}$$

Let us choose sets $T_1, \ldots, T_m$, $\bigcup_{j=1}^{m} T_j = T$ such that the functions $\beta_i(t)$ are constant on each T_j, $j = 1, \ldots, m$. These sets are measurable. We shall find functions $\bar{\beta}_i(t)$

assuming rational values on each set T_j such that $\bar{\beta}_i(t)$ are constant on T_j, $\sum_{i=1}^{p} \bar{\beta}_i(t) = 1$, $0 \leqq \bar{\beta}_i(t) \leqq 1$ on T' and, at the same time, the inequality

$$(35) \qquad |\beta_i(t) - \bar{\beta}_i(t)| < \frac{\varepsilon_0}{4pk\, \mu(T)}$$

holds on T. It is sufficient to define auxiliary functions $\beta_i^*(t) = \alpha_i(t_j)$ on each T_j for $i = 1, \ldots, p$, where t_j is any fixed point in each T_j. It holds

$$(36) \qquad \sum_{i=1}^{p} \beta_i^*(t) = 1\,, \quad 0 \leqq \beta_i^*(t) \leqq 1 \quad \text{on} \quad T' \cap T_j$$

and (33) implies the inequality

$$(37) \qquad |\beta_i(t) - \beta_i^*(t)| < \frac{1}{2} \cdot \frac{\varepsilon_0}{4pk\, \mu(T)} \quad \text{on} \quad T_j\,,$$

where $j = 1, \ldots, m$ and $i = 1, \ldots, p$.

If the function $\beta_i^*(t)$ assumes rational values on T_j then we define $\bar{\beta}_i(t) = \beta_i^*(t)$ on T_j. Let $e \in \{2, \ldots, p\}$ be the number of irrational values of $\beta_i^*(t)$ on a given T_j for $i = 1, \ldots, p$ and let us change the order of indices so that the values $\beta_i^*(t)$ are irrational for $i = 1, \ldots, e$. We shall find $\bar{\beta}_i(t)$ for these values $\beta_i^*(t)$, $i = 1, \ldots, e$. Let $\delta' = \max \{\beta_i^*(t) : i = 1, \ldots, e\}$. Then the inequality $0 < \delta' < 1$ follows from (36). Now we shall define rational values $\bar{\beta}_i(t)$ for each $i = 1, \ldots, e - 1$ so that the inequality

$$(38) \qquad 0 < \beta_i^*(t) - \bar{\beta}_i(t) < \min \left\{ \frac{1}{2} \frac{\varepsilon_0}{4pk(p-1)\, \mu(T)}\,, \frac{\delta'}{p-1} \right\}$$

holds. We shall construct $\bar{\beta}_e(t)$ on $T' \cap T_j$ in the form $\bar{\beta}_e(t) = 1 - \sum_{\substack{i=1 \\ i \neq e}}^{p} \bar{\beta}_i(t)$. It holds $\beta_e^*(t) = 1 - \sum_{\substack{i=1 \\ i \neq e}}^{p} \beta_i^*(t)$ on $T' \cap T_j$. Further,

$$\bar{\beta}_e(t) - \beta_e^*(t) = \sum_{\substack{i=1 \\ i \neq e}}^{p} (\beta_i^*(t) - \bar{\beta}_i(t)) < \sum_{\substack{i=1 \\ i \neq e}}^{p} \min \left\{ \frac{1}{2} \frac{\varepsilon_0}{4pk(p-1)\, \mu(T)}\,, \frac{\delta'}{p-1} \right\} =$$

$$= \min \left\{ \frac{1}{2} \frac{\varepsilon_0}{4pk\, \mu(T)}\,, \delta' \right\} \quad \text{on} \quad T' \cap T_j\,.$$

We shall define the function $\beta_e(t)$ on $T_j \cap (T - T')$ so that $\beta_e(t)$ assumes a rational value and satisfies the inequality (38). Consequently, the inequality

$$(39) \qquad |\bar{\beta}_i(t) - \beta_i^*(t)| < \frac{1}{2} \frac{\varepsilon_0}{4pk\, \mu(T)}$$

is satisfied on each T_j, $j = 1, \ldots, m$ and for each $i = 1, \ldots, p$, and $\sum_{i=1}^{p} \bar{\beta}_i(t) = 1$, $0 \leqq \bar{\beta}_i(t) \leqq 1$ hold on $T' \cap T_j$, $j = 1, \ldots, m$. The inequalities (37) and (39) yield the inequality (35) for an arbitrary $t \in T$. Then it holds

$$(40) \qquad \left\| \int_{t_1}^{t} \sum_{i=1}^{p} (\beta_i(\tau) - \bar{\beta}_i(\tau))\, g_i(\tau)\, d\tau \right\| \leqq \sum_{i=1}^{p} k \int_{t_1}^{t} |\beta_i(\tau) - \bar{\beta}_i(\tau)|\, d\tau < \frac{\varepsilon_0}{4}.$$

From (30, (32), (34), (40) we derive that (29) is satisfied for an arbitrary $t \in T$.

