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MINIMAL SETS OF VECTORS WHICH GENERATE R_n WITH EXCESS k

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It is the purpose of this note to give a simple proof of the fact that, for fixed integers $n \geqq 1$ and $k \geqq 0$, the smallest set of vectors in an n-dimensional real vector space R_n which generates R_n as its convex hull and preserves this property even after removing any k of the vectors, has cardinality $n + 2k + 1$. An equivalent result is proved in [1].

1. Preliminaries. In the whole paper, R_n will denote an n-dimensional real vector space. The cardinality of a set S will be denoted by card S.

(1,1) Definition. We shall say that a finite set S of vectors in R_n *generates* R_n if any vector in R_n is a linear combination of vectors in S with nonnegative coefficients.

The following lemma is well known:

(1,2) Lemma. *Let $S = \{v_1, \ldots, v_N\}$ be a set of vectors in R_n. Then the following are equivalent*:

(i) *S generates R_n*;

(ii) *S contains a basis of R_n and, there exist positive numbers $\alpha_1, \ldots, \alpha_N$ such that*

$$\sum_{i=1}^{N} \alpha_i v_i = 0\,;$$

(iii) *any open halfspace of R_n contains at least one vector from S.*

(1,3) Definition. Let $n \geqq 1$, $k \geqq 0$ be integers. We shall say that a finite set S of vectors in R_n *generates R_n with excess k* if for any subset $S' \subset S$ with k elements, $S \setminus S'$ generates R_n.

The following assertion is an easy consequence of (1,2):

(1,4) A finite set S of vectors in R_n generates R_n with excess k iff every open halfspace of R_n contains at least $k + 1$ vectors from S.

(1,5) Lemma. *If* $y_1, \ldots, y_m$ $(m \geqq 2)$ *are mutually different numbers then*

$$\sum_{i=1}^{m} \frac{y_i^s}{\prod\limits_{\substack{j=1\\ j\neq i}}^{m} (y_i - y_j)} = 0 \quad \text{for} \quad s = 0, \ldots, m-2\,.$$

Proof. By the Lagrange interpolation formula [2], any polynomial $f(x)$ of degree at most $m-1$ satisfies the identity

$$f(x) \equiv \sum_{i=1}^{m} \frac{f(y_i)}{\prod\limits_{\substack{j=1\\ j\neq i}}^{m} (y_i - y_j)} \prod_{\substack{j=1\\ j\neq i}}^{m} (x - y_j)\,.$$

Choosing $f(x) = x^s$, $s \in \{0, \ldots, m-2\}$ and comparing the coefficients at x^{m-1} on both sides, we obtain the desired equalities.

2. Results. (2,1) Theorem. *Let* $n \geqq 1$, $k \geqq 0$ *be integers. Let* S *be a set of vectors in* R_n *which generates* R_n *with excess* k. *Then*

$$\operatorname{card} S \geqq n + 2k + 1$$

and this bound is sharp for all k *and* n.

Proof. Let S be a set of vectors in R_n which generates R_n with excess k and assume that

$$\operatorname{card} S \leqq n + 2k\,.$$

Then there exists a subset $S_0 \subseteqq S$ consisting of $n-1$ linearly independent vectors. Let R_0 be the hyperplane spanned by the vectors in S_0. Since $S \setminus S_0$ contains at most $2k+1$ vectors, at least one of the open halfspaces of R_n with boundary R_0 contains at most k vectors from S. By (1,4), S does not generate R_n with excess k, a contradiction.

It remains to find, for any $n \geqq 1$ and $k \geqq 0$, a set of $n + 2k + 1$ vectors which generates R_n with excess k. This will be done in the following theorem.

(2,2) Theorem. *Let* $n \geqq 1$, $k \geqq 0$ *be integers, let* $x_1 > x_2 > \ldots > x_{n+2k+1}$ *be real numbers. Then the* $2k+1$ *row vectors*

$$(-1)^{s-1}\left((x_s - x_{2k+2})^{-1},\ (x_s - x_{2k+3})^{-1}, \ldots, (x_s - x_{2k+n+1})^{-1}\right),$$
$$s = 1, \ldots, 2k+1\,,$$

together with the n *unit vectors* $(1, 0, \ldots, 0)$, $(0, 1, 0, \ldots, 0)$, $\ldots$, $(0, 0, \ldots, 1)$, *form a set which generates the space* R_n' *of all row vectors with excess* k.

