Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 29 (1979), No. 1, 97–104

Persistent URL: `http://dml.cz/dmlcz/101581`

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Czechoslovak Mathematical Journal, 29 (104) 1979, Praha

QUASI-INJECTIVE S-SYSTEMS AND THEIR S-ENDOMORPHISM SEMIGROUP

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(Received March 14, 1977)

Patterned after the theory of modules over a ring, P. BERTHIAUME [1] introduced the concepts of injective and weakly-injective S-systems. He exhibited examples of such S-systems and showed that properties holding true for a right ring module need not hold for a right S-system. For example, a weakly injective S-system need not be injective; in ring theory, this is part of Baer's Theorem. In this paper, we study another weak form of injectivity called *quasi-injectivity*. Quasi-injective modules have been studied by JOHNSON and WONG [6], FAITH and UTUMI [3], and B. OSOFSKY [8], among others. Recently M. SATYANARAYANA [9] investigated quasi- and weakly-injective S-systems. Our paper is a study of quasi-injective S-systems and their S-endomorphism semigroup. We characterize the smallest quasi-injective essential extension of an S-system M_S contained in $I(M_S)$, its injective hull. Further we give conditions for $\mathrm{Hom}_S(M, M)$ to be (VON NUEMAN) regular and obtain as corollaries a result of M. BOTERO DE MEZA [2] dealing with the regularity of the maximal right quotient semigroup $Q(S)$ of a semigroup S, and a generalization for S-systems of Faith and Utumi's result on the regularity of the endomorphism ring of a quasi-injective module.

1. PRELIMINARIES

Definition 1.1. *A right S-system M with zero,* denoted M_S, is a set M, a semigroup S with zero, and a function $M \times S \to M$ such that $(m, s) \to ms$ and the following properties hold:

(i) $(ms)\, t = m(st)$ for $m \in M$ and $s, t \in S$.

(ii) M contains an element $\mathcal{O}$ (necessarily unique) such that $\mathcal{O}s = \mathcal{O}$ for all $s \in S$.

(iii) for all $m \in M$, $m0 = \mathcal{O}$, where 0 is the zero of S.

Dually we can define *a left S-system with zero*. In this paper all our S-system will be right S-systems with zero.

Definition 1.2. *A subsystem N of M_S*, denoted $N_S \subseteq M_S$, is a subset of M such that $ns \in N$ for all $n \in N$ and $s \in S$.

Definition 1.3. *A (right) congruence α on M_S* is an equivalence relation defined on M such that if $a \alpha b$ then $(as) \alpha (bs)$ for $a, b \in M$ and all $s \in S$.

Definition 1.4. *An S-homomorphism $f : A_S \to B_S$* is a mapping from A to B such that for any $a \in A$ and $s \in S$, $f(as) = f(a)s$.

The set of all S-homomorphisms from A_S to B_S is denoted by $\mathrm{Hom}_S(A, B)$. Under composition of functions $\mathrm{Hom}_S(M, M)$ is a semigroup called *the S-endomorphism semigroup of M_S*. If the elements of $K = \mathrm{Hom}_S(M, M)$ are regarded as left operators then M is a (K, S)-bisystem; that is to say, M is a right S-system and a left K-system such that $h(ms) = (hm)s$ for $h \in K$, $m \in M$, and $s \in S$.

Definition 1.5. An S-system M_S is *injective* if for each one-to-one S-homomorphism $g : P_S \to R_S$ and each S-homomorphism $h : P_S \to M_S$, there exists an S-homomorphism $\bar{h} : R_S \to M_S$ such that $\bar{h}g = h$.

Definition 1.6. An S-system M_S is *weakly-injective* if for any right ideal R of S and $f \in \mathrm{Hom}_S(R, M)$ there exists an element $m \in M$ such that $f(r) = mr$ for all $r \in R$.

Definition 1.7. An S-system M_S is *quasi-injective* if for $N_S \subseteq M_S$ and $f \in \mathrm{Hom}_S(N, M)$ there exists an S-homomorphism $\bar{f} : M_S \to M_S$ such that $\bar{f}|_N = f$.

