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TOTALLY INHOMOGENEOUS LATTICE ORDERED GROUPS

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1. INTRODUCTION

The considerations in this paper are based on an analogy between Boolean algebras and lattice ordered groups.

Let B be a Boolean algebra. B is called *homogeneous* if for each $0 < b \in B$ the interval $[0, b]$ is isomorphic with B. If for each $0 < b \in B$ there is $b_1 \in B$ with $0 < b_1 < b$ such that the intervals $[0, b]$ and $[0, b_1]$ fail to be isomorphic, then B is said to be *totally inhomogeneous*.

R. S. PIERCE [10] proposed the question whether each complete Boolean algebra is a direct product of homogeneous Boolean algebras. This is equivalent with the question whether there exist totally inhomogeneous complete Boolean algebras. The answer to this question is affirmative (cf. BUKOVSKÝ [2], MACALOON [9]).

If $0 < b \in B$, then the interval $[0, b]$ is

(a) a principal ideal of the lattice B;

(b) a direct factor of the lattice B.

(In fact, the mapping $\varphi(x) = (x \wedge b, x \wedge b')$ $(x \in B)$ is an isomorphism of the lattice B onto the direct product $[0, b] \times [0, b']$.)

Thus the homogeneity of B can be expressed either in terms of principal ideals or in terms of direct factors as follows:

(a_1) Each principal ideal of B distinct from $\{0\}$ is isomorphic with B.

(b_1) Each direct factor of B distinct from $\{0\}$ is isomorphic with B.

Similarly we can characterize the total inhomogeneity of B in terms of principal ideals or in terms of direct factors of B.

Let us now replace the Boolean algebra B by a lattice ordered group G and the ideals of B by l-ideals of G. We arrive at the following definitions:

(a_2) G is called *a-homogeneous* if each principal l-ideal of G distinct from $\{0\}$ is isomorphic with G. If for each principal l-ideal $B \neq \{0\}$ of G there exists a principal l-ideal $B_1 \neq \{0\}$ of G such that $B_1 \subset B$ and B_1 is not isomorphic with B, then G is said to be *totally a-inhomogeneous*.

(b_2) G is called *b-homogeneous* if each direct factor of G distinct from $\{0\}$ is isomorphic with G. If for each direct factor $B \neq \{0\}$ of G there exists a direct factor $B_1 \neq \{0\}$ of G with $B_1 \subset B$ such that B_1 is not isomorphic with B, then G is called *totally b-inhomogeneous.*

A lattice ordered group G is both a-homogeneous and totally a-inhomogeneous if and only if $G = \{0\}$. An analogous assertion holds for the b-homogeneity.

In this note the question of existence of complete lattice ordered groups $G \neq \{0\}$ that are either totally a-inhomogeneous or totally b-inhomogeneous will be dealt with and the relations between these types of lattice ordered groups and totally inhomogeneous Boolean algebras will be examined.

2. PRELIMINARIES AND RESULTS

We shall use the standard notation for lattice ordered groups (cf. BIRKHOFF [1], FUCHS [4]).

Let G be a lattice ordered group, $X \subseteq G$. The set

$$X^\delta = \{g \in G : |g| \wedge |x| = 0 \text{ for each } x \in X\}$$

is called a *polar* of G. We denote by $P(G)$ the set of all polars of G. If $P(G)$ is partially ordered by the inclusion and $G \neq \{0\}$, then it turns out to be a complete Boolean algebra (cf. ŠIK [12]). For each complete Boolean algebra B there exists a complete lattice ordered group G such that $P(G)$ is isomorphic with B (cf., e.g., VULICH [14], Thm. V. 2.3, and Lemma 3.5 below).

Let $a, b \in G$, $a < b$. The set

$$[a, b] = \{x \in G : a \leqq x \leqq b\}$$

is called a *nontrivial interval* of G. The center $C([a, b])$ of $[a, b]$ consists of those elements $x \in [a, b]$ that have a relative complement in the interval $[a, b]$. If G is complete, then (since G is infinitely distributive) it follows from [5] that $C([a, b])$ is a closed sublattice of G; thus $C([a, b])$ is a complete Boolean algebra.

The main results of this note are as follows. Let $G \neq \{0\}$ be a complete lattice ordered group.

The following conditions for G are equivalent:

(a) G is totally a-inhomogeneous.

(b) The Boolean algebra $P(G)$ is totally inhomogeneous.

(c) The center of each nontrivial interval of G is a totally inhomogeneous Boolean algebra.

(d) For each $0 < g \in G$ there is $g_1 \in G$ with $0 < g_1 \leqq g$ such that the center of the interval $[0, g_1]$ is totally inhomogeneous.

