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PRODUCTS OF UNIFORM SPACES

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Coreflective subcategories (we always mean full subcategories) of **Unif** are those classes of uniform spaces which are closed under sums and quotients. The dual notions to sums and quotients are products and subspaces, respectively. It is proved in [Hušek$_3$] that if a coreflective subcategory of **Unif** is productive and hereditary, then it must be the whole **Unif**; but it may happen that a proper coreflective subcategory of Unif is either productive or hereditary. Here we shall investigate the productivity of coreflective subcategories of **Unif**. Productivity of several special coreflective classes were investigated e.g. in [Isbell$_1$] (topologically fine spaces, locally fine spaces), in [Poljakov$_{1,2}$], [Isbell$_2$] (proximally fine spaces).

The first part of this paper deals with infinite products, namely with finding the least cardinal of indices determining whether a product belongs to a class. In the second part, finite products in coreflective classes are investigated.

By a space we always mean a uniform space. We shall use terms from [Čech] and [Isbell$_1$]. Two nets $\{M_a\}_A$, $\{N_a\}_A$ are said to be adjacent in a space X if the net $\{\langle M_a, N_a\rangle\}_A$ is eventual in any uniform neighborhood of the diagonal 1_X. By DX we mean the uniformly discrete modification of X, by pX the precompact modification of X. A space X is said to be $\mathscr{F}$-fine, where $\mathscr{F}$ is a concrete functor defined on **Unif** into $\mathscr{K}$, if **Unif** $(X, Y) = \mathscr{K}(\mathscr{F}X, \mathscr{F}Y)$ for all spaces Y. A metric space is usually denoted by $\langle M, d\rangle$, a cover composed of ε-balls by $\mathscr{S}_\varepsilon$. In a product $\prod_I X_i$, the symbol pr_i means the projection $\prod X_i \to X_i$; f defined in $\prod X_i$ depends on J if $fx = fy$ provided $\mathrm{pr}_i\, x = \mathrm{pr}_i\, y$ for each $i \in J$ (i.e., if $\mathrm{pr}_J\, x = \mathrm{pr}_J\, y$).

By $P_u(\lambda, \varkappa)$, λ and $\varkappa$ cardinals, we denote the following uniform space: the underlying set is $X \times (0, 1)$, where X is the set of all subsets $A \subset \varkappa$, $|A| < \lambda$; a base of uniform covers consists of $\mathfrak{A}_B$, $B \in X$,

$$\mathfrak{A}_B = \{(\langle A, 0\rangle) \cup (\langle A, 1\rangle) \mid A \in X\} \cup \{(\langle A, 0\rangle, \langle A, 1\rangle) \mid A \in X,\ A \supset B\}$$

(roughly speaking, $P_u(\lambda, \varkappa)$ is an adjacent pair of nets on the set $\{A \subset \varkappa \mid |A| < \lambda\}$ ordered by inclusion — we shall identify $P_u(\omega_0, \omega_0)$ with adjacent sequences).

By $P_t(\lambda, \varkappa)$ we denote the quotient of $P_u(\lambda, \varkappa)$ when $X \times (1)$ is contracted to a point ∞ (in other words, $P_t(\lambda, \varkappa)$ is a convergent net $\{A \subset \varkappa \mid |A| < \lambda\}$ together with its limit ∞ and with the topologically fine uniformity. We may identify $P_t(\varkappa, \varkappa)$ for regular $\varkappa$ with $\varkappa + 1$ (all points except $\varkappa$ are isolated).

Several results of this paper were published in [Hušek$_{4,5,6}$] (the paper [Hušek$_5$] contains a survey of adjacent nets and their applications) but mostly in weaker formulations; besides, some methods of the present paper are simpler than in [Hušek$_4$], where only factorizations of mappings were used (mainly those of proximal character — for a survey of topological factorizations see e.g. [Hušek$_2$]).

Theorem 1. *The following conditions are equivalent for a coreflective subcategory $\mathscr{C}$ of* **Unif**:

(a) *$\mathscr{C}$ contains all products (or powers) of uniformly discrete spaces.*

(b) *A product belongs to $\mathscr{C}$ if all finite subproducts belong to $\mathscr{C}$.*