Proof. Denote by $B = (b_{pi})$ the $(n + 2k + 1) \times n$ matrix with entries

$$b_{pi} = \Big((x_p - x_{2k+1+i}) \prod_{\substack{q=1 \\ q \neq p}}^{2k+1} (x_p - x_q)\Big)^{-1}$$

if

$$p = 1, \ldots, 2k + 1\,, \quad i = 1, \ldots, n\,,$$

$$b_{pi} = \Big(\prod_{q=1}^{2k+1} (x_p - x_q)\Big)^{-1} \delta_{i,p-2k-1}$$

if

$$p = 2k + 2, \ldots, n + 2k + 1\,, \quad i = 1, \ldots, n\,,$$

where δ_{ik} are the Kronecker symbols.

It is easily seen that the rows of the matrix B are positive multiples of the vectors defined above.

By Lemma (1,5), the product

$$VB = 0 \tag{1}$$

where $V = (v_{\alpha p})$ is the $(2k + 1) \times (n + 2k + 1)$ "Vandermonde matrix" with entries

$$v_{\alpha p} = x_p^{2k+1-\alpha}\,, \quad \begin{aligned} p \in M &= \{1, 2, \ldots, n + 2k + 1\}\,, \\ \alpha \in K &= \{1, \ldots, 2k + 1\}\,. \end{aligned}$$

On the other hand, there exists an $\binom{n + 2k + 1}{k} \times (2k + 1)$ matrix

$$Y = (y_{(j_1,\ldots,j_k)\alpha})$$

where $(j_1, \ldots, j_k)$ is a combination of k elements of the set of indices M and $\alpha \in K$, such that

$$YV = Z = (z_{(j_1,\ldots,j_k)p}) \tag{2}$$

with

$$z_{(j_1,\ldots,j_k)p} = \prod_{s=1}^{k} (x_p - x_{j_s})^2\,, \quad p \in M\,. \tag{3}$$

Indeed, the numbers $y_{(j_1,\ldots,j_k)\alpha}$ are the coefficients in the polynomial

$$\prod_{s=1}^{k} (x - x_{j_s})^2 = \sum_{\alpha=1}^{2k+1} x^{2k+1-\alpha} y_{(j_1,\ldots,j_k)\alpha}\,.$$

From (1) and (2), we have

$$ZB = 0\,.$$

Since any matrix $((p_i - q_j)^{-1})$, $i, j = 1, \ldots, s$, is nonsingular whenever $p_1, \ldots, p_s$, $q_1, \ldots, q_s$ are different from each other, any n rows of the matrix B are linearly independent. By (ii) of (1,2) and (3), for every subset $M' = \{j_1, \ldots, j_k\}$ of M having k elements the rows $b_{(p)}$ of the matrix B with $p \in M \setminus M'$ generate R'_n. Consequently, the rows of B, and hence also the vectors given in the theorem, generate R'_n with excess k. The proofs of both Theorems (2,2) and (2,1) are complete.

(2,3) Corollary. *Let $n \geqq 1$. Then there exists a sequence of systems $S_0, S_1, S_2, \ldots$ of vectors in R_n with the following properties:*

1° card $S_k = n + 2k + 1$;

2° $S_k \subset S_{k+1}$, $k = 0, 1, \ldots$;

3° *S_k generates R_n with excess k.*

We shall add four more theorems which are, by (1,4), equivalent to (2,1).

(2,4) Theorem. *Let $n \geqq 1$, $l \geqq 1$ be integers. Let N non-zero vectors in R_n have the property that every open halfspace of R_n contains at least l of these vectors. Then*

$$N \geqq n + 2l - 1$$

and this bound is sharp for all n and l.

(2,5) Theorem. *Let $n \geqq 2$, $k \geqq 1$ be integers. Let N non-zero vectors in R_n have the property that any non-zero vector of R_n forms an acute angle with at least k of the given vectors. Then $N \geqq n + 2k - 1$ and this bound is sharp for all n and k.*

Remark. This theorem can also be reformulated in terms of distance graphs introduced in [3].

(2,6) Theorem. *Let $p \geqq 1$, $q \geqq 1$ be integers. Let $\mathscr{H}$ be a finite system of N open halfspheres on a p-dimensional sphere S_p which covers S_p q-times, i.e. every point of S_p is contained in at least q halfspheres from $\mathscr{H}$. Then $N \geqq p + 2q$ and this bound is sharp for all p and q.*

(2,7) Theorem. *Let $m \geqq 1$, $l \geqq 1$ be integers. Let N points on an m-dimensional sphere S_m have the property that every open halfsphere of S_m contains at least l of the given N points. Then*

$$N \geqq m + 2l + 2$$

and this bound is sharp for all m and l.

Considering the Gram matrix of such systems of vectors, we obtain the following formulation:

(2,8) Theorem. *Let a positive semi-definite real n by n matrix A have the property that whenever one row aud column is added in such a way that the resulting matrix remains positive semidefinite and of the same rank as A then the new row contains at least k positive entries. Then the rank of A is at least $n - 2k + 3$ and this bound is sharp.*

References

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