We have found an approximation of the function $\dot{x}(t)$ on T' in the form $\sum_{i=1}^{p} \bar{\beta}_i(t)\, g_i(t)$ defined on T. The functions $g_i(t)$ are step functions on T, $\bar{\beta}_i(t)$ are simple measurable functions defined on T and assuming rational values and $0 \leqq \bar{\beta}_i(t) \leqq 1$, $\sum_{i=1}^{p} \bar{\beta}_i(t) = 1$ hold on T'. Further, we must prove the inequality

$$(41) \qquad \left\| \int_{t_1}^{t} (\dot{x}(\tau) - \sum_{i=1}^{p} \gamma_i(\tau)\, f(\tau, \psi^i(\tau)))\, d\tau \right\| < \varepsilon$$

for a certain $t \in T$, where the functions $\gamma_i(t)$ are defined and measurable on T. The functions $\gamma_i(t)$ satisfy the following condition: for every $t \in T$ there exists a single index $i_t \in \{1, \ldots, p\}$ such that $\gamma_{i_t}(t) = 1$ and $\gamma_i(t) = 0$ for each $i \in \{1, \ldots, p\} - \{i_t\}$. The inequality (41) can be expressed in the form

$$\left\| \int_{t_1}^{t} (\dot{x}(\tau) - \sum_{i=1}^{p} \gamma_i(\tau)\, f(\tau, \psi^i(\tau)))\, d\tau \right\| \leqq \left\| \int_{t_1}^{t} (\sum_{i=1}^{p} \gamma_i(\tau)\, f(\tau, \psi^i(\tau)) - \sum_{i=1}^{p} \gamma_i(\tau)\, g_i(\tau))\, d\tau \right\| +$$

$$+ \left\| \int_{t_1}^{t} (\sum_{i=1}^{p} \gamma_i(\tau)\, g_i(\tau) - \sum_{i=1}^{p} \bar{\beta}_i(\tau)\, g_i(\tau))\, d\tau \right\| + \left\| \int_{t_1}^{t} (\sum_{i=1}^{p} \bar{\beta}_i(\tau)\, g_i(\tau) - \dot{x}(\tau))\, d\tau \right\| < \varepsilon.$$

The first member on the right hand side of this inequality satisfies

$$\left\| \int_{t_1}^{t} (\sum_{i=1}^{p} \gamma_i(\tau)\, f(\tau, \psi^i(\tau)) - \sum_{i=1}^{p} \gamma_i(\tau)\, g_i(\tau))\, d\tau \right\| < \varepsilon_0 = \frac{\varepsilon}{3}$$

on T. To prove it, we proceed as in (32). The third member is smaller than ε_0 for every $t \in T$ (cf. (29)). This assertion has been already proved. Now it is sufficient to construct the functions $\gamma_i(t)$ on T such that

$$(42) \qquad \left\| \int_{t_1}^{t} (\sum_{i=1}^{p} (\gamma_i(\tau) - \bar{\beta}_i(\tau))\, g_i(\tau))\, d\tau \right\| < \varepsilon_0$$

holds on T. Then the inequality (41) will hold on T. There exist disjoint intervals I_z, $z = 1, \ldots, s$, $T = \bigcup_{z=1}^{s} I_z$ such that the step functions $g_i(t)$, $i = 1, \ldots, p$ are constant on each I_z, $z = 1, \ldots, s$.

The inequality (42) can be expressed in the form

$$\left\| \int_{t_1}^{t} \left(\sum_{i=1}^{p} (\gamma_i(\tau) - \bar\beta_i(\tau))\, g_i(\tau) \right) d\tau \right\| \leqq$$

$$\leqq \sum_{j=1}^{m} \sum_{z=1}^{s} \left\| \int (\sum_{\substack{i=1 \\ T_j \cap I_z \cap \langle t_1, t \rangle}}^{p} (\gamma_i(\tau) - \bar\beta_i(\tau))\, g_i(\tau)) \, d\tau \right\| < \varepsilon_0 \, .$$

We shall find the functions $\gamma_i(t)$, $i = 1, \ldots, p$ on each $T_j \cap I_z \cap T$, $j = 1, \ldots, m$; $z = 1, \ldots, s$ so that

$$\left\| \int (\sum_{\substack{i=1 \\ T_j \cap I_z \cap \langle t_1, t \rangle}}^{p} (\gamma_i(\tau) - \bar\beta_i(\tau))\, g_i(\tau)) \, d\tau \right\| < \frac{\varepsilon_0}{ms} \, . \tag{43}$$

Then the inequality (42) will hold.