Proof. The only nontrivial implication: if (a) is fulfilled and all finite subproducts of $\prod_I X_i$ belong to $\mathscr{C}$, then $\prod_I X_i \in \mathscr{C}$. Suppose first that I is countable, $I = \omega_0$. We know that $\mathscr{C}$ contains 2^{ω_0}, hence it contains $\omega_0 \times (\omega_0 + 1)$ (a quotient of $\omega_0 \times 2^{\omega_0}$) and hence also its retract $P_u(\omega_0, \omega_0) = \{\langle x, y\rangle \mid x = y \text{ or } y = \omega_0\}$. Let $f : \prod_{\omega_0} X_n \to \langle M, d\rangle$ be such that $f : c\prod_{\omega_0} X_n \to M$ is uniformly continuous (c is the coreflection onto $\mathscr{C}$). If f is not uniformly continuous on $\prod_{\omega_0} X_n$, then for an $\varepsilon > 0$ and any uniform cover $\mathscr{U}$ of $\prod_{\omega_0} X_n$ there are $x_{\mathscr{U}}, y_{\mathscr{U}}$ lying in a member of $\mathscr{U}$, with $d\langle fx_{\mathscr{U}}, fy_{\mathscr{U}}\rangle \geqq \varepsilon$. There is a $k \in \omega_0$ such that $d\langle fx, fy\rangle < \varepsilon/3$ provided $\mathrm{pr}_n x = \mathrm{pr}_n y$ for all $n \leqq k$ (if not, then for any $k \in \omega_0$ we could find a_k, b_k with $\mathrm{pr}_n a_k = \mathrm{pr}_n b_k$ for all $n \leqq k$ and $d\langle fa_k, fb_k\rangle \geqq \varepsilon/3$; but $\{a_k\}, \{b_k\}$ are adjacent in the product $\prod_{\omega_0} DX_n$ of discrete spaces). Pick out $a_n \in X_n$ for $n > k$ and put $Z = \prod_{n \leqq k} X_n \times (\{a_n \mid n > k\})$, $g = f/Z$. Since g is uniformly continuous (Z is a retract of $c\prod_{\omega_0} X_n$), there is a uniform cover $\mathscr{V}$ of Z refining $f^{-1}[\mathscr{S}_{\varepsilon/3}]$. Let $\mathscr{U}$ be the preimage of $\mathscr{V}$ along the projection $\mathrm{pr}_Z : \prod_{\omega_0} X_n \to Z$ and put $x'_{\mathscr{U}} = \mathrm{pr}_Z x_{\mathscr{U}}$, $y'_{\mathscr{U}} = \mathrm{pr}_Z y_{\mathscr{U}}$. Then $d\langle fx_{\mathscr{U}}, fy_{\mathscr{U}}\rangle \leqq d\langle fx_{\mathscr{U}}, fx'_{\mathscr{U}}\rangle + d\langle fx'_{\mathscr{U}}, fy'_{\mathscr{U}}\rangle + d\langle fy'_{\mathscr{U}}, fy_{\mathscr{U}}\rangle < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$ because $\mathrm{pr}_n x_{\mathscr{U}} = \mathrm{pr}_n x'_{\mathscr{U}}$, $\mathrm{pr}_n y_{\mathscr{U}} = \mathrm{pr}_n y'_{\mathscr{U}}$ for all $n \leqq k$ and $x'_{\mathscr{U}}, y'_{\mathscr{U}}$ lie in a member of $\mathscr{V}$. This contradiction implies that $\prod_{\omega_0} X_n = c\prod_{\omega_0} X_n$. Now we know that for any set I, all countable subproducts of $\prod_I X_i$ belong to $\mathscr{C}$. If $f : c\prod_I X_i \to M$ is uniformly continuous, then $f : \prod_I DX_i \to M$ is uniformly continuous (since $\prod_I DX_i$ is finer than $c\prod_I X_i$). Hence f depends on countably many coordinates J and thus it is uniformly continuous on $\prod_I X_i$ because $\prod_J X_i$ is a retract of $c\prod_I X_i$. Again we have $\prod_I X_i = c\prod_I X_i$.

Corollary. *A coreflective subcategory of* **Unif** *is productive iff it is finitely productive and contains all products of uniformly discrete spaces.*

Using the fact that finite products of uniform quotients are uniform quotients (this result was communicated to me by M. D. Rice), one can esaily see that the coreflective hull of all products of uniformly discrete spaces is the least productive coreflective subcategory of **Unif**. The details and a comparison of various productive coreflective subcategories of Unif will appear in a joint paper with M. D. Rice. The first nontrivial productive coreflective subcategory in **Unif** was constructed in [Rice] (a subclass of proximally fine spaces).

In the first part of the proof of Theorem 1 we used only the fact that the space $P_u(\omega_0, \omega_0)$ belongs to $\mathscr{C}$, so that the following two conditions are equivalent for a coreflective $\mathscr{C}$ in **Unif**:

(a) *A countable product belongs to* $\mathscr{C}$ *if all finite subproducts belong to* $\mathscr{C}$.

(b) $P_u(\omega_0, \omega_0) \in \mathscr{C}$.

Clearly, in (b) we may replace $P_u(\omega, \omega)$ by any one of the spaces $\omega_0 \times (\omega_0 + 1)$, $\omega_0 \times 2^{\omega_0}$, $\omega_0^{\omega_0}$. Also, it is easy to see that condition (b) is equivalent to the condition that $\mathscr{C}$ contains all metric spaces.