In [1], Berthiaume showed that a weakly-injective S-system need not be injective. However, the converse is true. Also, it is clear that M_S being injective implies that M_S is quasi-injective, but the converse here is false. In fact, quasi-injective does not imply weakly-injective, as shown by the following example adapted from [9].

Example 1.8. Let S be the semigroup $\{0, a, b\}$ with $ab = a^2 = a$ and $ba = b^2 = b$. Now S considered as an S-system over itself is quasi-injective but it is not weakly injective since the identity map is not determined by left multiplication by an element of S. Consequently, it is not injective.

Definition 1.9. A subsystem N is *large* (or *essential*) in M_S if for any P_S and any S-homomorphism $f : M_S \to P_S$ whose restriction to N is one-to-one, then f is itself one-to-one. In such a case, we say that M_S is an essential extension of N_S.

The main result of Berthiaume's work in [1] is that every S-system has a maximal essential extension which is injective and unique up to S-isomorphism over M_S. This maximal essential extension which is injective is called *the injective hull* of M_S and is denoted by $I(M_S)$.

Definition 1.10. A nonzero subsystem N of M_S is *intersection large* ($\bigcap$-large) if for all nonzero subsystems X of M, $X \cap N \neq \mathcal{O}$. This will be denoted by $N_S \subseteq' M_S$.

Equivalently, a nonzero subsystem $N_S \subseteq' M_S$ if and only if for all $\mathcal{O} \neq m \in M$ there exists $s \in S^1$ (an identity adjoined) such that $\mathcal{O} \neq ms \in N$. FELLER and GANTOS in [4] proved that every large subsystem of M_S is $\bigcap$-large. The converse is false.

Definition 1.11. The *singular congruence* ψ_M on M_S is *a* right congruence defined by $a\psi_M b$ if and only if $ax = bx$ for all x in some $\bigcap$-large right ideal of S.

In [5], HINKLE showed that when $\psi_M = i$, the identity congruence on M, the concepts of large and $\bigcap$-large are the same. He also showed that M_S being weakly--injective and $\psi_M = i$ imply that M_S is injective. Example 1.8 shows that M_S being quasi-injective and $\psi_M = i$ does not imply that M_S is itself injective.

2. THE INJECTIVE HULL

Let M_S be an S-system with zero, let $I = I(M_S)$, its injective hull, and let $H = \mathrm{Hom}_S(I, I)$ the S-endomorphism semigroup of I. We know that I is the minimal injective essential extension containing M_S. Is there a minimal quasi-injective essential extension of M_S contained in I as in ring theory?

Lemma 2.1. *If M is an (H, S)-bisubsystem of I, then M is quasi-injective.*

Proof. Let $N_S \subseteq M_S$ and $f : N_S \to M_S$ an S-homomorphism. Since $M_S \subseteq I$, f can be extended to an S-homomorphism $\bar{f} \in H$. But $\bar{f}(M) \subseteq M$ so f can be extended to an S-homomorphism of M into M, namely $\bar{f}|_M$.

Lemma 2.2. *If $\psi_M = i_M$ then $\psi_I = i_I$.*

Proof. This follows immediately from the fact that M_S is large in I and Theorem 7 in [1].

Lemma 2.3. *Let f, $g \in \mathrm{Hom}_S(M, M)$ and suppose f and g agree on an $\bigcap$-large subsystem N_S of M_S. If $\psi_M = i$, then $f = g$.*

Proof. Let $x \in M$, then for $c \in x^{-1}N = \{s \in S : xs \in N\}$, an $\bigcap$-large right ideal of S, we have $f(x)\, c = g(x)\, c$. Since $\psi_M = i$ then $f(x) = g(x)$.

Theorem 2.4. *If M_S is quasi-injective and $\psi_M = i$, then M is an (H, S)-bisubsystem of I.*

Proof. Let $h \in H$. Since $M_S \subseteq' I$ then $h^{-1}(M) \subseteq' I$ and so $\mathcal{O} \neq h^{-1}(M) \cap M \subseteq' I$. Let $N = h^{-1}(M) \cap M$ and define an S-homomorphism $a : N_S \to M_S$ by $x \to h(x)$. Since M_S is quasi-injective there exists $b \in \mathrm{Hom}_S(M, M)$ such that $b(x) = a(x)$ for all $x \in N$. Since I is injective, there exists $c \in H$ such that $c(x) = b(x)$ for all $x \in M$. Hence $c(n) = b(n) = a(n) = h(n)$ for all $n \in N$. Since $\psi_M = i$ then $\psi_I = i$ by Lemma

2.2, and so $c = h$ by Lemma 2.3. But $c(M) \subseteq M$ so $h(M) \subseteq M$. Hence M is an (H, S)-bisubsystem of I.