From the equivalence of the conditions (a) and (b) and from the existence of totally inhomogeneous complete Boolean algebras we obtain that totally a-inhomogeneous complete lattice ordered groups distinct from $\{0\}$ do exist.

If G is totally a-inhomogeneous then it is totally b-inhomogeneous. If G is totally b-inhomogeneous and orthogonally complete, then G is totally a-inhomogeneous.

Let α be an infinite cardinal. There exists a complete lattice ordered group G_α with card $G_\alpha \geqq \alpha$ such that

(a_1) G_α is totally b-inhomogeneous;

(b_1) the Boolean algebra $P(G_\alpha)$ is homogeneous.

Hence the notions of the total a-inhomogeneity and the total b-inhomogeneity for complete lattice ordered groups are not equivalent.

Further, it will be shown that totally a-inhomogeneous complete vector lattices G can be characterized by using the representation of G as a system of extended real valued functions. Let us recall some relevant notions.

Let R be the set of all reals and let $R_1 = R \cup \{-\infty, \infty\}$. The set R_1 is linearly ordered and topologized in the natural way. Let B be a complete Boolean algebra. We denote by $S(B)$ the Stone space of B. Let $F_\infty(B)$ be the set of all continuous functions $f: S(B) \to R_1$ such that the set

$$\{x \in S(B) : f(x) \notin R\}$$

is nowhere dense in $S(B)$. Then $F_\infty(B)$ is an additive complete lattice ordered group (for more details, cf. Vulich [14], Chap. V, § 2). Let $F_b(B)$ be the set of all bounded functions belonging to $F_\infty(B)$.

The following assertions will be proved:

(A) Let $G \neq \{0\}$ be a complete vector lattice. Then G is totally a-inhomogeneous if and only if G is isomorphic with a completely subdirect product of vector lattice G_k ($k \in K$) such that for each $k \in K$ there is a totally inhomogeneous complete Boolean algebra B_k having the property that G_k is an l-subgroup of $F_\infty(B_k)$ with $F_b(B_k) \subseteq G_k$.

(B) Let $G \neq \{0\}$ be a complete vector lattice. Suppose that G is orthogonally complete. Then G is totally b-inhomogeneous if and only if G is isomorphic with a completely subdirect product of lattice ordered groups G_k ($k \in K$) such that for each $k \in K$ there is a totally inhomogeneous complete Boolean algebra B_k with $F_\infty(B_k) = G_k$.

3. TOTAL a-INHOMOGENEITY

Let us recall some notions and results we shall need in the sequel. Let G be a lattice ordered group.

Each polar of G is a closed convex l-subgroup of G. If $\emptyset \neq \{A_i\} \subseteq P(G)$, then $A = \bigcap A_i \in P(G)$ and $A = \bigwedge A_i$ in the Boolean algebra $P(G)$ (cf. Šik [12]). If X is a one-element subset of G, then $X^{\delta\delta}$ is said to be a principal polar of G.

Let A, B be l-subgroups of G such that (i) the group G is a direct product of its subgroups A, B and (ii) if $g \in G$, $a \in A$, $b \in B$, $g = a + b$, then $g \geqq 0$ only if $a \geqq 0$ and $b \geqq 0$. Under these assumptions the lattice ordered group G is said to be a direct product of its l-subgroups A and B; we write $G = A \times B$. The l-subgroups A and B are called direct factors of G. Each direct factor of G is a closed convex l-subgroup of G. Under the above notation, the element a will be called the component of g in A and it will be denoted by $g(A)$. A convex l-subgroup H of G is a direct factor of G if and only if for each $0 \leqq g \in G$ the set

$$H_1 = \{h \in H : 0 \leqq h \leqq g\}$$

possesses the greatest element (if this is the case, then the greatest element of H_1 is the component of g in H). G is called strongly projectable (projectable) if each polar (each principal polar) of G is a direct factor of G. If G is complete, then it is strongly projectable.

An element $0 \leqq s \in G$ is called singular if $x \wedge (s - x) = 0$ for each $x \in G$ with $0 \leqq x \leqq s$. The lattice ordered group G is said to be singular if for each $0 < g \in G$ there exists a singular element $s \in G$ with $0 < s \leqq g$.

Let $g \in G$. The smallest l-ideal of G containing the element g will be called the principal l-ideal of G generated by g and it will be denoted by $[g]$. For each $g \in G$ we have $[g] = [|g|]$; hence each principal l-ideal is generated by a positive element. If G is abelian (in particular, if G is complete) and $0 \leqq g \in G$, then

$$[g] = \bigcup [-ng, ng] \quad (n = 1, 2, \ldots).$$

Let $0 \leqq e \in G$. If $e \wedge g > 0$ for each $0 < g \in G$, then e is called a weak unit of G. If $[e] = G$, then e is said to be a strong unit of G.