The condition (a) in Theorem 1, as formulated, may be difficult to decide. We shall try to find a better condition. There are two "disjoint" approaches – one via countable sequences, the other via countably directed nets.

Define transfinitely cardinals $s_\alpha : s_\alpha$ is the first cardinal such that there is a non-continuous real-valued f on 2^{s_α} which is continuous on all images of $P_t(\omega_0, s_\beta)$ in 2^{s_α}, for all $\beta < \alpha$. Clearly, $s_0 = \omega_0$, s_1 is Mazur's sequential cardinal ([Mazur], [Noble]). The cardinal s_1 is not smaller than the first uncountable weakly inaccessible cardinal [Mazur]; in fact, there are many weakly inaccessible cardinals before s_1 and, under MA, s_1 is the first (two-valued) measurable cardinal [Čudnovskii].

Theorem 2. *Any one of the following conditions implies that a product belongs to a coreflective* $\mathscr{C}$ *in* **Unif** *whenever all countable subproducts belong to* $\mathscr{C}$:

(i) $\mathscr{C}$ *contains all spaces* $P_u(\omega_1, \varkappa)$;

(ii) $\mathscr{C}$ *contains all spaces* $P_t(\omega_0, s_\alpha)$.

Proof. Suppose that all countable subproducts of $\prod_I X_i$ belong to $\mathscr{C}$ and let f be uniformly continuous on the coreflection $c \prod_I X_i$ in $\mathscr{C}$ into a metric space $\langle M, d\rangle$. To prove that f is uniformly continuous on $\prod_I X_i$, it suffices to show that it factorizes via a countable subproduct.

If f does not depend on countably many coordinates, then for any countable $J \subset I$ there are x^J, y^J in $\prod_I X_i$ with $\mathrm{pr}_J\, x = \mathrm{pr}_J\, y$, $d\langle fx^J, fy^J\rangle \geqq \varepsilon$ for a fixed $\varepsilon > 0$.

If we assume (i), then we obtain a contradiction because $\{x^J\}$, $\{y^J\}$ are adjacent in $\prod_I X_i$ of the type $P_u(\omega_1, |I|)$.

Now pick out a point $a \in \prod_I X_i$ and put

$$X = \{x \in \prod_I X_i \mid |\{i \mid \mathrm{pr}_i x \neq \mathrm{pr}_i a\}| \leqq \omega_0\} .$$

The sets $J_\varepsilon = \{i \in I \mid \text{there are } x, y \text{ in } X,\ \mathrm{pr}_{I-(i)} x = \mathrm{pr}_{I-(i)} y,\ d\langle fx, fy\rangle \geqq \varepsilon\}$ are finite (otherwise we obtain two adjacent sequences in a "countable" subproduct $\prod_J X_i \times (\mathrm{pr}_{I-J} a)$ with non-adjacent f-images). Thus the restriction f_X depends on a countable set $J = \bigcup_n J_{1/n}$. It remains to prove that under (ii), f depends on J. i.e., if x is a fixed point in $\prod_I X_i$, $y \in X$, $\mathrm{pr}_J\, y = x$, $\mathrm{pr}_{I-J}\, y = a$, then $fx = fy$. If we define g on $2^{|I|}$ by $gz = d\langle f(zx), f \circ s_J \circ \mathrm{pr}_J\,(zx)\rangle$, where s_J is the canonical embedding of $\prod_J X_i$ onto $\prod_J X_i \times (\mathrm{pr}_{I-J}\, a)$ and $\mathrm{pr}_i\,(zx)$ is $\mathrm{pr}_i\, x$ if $\mathrm{pr}_i\, z = 1$ and is $\mathrm{pr}_i\, a$ otherwise. If $\{z_t\} \to z$ in $2^{|I|}$, then $\{z_t x\} \to zx$ in $\prod_I X_i$, so that g is continuous if we suppose (ii). Consequently, $g\ \{1\} = 0$ which means that f depends on J.

The proof of sufficiency of (ii) is a uniform analogue of Mazur's proof for factorizing sequentially continuous mappings on products of separable metrizable spaces.

The condition (i) is satisfied e.g. if $\mathscr{C}$ contains all spaces admitting ω_0, and (ii) e.g. if $\mathscr{C}$ contains all topologically fine spaces. The result of Theorem 2 holds also if $\mathscr{C}$ contains all Baire-fine spaces ([Tashjian], [Hager]); in this case $\mathscr{C}$ need not contain all $P_u(\omega_1, \varkappa)$ but to prove the assertion, one needs only (see our proof) that a Baire map f on $\prod_I X_i$ preserves adjacent nets of types $P_u(\omega_1, \varkappa)$, which is true. Consequently, our selection of $\mathscr{C}$'s in Theorem 2 is not the most general one – of course, also $\mathscr{C}$ consisting of all uniformly discrete spaces has the property that a product belongs to $\mathscr{C}$ if all countable subproducts belong to $\mathscr{C}$. It will follow from Theorem 4 that there is no $\mathscr{A}$ such that an analogue of Theorem 1 holds for countable subproducts instead of finite ones in (b) and $\mathscr{A}$ instead of powers of uniformly discrete spaces in (a).