Corollary 2.5. *Let M_S be an S-system for which $\psi_M = i$. Then M_S is quasi-injective if and only if $M = HM$ where $HM = \{f(m) \in I \mid f \in H \text{ and } m \in M\}$.*

Proof. We note that HM is the smallest fully invariant (H, S)-bisubsystem of I containing M and it is quasi-injective.

Note that if M_S is quasi-injective and $K = \mathrm{Hom}_S(M, M)$, then any K-invariant subsystem of M_S is also quasi-injective.

Theorem 2.6. *Let M_S be an S-system for which $\psi_M = i$. Then M_S is quasi-injective if and only if* $\mathrm{Hom}_S(M, M) \approx \mathrm{Hom}_S(I, I)$.

Proof. Let $K = \mathrm{Hom}_S(M, M)$. If $H \approx K$ then M is an (H, S)-bisubsystem of I and so by Lemma 2.1 must be quasi-injective. Conversely, consider $\phi : K \to H$ defined by $a \to \bar{a}$ where $\bar{a} : I \to I$ is the quasi-injective extension of $a : M \to M \subseteq I$. Since $\psi_M = i$ this mapping is well defined, one-to-one and a semigroup homomorphism. Furthermore, M_S being quasi-injective implies by Theorem 2.4 that M is an (H, S)-bisubsystem of I.

We now show that HM is the smallest quasi-injective essential extension of M contained in I.

Theorem 2.7. *Let M_S be an S-system with $\psi_M = i$. Then HM is the intersection of all quasi-injective S-subsystems of I containing M.*

Proof. Let P be a quasi-injective subsystem of I containing M. We must show that $HM \subseteq P$, but it is sufficient to show that $aP \subseteq P$ for all $a \in H$. To this end then let $a \in H$. Since $M \subseteq' I$ and $M \subseteq P \subseteq I$ then both P and $a^{-1}(P)$ are $\bigcap$-large S-subsystems of I and so $\emptyset \neq a^{-1}(P) \cap P$ is an $\bigcap$-large S-subsystem of P. Consider the mapping $a^{-1}(P) \cap P \to P$ defined by $x \to a(x)$. Since P is quasi-injective then there exists an $\hat{a} \in \mathrm{Hom}_S(P, P)$ such that $\hat{a}(x) = a(x)$ for all $x \in a^{-1}(P) \cap P$. Since I is injective there exists $\bar{a} \in H$ such that $\bar{a}(y) = \hat{a}(y)$ for all $y \in P$. Thus $\bar{a}P \subseteq P$. But by Lemma 2.2 and 2.3, $\bar{a}(x) = \hat{a}(x) = a(x)$ for all $x \in a^{-1}(P) \cap P \subseteq' I$ implies that $\bar{a} = a$, and so $aP \subseteq P$.

Since there are S-systems which are quasi-injective but not injective (Example 1.8) we can have $HM \subset I$, $HM \neq I$. The condition that $\psi_M = i$ cannot be omitted in the previous theorem as the following example demonstrates.

Example 2.8. Let Q^* represent the noncomplete chain of rationals with largest element $+\infty$ and $q \,.\, q' = q$ if and only if $q \leqq q'$. Thus $Q^*_{Q^*}$ has for its injective hull the chain of extended reals R^*. Berthiaume [1] showed that every noncomplete chain is weakly injective. Satyanarayana [9] showed that since $Q^*_{Q^*}$ has an identity it must also be quasi-injective. Here $\psi_{Q^*} \neq i$ because if $\psi_{Q^*} = i$ then the maximal right quotient semigroup $Q(Q^*) \approx B(Q^*)$, the bicommutator of the injective hull of Q^*, [7; Corollary 3.1] which is a contradicition since $Q^* = Q(Q^*)$ and $R^* = B(Q^*)$. Hence $Q^*_{Q^*}$ is quasi-injective and $\psi_{Q^*} \neq i$. In this case, $H = \mathrm{Hom}_{Q^*}(R^*, R^*)$ and considering the mapping $f : R^* \to R^*$ defined by $r \to (\sqrt{2}) \, . \, r$, we say that $HQ^* \nsubseteq Q^*$. Hence HQ^* is not the smallest quasi-injective essential extension contained in R^*.