Let us choose a certain set T_j from the sequence $\{T_1, \ldots, T_m\}$. Let $k \leqq p$ be the number of the indices i such that $\bar\beta_i(t) \neq 0$. Let us change the order of indices so that $\bar\beta_i(t) \neq 0$ for each $i = 1, \ldots, k$. For $k = 1$ we define $\gamma_1(t) = \bar\beta_1(t)$ on T_j. Let $k > 1$. The functions $\bar\beta_i(t)$ are constant on T_j. We can write these functions $\bar\beta_i(t)$ without the variable t. Then $\sum_{i=1}^{k} \bar\beta_i\, g_i(t)$ is an approximation of the function $\dot x(t)$ on the set $T_j \cap T'$. The function $\sum_{i=1}^{k} \bar\beta_i\, g_i(t)$ is defined on the interval T, where $\sum_{i=1}^{k} \bar\beta_i = 1$, $0 < \bar\beta_i < 1$, $\bar\beta_i$ are rational values and $g_i(t)$ are step functions on T.

Now, let us choose a certain I_z from the sequence $\{I_1, \ldots, I_s\}$ and $g_i(t) = g_i$ on I_z for each i. There exists a constant $K_0 > 0$ such that

$$\max \left\{ \left\| g_v(t) - \sum_{i=1}^{k} \beta_i\, g_i(t) \right\| : v = 1, \ldots, k \right\} \leqq K_0$$

holds for every $t \in T$. The last inequality implies

$$\left\| \int_{T_j \cap I_z \cap \langle t_0, t \rangle} (g_{v\tau}(\tau) - \sum_{i=1}^{k} \bar\beta_i\, g_i(\tau)) \, d\tau \right\| \leqq \tag{44}$$

$$\leqq \int_{T_j \cap I_z \cap \langle t_0, t \rangle} \left\| g_{v\tau}(\tau) - \sum_{i=1}^{k} \bar\beta_i\, g_i(\tau) \right\| d\tau \leqq K_0(t - t_0)$$

for $t_1 \leqq t_0 < t \leqq t_2$, where v_τ is an arbitrary simple measurable function on T and $v_\tau \in \{1, \ldots, k\}$. Further, we choose $\delta_2 > 0$ such that

$$K_0 \delta_2 < \frac{\varepsilon_0}{ms} \tag{45}$$

holds. The interval $T = \langle t_1, t_2 \rangle$ can be divided into a finite system of intervals

$$(46)\qquad \langle t_1, t_1 + \delta_2), \langle t_1 + \delta_2, t_1 + 2\delta_2), \ldots, \langle t_1 + (l - 1)\,\delta_2, t_1 + l\delta_2)\,,$$

where the last interval contains the point t_2. We divide each interval $\langle t_1 + (u - 1)\,\delta_2, t_1 + u\delta_2)$, $u = 1, \ldots, l$ from (45) into the following parts:

If $\mu\{T_j \cap I_z \cap \langle t_1 + (u - 1)\,\delta_2, t_1 + u\delta_2)\} = \mu_{j,z,u}$, then we divide the interval $\langle t_1 + (u - 1)\,\delta_2, t_1 + u\delta_2)$ into k parts

$$(47)\qquad \langle t_1 + (u - 1)\,\delta_2, t_1 + (u - 1)\,\delta_2 + \Delta_1^u), \ldots,$$
$$\langle t_1 + (u - 1)\,\delta_2 + \sum_{i=1}^{r-1} \Delta_i^u, t_1 + (u - 1)\,\delta_2 + \sum_{i=1}^{r} \Delta_i^u), \ldots,$$
$$\langle t_1 + (u - 1)\,\delta_2 + \sum_{i=1}^{k-1} \Delta_i^u, t_1 + u\delta_2)\,.$$

The values Δ_r^u, $r = 1, \ldots, k$; $u = 1, \ldots, l$ are defined by the equations

$$\mu\{T_j \cap I_z \cap \langle t_1 + (u - 1)\,\delta_2 + \sum_{i=1}^{r-1} \Delta_i^u, t_1 + (u - 1)\,\delta_2 + \sum_{i=1}^{r} \Delta_i^u)\} =$$
$$= \beta_r \mu_{j,z,u}\,.$$