Theorem 2 may be given another form by means of inductive generation, e.g., for (ii): *Any product of uniform spaces is inductively generated by canonical embeddings of countable subproducts and by spaces $P_t(\omega_0, s_\alpha)$, or $\prod_I X_i$ is inductively generated by the topological modification of $\prod_I DX_i$ and by a (Corson) Σ-product of X_i's.*

Combining Theorems 1 and 2 we obtain

Theorem 3. *The following conditions are equivalent for a coreflective subcategory $\mathscr{C}$ of* **Unif** *containing all spaces $P_u(\omega_1, \varkappa)$ or all spaces $P_t(\omega_0, s_\alpha)$:*

(a) *$\mathscr{C}$ contains all countable powers of uniformly discrete spaces;*

(b) *$\mathscr{C}$ contains all metrizable spaces;*

(c) *a product belongs to $\mathscr{C}$ if all finite subproducts belong to $\mathscr{C}$.*

The assumption on $\mathscr{C}$ is superfluous if s_1 does not exist.

Corollary. *A product is proximally fine iff all finite subproducts are proximally fine.*

In Corollary, the proximally fine spaces may be replaced by any coreflective subcategory of **Unif** containing metrizable spaces and topologically fine spaces, e.g., by $\mathscr{F}$-fine spaces for functors $\mathscr{F}$ preserving proximities.

The last corollary was proved in [Hušek$_4$] by means of factorization theorems; one of them is also a consequence of the proof of Theorem 2 but the direct proof is simpler:

Any proximally continuous f on $\prod_I X_i$ into a metric space depends on countably many coordinates (see also [Tashjian]). As is shown in [Hušek$_4$], we cannot assume f to be defined only on a subspace of $\prod_I X_i$ as in the case of uniformly continuous mappings, [Vidossich] – in such a case *f depends on less than* card I *coordinates provided* cof card $I \neq \omega_0$, but not generally on countably many coordinates.

We shall show in Theorem 7 that if $\mathscr{C}$ is finitely productive coreflective and contains all topologically fine spaces (i.e., all topologically fine spaces with unique accumulation points), then $\mathscr{C}$ = **Unif**. Of course, there are productive coreflective $\mathscr{C} \neq$ **Unif** containing all $P_t(\omega_0, s_\alpha)$.

It follows from the proof of Theorem 2 that we may replace countable products in the condition (a) in the remarks following Theorem 1 by products $\prod_I X_i$ with card $I < s_1$. Thus, in the case that s_1 does not exist, a coreflective subcategory is productive iff it is countably productive iff it is finitely productive and contains a converging sequence; in this case the coreflective hull of metrizable spaces is productive and thus coincides with the coreflective hull of all powers of uniformly discrete spaces (see [Hušek, Rice]).

The following example shows that we cannot omit (i), (ii) from Theorem 2.

Example 1. Let S be a strongly rigid Hausdorff compact space of infinite cardinality at least $\varkappa$ (for the existence see e.g. [Trnková]). Put

$$A = \{\{x_\xi\} \in S^\varkappa \mid \operatorname{card} \{x_\xi\} < \varkappa\}, \quad B = S^\varkappa - A .$$

Then $A \neq \emptyset$, $B \neq \emptyset$ and if f is a continuous mapping on S^λ into $S^\varkappa$, $\lambda < \varkappa$, such that the image meets A, then it is contained in A (this follows from the fact [Herrlich] that f must be a product of constant maps or projections). Thus the noncontinuous mapping $f : S^\varkappa \to 2$ equal to 0 on A and to 1 on B has the property that all compositions $f \circ g$, $g : S^\lambda \to S^\varkappa$ being continuous, $\lambda < \varkappa$, are continuous.

Thus a product need not be inductively generated by all uniformly continuous mappings on countable subproducts into the product. Moreover, we have proved

Theorem 4. *For any infinite cardinal $\varkappa$ there are coreflective subcategories $\mathscr{C}$ in* **Unif** *and spaces $X_\xi \in \mathscr{C}$, $\xi < \varkappa$, such that $\prod_{\xi<\varkappa} X_\xi \notin \mathscr{C}$ but $\prod_{\xi<\eta} X_\xi \in \mathscr{C}$ for any $\eta < \varkappa$.*

There is a simpler example showing that a product need not be inductively generated by canonical embeddings of countable subproducts:

Example 2. Put $f : 2^{\omega_1} \to 2, f\{x_\xi\} = 0$ if all but countably many x_ξ are $0, f\{x_\xi\} = 1$ otherwise. Then f is not continuous on 2^{ω_1} but it is continuous on all countable canonical embeddings of 2^{ω_0} into 2^{ω_1}.