3. THE S-ENDOMORPHISM SEMIGROUP OF A QUASI-INJECTIVE S-SYSTEM

In addition to the notation of the previous section we let $K = \mathrm{Hom}_S(M, M)$ and define for $m \in M$ the mapping $\lambda_m : S_S \to M_S$ by $s \to ms$. Let

$J(M_S) = \{m \in M : \lambda_m$ is one-to-one only on one element right ideals of $S\}$.

Lemma 3.1. *$J(M_S)$ is an S-subsystem of M_S.*

Proof. It is clear that $J(M_S)$ is not empty since $\mathcal{O} \in J(M_S)$. Let $m \in J(M_S)$ and $s \in S$, we must show that $ms \in J(M_S)$. Let A be a right ideal of S with more than one element, denoted $|A| \geqq 2$. Consider the right ideal sA of S. Either $sA = 0$ or $|sA| \geqq 2$.

Case 1. Suppose $sA = 0$ then for all $a_1 \neq a_2 \in A$, $sa_1 = sa_2 = 0$ and so $m(sa_1) = m(sa_2) = \mathcal{O}$. Consequently λ_{ms} is not one-to-one on A and thus $ms \in J(M_S)$.

Case 2. Suppose $|sA| \geqq 2$ then there exists $sa_1 \neq sa_2 \in sA$ such that $m(sa_1) = m(sa_2)$ because $m \in J(M_S)$. Hence λ_{ms} is not one-to-one on A and $ms \in J(M_S)$.

Lemma 3.2. *$J(M_S)$ is K-invariant.*

Proof. Let $f \in K$ and $m \in J(M_S)$. Since f is an S-homomorphism then $f(m)\, s = f(ms) = f(\lambda_m(s)) = f \circ \lambda_m(s)$. Suppose $f(m) \notin J(M_S)$ then $\lambda_{f(m)}$ is one to one on a right ideal R of S with $|R| \geqq 2$. Since $m \in J(M_S)$ then there exists $r_1 \neq r_2 \in R$ such that $\lambda_m(r_1) = \lambda_m(r_2)$. But then $f(\lambda_m(r_1)) = f(\lambda_m(r_2))$ and so $f \circ \lambda_m(r_1) = f \circ \lambda_m(r_2)$. Thus $\lambda_{f(m)}$ is not one-to-one on R; a contradiction.

Thus $J(M_S)$ is a (K, S)-bisubsystem of M_S and when M_S is quasi-injective, $J(M_S)$ is also. Furthermore, when $\psi_M = i$ and M_S is quasi-injective, $J(M_S)$ is an (H, S)-bisubsystem of M_S. We now define the set

$$T(M_S) = \{f \in K : f^{-1}(J(M_S)) \subseteq' M_S\}.$$

Clearly the zero mapping $\theta \in K$ is in $T(M_S)$ and $\{f \in K : f^{-1}(\mathcal{O}) \subseteq' M_S\} \subseteq T(M_S)$.

Lemma 3.3. *If $J(M_S) = \{\mathcal{O}\}$, then*

$$T(M_S) = \{f \in K : f^{-1}(\mathcal{O}) \subseteq' M_S\} = \{\theta\}\,.$$

Proof. Let $\theta \neq f \in T(M_S)$, then there exists $\mathcal{O} \neq m \in M_S$ such that $f(m) \neq \mathcal{O}$. Since $J(M_S) = \{\mathcal{O}\}$ then $f(m) \notin J(M_S)$ so there exists a right ideal R of S with $|R| \geqq 2$ such that $\lambda_{f(m)}$ is one-to-one on R. Consider now the S-subsystem mR and note that $|mR| \geqq 2$. Now f is one-to-one on mR and since $f^{-1}(\mathcal{O}) \subseteq' M_S$ then $f^{-1}(\mathcal{O}) \cap mR \neq \mathcal{O}$. This is a contradiction since if $x \in f^{-1}(\mathcal{O}) \cap mR$, $f(x) = \mathcal{O}$ and since $x \in mR$, then $f(x) = f(\mathcal{O})$ which implies that $x = \mathcal{O}$ since f is one-to-one on mR. Hence $T(M_S) = \{\theta\}$.