We define functions $\hat{\gamma}_r(t)$ on $T_j \cap I_z$ by

$$\hat{\gamma}_r(t) = 0 \quad \text{for} \quad t \in (T_j \cap I_z) - \langle t_1 + (u - 1)\,\delta_2 +$$
$$+ \sum_{i=1}^{r-1} \Delta_i^u, t_1 + (u - 1)\,\delta_2 + \sum_{i=1}^{r} \Delta_i^u)\,,$$
$$\hat{\gamma}_r(t) = 1 \quad \text{for} \quad t \in T_j \cap I_z \cap \langle t_1 + (u - 1)\,\delta_2 +$$
$$+ \sum_{i=1}^{r-1} \Delta_i^u, t + (u - 1)\,\delta_2 + \sum_{i=1}^{r} \Delta_i^u)\,,$$

where $r = 1, \ldots, k$. We define $\hat{\gamma}_{k+1}(t) = \ldots = \hat{\gamma}_p(t) = 0$ on $T_j \cap I_z$. We shall prove that these functions $\hat{\gamma}_r(t)$ satisfy the inequality

$$(48)\qquad \left\| \int_{T_j \cap I_z \cap \langle t_1, t \rangle} \left(\sum_{i=1}^{k} \hat{\gamma}_i(\tau)\, g_i(\tau) - \sum_{i=1}^{k} \beta_i\, g_i(\tau) \right) \mathrm{d}\tau \right\| < \frac{\varepsilon_0}{ms}$$

for every $t \in T$. First of all we prove the identity

$$(49)\qquad \left\| \int_{T_j \cap I_z \cap \langle t_1, t \rangle} \left(\sum_{i=1}^{k} \hat{\gamma}_i(\tau)\, g_i(\tau) - \sum_{i=1}^{k} \beta_i\, g_i(\tau) \right) \mathrm{d}\tau \right\| = 0$$

at the points $t = t_1 + \delta_2, t_2 + 2\delta_2, \ldots, t_1 + (l-1)\delta_2$. It holds

$$\int_{T_j \cap I_z \cap \langle t_1, t_1 + u\delta_2 \rangle} \sum_{i=1}^{k} \hat{\gamma}_i(\tau)\, g_i(\tau)\, \mathrm{d}\tau = \sum_{v=1}^{u} \sum_{i=1}^{k} g_i\, \mu_{j,z,v} \beta_i\,,$$

$$\int_{T_j \cap I_z \cap \langle t_1, t_1 + u\delta_2 \rangle} \sum_{i=1}^{k} \beta_i\, g_i(\tau)\, \mathrm{d}\tau = \sum_{v=1}^{u} \sum_{i=1}^{k} g_i\, \mu_{j,z,v} \beta_i$$

for each $u = 1, \ldots, l - 1$.

This implies the validity of (49) for each $t = t_1 + u\delta_2$, $u = 1, \ldots, l - 1$. From (44), (45), (46) and (49) we obtain (48).

The functions $\hat{\gamma}_i(t)$, $i = 1, \ldots, p$ are defined analogously on $T_j \cap I_z \cap T$ for each $j = 1, \ldots, m$; $z = 1, \ldots, s$. Then the inequality (42) holds. Further, we get the inequality (41) for $\hat{\gamma}_i(t)$ defined on T for $i = 1, \ldots, p$. Then it holds

$$\left\| \int_{t_1}^{t} \left(\dot{x}(\tau) - \sum_{i=1}^{p} \hat{\gamma}_i(\tau)\, f(\tau, \psi^i(\tau)) \right) \mathrm{d}\tau \right\| < \varepsilon$$

for certain $t \in T$, where $\hat{\gamma}_i(t)$ are measurable simple functions with the following property:

For every $t \in T$ there exists an index $j \in \{1, \ldots, p\}$ such that $\hat{\gamma}_j(t) = 1$ and $\hat{\gamma}_i(t) = 0$ for $i \in \{1, \ldots, p\} - \{j\}$.

Now we define a function $\hat{\psi}(t)$ on T: $\hat{\psi}(t) = \psi^i(t)$, where i is the index for which $\hat{\gamma}_i(t) = 1$. We have constructed a function $\hat{\psi}(t)$ on T with the properties (24), (25), (26), (28). Finally it holds

$$\left\| x(t) - \left(x(t_1) + \int_{t_1}^{t} f(\tau, \hat{\psi}(\tau))\, \mathrm{d}\tau \right) \right\| = \left\| \int_{t_1}^{t} \left(\dot{x}(\tau) - f(\tau, \hat{\psi}(\tau)) \right) \mathrm{d}\tau \right\| =$$

$$= \left\| \int_{t_1}^{t} \left(\dot{x}(\tau) - \sum_{i=1}^{p} \hat{\gamma}_i(\tau)\, f(\tau, \psi^i(\tau)) \right) \mathrm{d}\tau \right\| < \varepsilon$$

for every $t \in T$. It means that the inequality (27) holds for the function $\hat{\psi}(t)$ on T. This completes the proof.

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