The next part is devoted to finding special cases when a finite product belongs to a coreflective subcategory $\mathscr{C}$. Theorem 5 is basic for a general consideration.

Theorem 5. *Let $f : X \times Y \to Z$ be proximally continuous and separately uniformly continuous, Y being a precompact space. Then f is uniformly continuous.*

Proof. We may assume that Z is complete metrizable and Y is Hausdorff. The equalities and inclusion in the next formula are taken in Set and $\mathbf{U}(X, pZ)$, $\mathbf{U}(X, Z)$ have uniformities of uniform convergence (see [Isbell$_1$]):

$$\mathbf{U}(X \times Y, pZ) = \mathbf{U}(Y, \mathbf{U}(X, pZ)) \supset \mathbf{U}(Y, \mathbf{U}(X, Z)) = \mathbf{U}(X \times Y, Z)\,.$$

We must prove that under the conditions imposed on f, the corresponding map $f^* : Y \to \mathbf{U}(X, Z)$ is uniformly continuous. To prove that, it suffices to show that the image $f^*[Y]$ is precompact in $\mathbf{U}(X, Z)$ (then the uniformities of $\mathbf{U}(X, pZ)$, $\mathbf{U}(X, Z)$ coincide on $f^*[Y]$. We may suppose now that X is uniformly discrete and Y is compact (f can be uniformly continuously extended on the completion of $DX \times Y$ into the Samuel compactification of Z, but the image of the completion lies in Z). Suppose $f^*[Y]$ is not precompact in $\mathbf{U}(X, Z)$; there exist countable sets $\{y_n\} \subset Y$, $\{x_{mn}\} = X' \subset X$ and $\varepsilon > 0$ such that for any m, n we have $d\langle f\langle x_{mn}, y_n\rangle, f\langle x_{mn}, y_m\rangle\rangle \geqq \varepsilon$. Embed Y into a power of the closed interval $[0, 1]$. For any m, n the restriction $f/(x_{mn}) \times Y$ factorizes via a countable subpower [Vidossich] and hence there is a projection pr of Y into a countable subpower and a mapping $g : X' \times \operatorname{pr}[Y] \to Z$ such that $g\langle x, \operatorname{pr} y\rangle = f\langle x, y\rangle$ for all $x \in X'$, $y \in Y$. The mapping g is proximally continuous on metrizable $X' \times \operatorname{pr}[Y]$ (because $1_{X'} \times \operatorname{pr}$ is proximal quotient), thus uniformly continuous. Consequently, $f : X' \times Y \to Z$ is uniformly continuous, which is a contradiction with the properties of $\{y_n\}$, $\{x_{mn}\}$.

Theorem 5 was proved in [Hušek$_4$] by another method: by induction both on the density of X and the uniform character of Y, the mapping f is step by step factorized, finally via a metrizable product $X' \times Y'$; in that approach, the conditions on f in Theorem 5 may be weakened, but for our purposes such a generalization is irrelevant.

Theorem 6. *A product $\prod_I X_i$ of proximally fine spaces is proximally fine provided all but at most one of spaces X_i are precompact.*

Proof. By Corollary to Theorem 3 we may suppose that I is finite. Thus it suffices to show that $X \times Y$ is proximally fine for X, Y proximally fine and Y precompact; this assertion follows directly from Theorem 5.

Theorem 6 for finite products and all X_i precompact was stated in [Poljakov$_1$], proved for infinite I and all X_i precompact in [Isbell$_2$] and for two factors, one proximally fine, the other countably compact, in [Kůrková] (by refining Poljakov's method of investigating $N \times \beta N$), and in the form stated here in [Hušek$_4$].

By the same proof as that of Theorem 6 we can prove that if $\mathscr{C}$ is a coreflective subcategory of **Unif** generated by a class $\mathscr{L}$ of proximally continuous mappings in the sense that $X \in \mathscr{C}$ provided the maps from $\mathscr{L}$ defined on X are uniformly continuous, then $\prod X_i \in \mathscr{C}$ if all $X_i \in \mathscr{C}$ and all but at most one of X_i's are precompact. Such $\mathscr{C}$ are just all $\mathscr{F}$-fine spaces where $\mathscr{F}$ is a functor **Unif** $\rightarrow \mathscr{K}$ preserving proximity (i.e., the canonical functor **Unif** $\rightarrow$ **Prox** factorizes via $\mathscr{F}$), e.g. when $\mathscr{F}$ is any upper modification in **Unif** preserving topology. Clearly, such $\mathscr{C}$ contain all proximally fine spaces; the next example shows that not all coreflective $\mathscr{C}$ containing all proximally fine spaces fulfil the above analogue of Theorem 6.