Theorem 3.4. *Let M_S be a quasi-injective S-system. If $\psi_M = i$ and $J(M_S) = \{\mathcal{O}\}$, then $K = \mathrm{Hom}_S(M, M)$ is regular.*

Proof. Let $\theta \neq f \in K$, then there exists $\mathcal{O} \neq x \in M_S$ such that $f(x) \neq \mathcal{O}$ and so $f(x) \notin J(M_S)$. Hence there exists a right ideal R of S with $|R| \geqq 2$ such that $\lambda_{f(x)}$ is one-to-one on R. Hence considering the S-subsystem xR we can say that f is one-to-one on xR and $|xR| \geqq 2$. By Zorn's Lemma, there is a maximal S-subsystem on which f is one-to-one, call it D_f. Define the S-homomorphism $g : f(D_f) \to D_f$ by $y = f(z) \to z$. Since M_S is quasi-injective then we can extend g to $\bar{g} \in K$ such that $\bar{g}|_{f(D_f)} = g$. Let $\bar{D}_f = f^{-1}(f(D_f))$, then for $t \in \bar{D}_f$, $f(t) = f(r)$ for some $r \in D_f$. Hence for $t \in \bar{D}_f$ we have

$$f\bar{g}\,f(t) = f(\bar{g}(f(t))) = f(\bar{g}(f(r))) = f(r) = f(t)\,.$$

Thus if $\bar{D}_f \subseteq' M_s$ we have by Lemma 2.3 that $f\bar{g}f = f$ on M_S. Hence suppose $\bar{D}_f$ is not an $\bigcap$-large subsystem of M_S, then there exists $A_S \subseteq M_S$ such that $|A_S| \geqq 2$ and $A_S \cap \bar{D}_f = \{\mathcal{O}\}$. Let $\mathcal{O} \neq a \in A_S$ such that $f(a) \neq \mathcal{O}$. Then $f(a) \notin J(M_S)$ so there exists a right ideal Y of S such that $|Y| \geqq 2$ and $f(a)\,y_1 \neq f(a)\,y_2$ for all $y_1 \neq y_2 \in Y$. Hence f is one-to-one on $aY \subseteq M_S$. But $D_f \subseteq \bar{D}_f$ so $A_S \cap D_f = \{\mathcal{O}\}$ implies $D_f \cap aY = \{\mathcal{O}\}$. Now $D_f \cup aY \supset D_f$ so f is not one-to-one on $D_f \cup aY$ by the maximality of D_f. Hence there exists $d \in D_f$ and $ay \in aY$ such that $d \neq ay$ but $f(d) = f(ay)$. Thus $ay \in f^{-1}(f(D_f)) = \bar{D}_f$. But $\bar{D}_f \cap aY = \{\mathcal{O}\}$ since $\bar{D}_f \cap A_S = \{\mathcal{O}\}$ and so $ay = \mathcal{O}$. Thus $f(d) = \mathcal{O} = f(\mathcal{O})$ and $d = \mathcal{O}$; a contradiction since $ay \neq d$. Thus $\bar{D}_f \subseteq' M_S$ and K is regular.

Corollary 3.5. *Let M_S be an S-system with $H = \mathrm{Hom}_S(I, I)$ where I is the injective hull of M_S. If $\psi_M = i$ and $J(M_S) = \{\mathcal{O}\}$, then H is regular.*

Proof. It suffices to show that $J(M_S) = \{\mathcal{O}\}$ implies $J(I) = \{\mathcal{O}\}$. Let $\mathcal{O} \neq t \in J(I)$. Since $M_S \subseteq' I$ then $t^{-1}M$ is an $\bigcap$-large right ideal of S and $|t^{-1}M| \geqq 2$. Hence there exists $0 \neq s \in S$ such that $\mathcal{O} \neq ts \in M$. We now show that $ts \in J(M_S)$ which gives a contradiction. Let R be any right ideal of S with $|R| \geqq 2$. Then either $sR = 0$ or $|sR| \geqq 2$.