Example 3. Let X be a precompact infinite space the proximally fine modification of which is uniformly discrete (e.g. the finest precompact uniformity on an infinite set) and let $\mathscr{C}$ be the coreflective hull of all proximally fine spaces and of X. Then $X \times \omega \notin \mathscr{C}$ because any uniformly continuous f on X into $X \times \omega$ is uniformly continuous into $\sum_\omega X$, and the proximally fine modification of $X \times \omega$ is $DX \times \omega$.

One can prove that if $\mathscr{C}$ is coreflective in Unif and contains all proximally fine spaces, X is proximally fine precompact and $Y \in \mathscr{C}$, then $X \times Y \in \mathscr{C}$ because $X \times Y$ is the least upper bound of $\sum_X Y$, $X \times DY$.

N. Noble in his review for Math. Reviews (MR 44 # 3280) pointed out that there is a gap in the proof of Theorem 2 in [Hušek$_1$], but no counterexample to Theorem 2 was known; the next example provides one. First, let us recall the assertion of Theorem 2 in [Hušek$_1$]: A space P is locally pseudocompact iff for any quotient mapping g onto a k'-space the product $1_P \times g$ is quotient. (Quotients are taken in the category of completely regular spaces.)

Example 4. It is known that Theorem 5 is not true for continuous f instead of proximally continuous, moreover, there are pseudocompact spaces X, Y with a non-pseudocompact product $X \times Y$. Take such X, Y separable Hausdorff with topologically fine uniformities, and let $f : X \times Y \to M$, M metric, be a continuous not uniformly continuous mapping. The mapping f factorizes via $1_X \times \mathrm{pr} : X \times Y \to X \times Y'$, Y' compact Hausdorff (Lemma 3 in [Hušek$_4$]). If $1_X \times \mathrm{pr}$ is a topological quotient, then the factorized map $f' : X \times Y' \to M$ is continuous, hence uniformly continuous, because $X \times Y'$ is topologically fine — a contradiction. Clearly, $1_X \times \mathrm{pr}$ is both a proximal and a uniform quotient map.

Theorem 7. *Let $\mathscr{C}$ be a coreflective subcategory of* **Unif** *containing all topologically fine spaces. If $X \times D \in \mathscr{C}$ for a uniformly discrete space D, then $X \times Y \in \mathscr{C}$ for any $Y \in \mathscr{C}$ admitting dX and with* card $Y \leqq$ card D.

Proof. The case of finite dX is trivial, so suppose that $dX \geqq \omega_0$. Let $c(X \times Y)$ be the coreflection of $X \times Y$, where X, Y have the properties from Theorem 7. We must prove that if $f : X \times Y \to M$, M metric, and f is uniformly continuous on $c(X \times Y)$, then f is uniformly continuous on $X \times Y$. For any s from a dense set S in X of cardinality dX, the uniformly continuous $f|(s) \times Y$ factorizes via a projection

$$(s) \times Y \xrightarrow{1_X \times \mathrm{pr}_s} (s) \times D_s$$

for a uniformly discrete D_s (since Y admits ω_0) and thus f factorizes via a projection

$$X \times Y \xrightarrow{1_X \times \mathrm{pr}} X \times \tilde{D}$$

(since Y admits dX and f is continuous), where $\tilde{D}$ is a uniformly discrete space, card $\tilde{D} \leqq$ card Y. It follows that $X \times \tilde{D} \in \mathscr{C}$ and that $1_X \times \mathrm{pr}$ is a retraction, hence a quotient map $c(X \times Y) \to X \times \tilde{D}$. Consequently, f is uniformly continuous on $X \times Y$.

By a similar procedure we can prove:

Any product of spaces having linearly ordered bases is proximally fine (Theorem 3 in [Hušek$_4$]). By Theorem 5, it suffices to prove the assertion for finite products $\prod_0^k X_n$, where we may suppose that the uniform character of X_n is less than that of X_{n+1}, $n = 0, \ldots, k - 1$. By virtue of the second factorization theorem following Corollary to Theorem 3, any proximally continuous mapping on $\prod_0^k X_n$ into a metric space factorizes proximally continuously via a product of X_0 and of a uniformly discrete space. If X_0 is metrizable, the proof is complete, if not, we can use the factorization theorem once more. As a consequence, we obtain:

If a coreflective $\mathscr{C}$ in **Unif** *contains all proximally fine spaces and $X \times D \in \mathscr{C}$ for* *a uniformly discrete space D, then* $X \times \prod X_i \in \mathscr{C}$ *provided* X_i *have linearly ordered bases and* $\operatorname{card} \prod X_i \leqq \operatorname{card} D$.

Indeed, $X \times \prod X_i$ is inductively generated by $X \times D$ and $D' \times \prod X_i$. This last result is not true e.g. for topologically fine spaces (a product of a uniformly discrete space and of a compact metrizable one need not be topologically fine.