Case 1. If $sR = 0$ then for $r_1 \neq r_2 \in R$, $t(sr_1) = t(sr_2)$ so λ_{ts} is not one-to-one on R.

Case 2. If $|sR| \geqq 2$ then there exists $sr_1 \neq sr_2 \in sR$ such that $t(sr_1) = t(sr_2)$ because $t \in J(I)$. Hence once again λ_{ts} is not one-to-one on R.

Thus in both cases $ts \in J(M_S)$.

The next corollary is similar to a result of M. BOTERO DE MEZA [2].

Corollary 3.6. *Let S be a monoid considered as a right S-system with zero over itself, and let $Q(S)$ be the maximal right quotient semigroup of S. If $\psi_S = i$ and $J(S) = 0$, then $Q(S)$ is regular.*

Proof. Corollary 3.5 and [7, Corollary 3.2].

We now link this work with a result of Faith and Utumi [3] by considering the following set:

$$X(K) = \{f \in K : f \text{ is one-to-one only on one element } S\text{-subsystems of } M_S\}$$

Lemma 3.7. $T(M_S) \subseteq X(K)$.

Proof. Let $f \in T(M_S)$ then $f^{-1}(J(M_S)) \subseteq' M_S$. Let $\mathcal{O} \neq N_S \subseteq M_S$, then $f^{-1}(J(M_S)) \cap N_S \neq \{\mathcal{O}\}$. Let $\mathcal{O} \neq n \in f^{-1}(J(M_S)) \cap N_S$ then $f(n) \in J(M_S)$. Consequently, $\lambda_{f(n)}$ is one-to-one on only one element right ideals of S. So there exists $s_1 \neq s_2 \in S$ such that $f(n)\, s_1 = f(n)\, s_2$. But then f is not one-to-one on $nS \subseteq N_S$ so $f \in X(K)$.

Lemma 3.8. *If $J(M_S) = \{\mathcal{O}\}$ then*

$$X(K) = T(M_S) = \{f \in K : f^{-1}(\mathcal{O}) \subseteq' M_S\} = \{\theta\}\,.$$

Proof. Let $f \in X(K)$ and suppose $f^{-1}(\mathcal{O})$ is not an $\bigcap$-large subsystem of M_S. Then there exists $\{\mathcal{O}\} \neq T_S \subseteq M_S$ such that $f^{-1}(\mathcal{O}) \cap T = \{\mathcal{O}\}$; that is, $\{m \in M : f(m) = \mathcal{O}\} \cap T = \{\mathcal{O}\}$. So there exists $\mathcal{O} \neq t \in T$ such that $f(t) \neq \mathcal{O}$ and so $f(t) \notin J(M_S)$. Furthermore, there exists a right ideal R of S with $|R| \geqq 2$ such that $r_1 \neq r_2 \in R$ implies $f(t)\, r_1 \neq f(t)\, r_2$. Hence f is one-to-one on $tR \subseteq T \subseteq M$. But this is a contradiction since $|tR| \geqq 2$ and $f \in X(K)$. Thus $f^{-1}(\mathcal{O}) \subseteq' M_S$ and so $X(K) = \{f \in K : f^{-1}(\mathcal{O}) \subseteq' M_S\}$.

Theorem 3.9. *If S is a ring and M_S is a quasi-injective right S-module then*

$$X(K) = \{f \in K : \ker f \subseteq' M_S\}\,.$$

Proof. If $f \in X(K)$ but $\ker f = \{m \in M : f(m) = 0\}$ is not $\bigcap$-large in M_S then there exists $\{0\} \neq T_S \subseteq M_S$ such that $\ker f \cap T_S = \{0\}$ so f is one-to-one on T_S. This is a contradiction since $f \in X(K)$ so $\ker f \subseteq' M_S$.

Faith and Utumi [3] showed that $K \setminus X(K)$ is a regular ring and when $X(K) = \{\theta\}$, K is a regular ring. Thus Theorem 3.4 generalizes the second half of Faith and Utumi's result to quasi-injective S-system whose singular congruence is the identity congruence.

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