The last part of the paper deals with cases which are opposite to those from the foregoing part: with finding spaces from $\mathscr{C}$ the product of which does not belong to $\mathscr{C}$. The following result gives a rather general answer (a weaker form appeared as Example 2 in [Hušek$_4$]).

Theorem 8. *Any uniform space X is a quotient of a product* $D \times P$ *where D is a uniformly discrete space and P is topologically fine with at most one accumulation point. Moreover,* $\operatorname{card} D = \operatorname{card} P = \operatorname{card} X$ *provided X is infinite.*

Proof. Put $Y = (X \times X - 1_X) \times (0, 1)$ with the base of uniform covers consisting of

$$\{(\langle x, y, i\rangle) \mid i \in (0, 1), \langle x, y\rangle \in X \times X - U\} \cup$$

$$\cup \{(\langle x, y, 0\rangle, \langle x, y, 1\rangle) \mid \langle x, y\rangle \in U - 1_X\},$$

U being a symmetric uniform neighborhood of the diagonal 1_X. Then the mapping $f : Y \to X$,

$$f\langle x, y, 0\rangle = x, \quad f\langle x, y, 1\rangle = y,$$

is quotient [Isbell$_1$]. Let D be the uniformly discrete space on the set $X \times X - 1_X$ and let P be the set $(X \times X - 1_X) \cup (\infty)$, $\infty \notin X \times X$, endowed with the topologically fine uniformity of the topology having at most one accumulation point ∞ with the base of neighborhoods $\{(U - 1_X) \cup (\infty) \mid U$ a symmetric uniform neighborhood of $1_X\}$. It is easy to see that the mapping $g : D \times P \to Y$,

$$g\langle z_1, z_2\rangle = \langle z_1, 1\rangle \quad \text{if} \quad z_1 \neq z_2, \quad g\langle z_1, z_2\rangle = \langle z_1, 0\rangle \quad \text{if} \quad z_1 = z_2$$

is a retraction. Consequently, the map $h = f \circ g : D \times P \to X$ is quotient.

Corollary. *Let C be a coreflective subcategory of* **Unif** *containing all topologically fine spaces. If* $\mathscr{C} \neq$ **Unif**, *then* $\mathscr{C}$ *is not finitely productive.*

Theorem 8 and its Corollary are true if we replace "topologically fine" by "proximally discrete" [Hušek, Rice].

We see that for any infinite uniformly discrete space D there is a topologically fine space P such that $D \times P$ is not proximally fine. We shall show now that the space D can be replaced by any (proximally fine) non-precompact space.

We shall denote by Cone the following functor **Unif** $\to$ **Unif**:

Cone X is the quotient of $X \times [0,1]$ along $X \times (0)$, for $f : X \to Y$, Cone $f =$ $= \{\langle x, y\rangle \to \langle fx, y\rangle\}$: Cone $X \to$ Cone Y.
One can easily prove that Cone X has the sets

$$\{U \times V_k \mid U \in \mathscr{U},\ k \neq 0\} \cup (X \times (V_0 - (0)) \cup (0)),$$

for a base, where $\mathscr{U}$ is a uniform cover of X, $\{V_k\}_0^n$ is a uniform cover of $[0, 1]$ such that $0 \in V_0 - \bigcup_1^n V_k$.

Formally, we shall use also Cone X, Cone f for a set X and a set-mapping f in the obvious meaning.

The components $i_X = \{x \to \langle x, 1\rangle\}$ of the natural transformation $i : 1_{\mathbf{Unif}} \to$ Cone are embeddings.

For any space P there is a natural transformation $l : P \times$ Cone $X \to$ Cone $(P \times X)$ with the components $l_P = \{\langle\langle x, i\rangle, y\rangle \to \langle\langle x, y\rangle, i\rangle\}$.

If D is a uniformly discrete subspace of X, then by k_D we shall denote a standard mapping on X into the hedgehog Cone D: first pick out a uniformly continuous pseudometric d on X with $d\langle x, y\rangle \geqq 2$ for all different x, y from D, and put

$$k_D x = 0 \quad \text{if} \quad d\langle x, y\rangle \geqq 1 \quad \text{for all} \quad y \in D\,,$$

$$k_D x = \langle y, 1 - d\langle x, y\rangle\rangle \quad \text{if} \quad d\langle x, y\rangle < 1 \quad \text{for a} \quad y \in D\,.$$

Theorem 9. *Let D be a uniformly discrete subspace of a space X. If $D \times P$ is not proximally fine, then $X \times P$ is not proximally fine.*

Proof. There is a proximally continuous $f : D \times P \to M$ which is not uniformly continuous. We shall prove that $f' =$ Cone $f \circ l_P \circ (k_D \times 1_P)$ on $X \times P$ into Cone M is proximally continuous (certainly it is not uniformly continuous because its restriction to $D \times P$ is f). It suffices to prove that Cone f is proximally continuous and this fact follows from the easy equality Cone $(pM) = p($Cone $M)$.

Corollary ([Hušek$_6$]). *A proximally fine space X is precompact iff $X \times Y$ is proximally fine for any proximally fine Y.*

In general, there is no proximally continuous extension f' of a proximally continuous $f : A \to C$ on a space B containing A into a space D containing C (as subspaces): let A be a non-proximally fine subspace of a proximally fine space B, and let f be proximally continuous, not uniformly continuous on A into a space C.

One can easily extend Theorem 9 to other coreflective subcategories. The first extension has the same proof:

Let $\mathscr{F}$ be an upper modification in **Unif** *and D a uniformly discrete subspace of X. If $D \times P$ is not $\mathscr{F}$-fine, then $X \times P$ is not $\mathscr{F}$-fine.*

(The case when $\mathscr{F}$ does not preserve topology is trivial because only uniformly discrete spaces are $\mathscr{F}$-fine; in the case when $\mathscr{F}$ preserves topology (i.e., when $\mathscr{F}$ is finer than p, or $\mathscr{F}[0, 1] = [0, 1]$) one uses e.g. Theorem 5 to prove that Cone $(\mathscr{F}M)$ is finer than $\mathscr{F}(\text{Cone } M)$, which is sufficient to our purposes). It is easy to find bireflective subcategories of **Unif** with Cone $\mathscr{F}M \neq \mathscr{F}$ Cone M (e.g., in that generated by the real line, $\mathscr{F}$ Cone $\omega_0 = p$ Cone $\omega_0 \neq$ Cone $\omega_0 =$ Cone $\mathscr{F}\omega_0$).

The second partial extensions of Theorem 9 use the following easy observation (we denote here $m\mathscr{C} = \min\{\text{card } X \mid X \in \mathbf{Unif} - \mathscr{C}\}$): Let $\mathscr{C} \supset \mathscr{C}'$ be coreflective subcategories of Unif containing topologically fine spaces and with $m\mathscr{C} = m\mathscr{C}'$; if $\mathscr{C}$ has the property that for any $X \in \mathscr{C}$ with uniform covering character bigger than $m\mathscr{C}$ there is a topologically fine P such that $X \times P \notin \mathscr{C}$, then $\mathscr{C}'$ has the same property. If we take e.g. $\mathscr{C}$ to be all proximally fine spaces, then for any non-precompact $X \in \mathscr{C}'$ there is a topologically fine P such that $X \times P \notin \mathscr{C}'$ (e.g. if $\mathscr{C}'$ are all coz-fine spaces or topologically fine spaces).

The next examples show that Theorem 9 is not valid in general.

Example 5. Let $\mathscr{C}$ be a coreflective hull of $[0, 1] \times$ Cone ω_0. Then $[0, 1] \times \omega_0 \notin \mathscr{C}$ (since $[0, 1] \times$ Cone ω_0 is connected, the coreflection of $[0, 1] \times \omega_0$ in $\mathscr{C}$ is $\sum_{\omega_0}[0, 1]$) and, clearly, $[0, 1] \times$ Cone $\omega_0 \in \mathscr{C}$.

Example 6. Let P be a 0-dimensional space such that $\omega_0 \times P \notin \mathscr{C}$, where $\mathscr{C}$ is coreflective in **Unif**. Then $\omega_0 \times P$ does not belong to the coreflective hull $\overline{\mathscr{C}}$ of all $X \times$ Cone α with $X \in \mathscr{C}$, α cardinal (indeed, any uniformly continuous $f : X \times \times \text{Cone } \alpha \to \omega_0 \times P$ factorizes via the projection $X \times \text{Cone } \alpha \to X$) and, moreover, the coreflections of $\omega_0 \times P$ in both $\mathscr{C}$, $\overline{\mathscr{C}}$ coincide.

If we define transfinitely coreflective subcategories $\mathscr{C}_\alpha = \overline{\bigcup_{\beta<\alpha} \mathscr{C}_\beta}$, where $\mathscr{C}_0$ is the category of all topologically fine spaces, then $\bigcup_\alpha \mathscr{C}_\alpha = \mathscr{C}_\infty$ is coreflective in **Unif** with the properties:

$\mathscr{C}_\infty$ contains all topologically fine spaces,
there is a countable 0-dimensional P such that $\omega_0 \times P \notin \mathscr{C}_\infty$, $m\mathscr{C}_\infty = \omega_0$,
$X \times$ Cone $\omega_0 \in \mathscr{C}_\infty$ for each $X \in \mathscr{C}_\infty$,
if $\mathscr{F}$ is an upper modification in **Unif** such that each $X \in \mathscr{C}_\infty$ is $\mathscr{F}$-fine, then $\mathscr{F}$ is the identity.

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