A system $\emptyset = \{a_i\}$ $(i \in I)$ of elements of G is called disjoint if $a_i > 0$ for each $i \in I$ and $a_i \wedge a_j = 0$ for each pair i, j of distinct elements of I. If each disjoint system of elements of G possesses the least upper bound in G, then G is called orthogonally complete.

Let $\{G_i\}$ $(i \in I)$ be a system of lattice ordered groups. Their (external) direct product or direct sum will be denoted by $\prod_{i \in I} G_i$ or $\sum_{i \in I} G_i$, respectively. Let H be an l-subgroup of $\prod_{i \in I} G_i$ such that $\sum_{i \in I} G_i \subseteq H$. Then H is called a completely subdirect product of the system $\{G_i\}$ $(i \in I)$. The notion of a completely subdirect product of lattice ordered groups has been introduced by Šik [13].

Let H be a completely subdirect product of the system $\{G_i\}$ $(i \in I)$. For each $i \in I$ let G_i^0 be the set of all $f \in H$ such that $f(j) = 0$ for each $j \in I$, $j \neq i$. Then for each $0 \leqq h \in H$ there are uniquely determined elements $h_i^0 \in G_i^0$ such that $h = \bigvee_{i \in I} h_i^0$.

Let G be a lattice ordered group and let $\{G_i\}$ $(i \in I)$ be a system of l-subgroups of G. Suppose that the following condition is fulfilled:

(a) For each $0 \leqq g \in G$ there are uniquely determined elements $g_i \in G_i$ such that $g = \bigvee_{i \in I} g_i$.

It is not hard to verify that then there exists an l-subgroup G' of $\prod_{i\in I} G_i$ and an isomorphism φ of G onto G' such that

(i) G' is a completely subdirect product of the system $\{G_i\}$ $(i \in I)$,

(ii) if g and $\{g_i\}$ are as above, then $(\varphi(g))(i) = g_i$ for each $i \in I$.

Hence we may take the condition (a) as an internal definition of a completely subdirect product; we say that G is a completely subdirect product of its l-subgroups G_i $(i \in I)$ if (a) is valid.

If G is a completely subdirect product of its l-subgroups G_i $(i \in I)$, then clearly each G_i is a direct factor of G.

3.1. Lemma. *Suppose that G is a completely subdirect product of its l-subgroups G_i $(i \in I)$. Then G is totally a-inhomogeneous if and only if each G_i is totally a-inhomogeneous.*

Proof. Assume that G is totally a-inhomogeneous. Since G_i are convex l-subgroups of G, they are totally a-inhomogeneous as well. Conversely, assume that all G_i are totally a-inhomogeneous. Let $D \neq \{0\}$ be a principal l-ideal of G. Then D is a completely subdirect product of its l-subgroups $D \cap G_i$ $(i \in I)$ and each $D \cap G_i$ is principal. There exists $i \in I$ with $D \cap G_i \neq \{0\}$. Since G_i is totally a-inhomogeneous, there is a principal l-ideal $D_1 \neq \{0\}$ in $D \cap G_i$ such that D_1 is not isomorphic with $D \cap G_i$. Thus $D \cap G_i$ and D_1 are principal l-ideals in D such that either $D \cap G_i$ or D_1 fails to be isomorphic with D. Hence G is totally a-inhomogeneous.

3.2. Corollary. *Let $G = A \times A'$. Then G is totally a-inhomogeneous if and only if both A and A' are totally a-inhomogeneous.*

In the remainder of this paragraph we assume that G is a complete lattice ordered group (unless otherwise stated).

3.3. Lemma. *Suppose that $G \neq \{0\}$ is totally a-inhomogeneous. Then there is a system $\{G_i\}$ $(i \in I)$ of convex l-subgroups of G such that*

(i) *for each $i \in I$, G_i is a totally a-inhomogeneous lattice ordered group with a weak unit;*

(ii) *G is a completely subdirect product of the system $\{G_i\}$ $(i \in I)$.*

Proof. From the Axiom of Choice it follows that there exists a maximal disjoint system $\{e_i\}$ $(i \in I)$ in G. For each $i \in I$ we put

$$G_i = \{e_i\}^{\delta\delta}.$$

Then, since G is complete, each G_i is a direct factor of G. If $i, j \in I$, $i \neq j$, $0 \leqq a \in G_i$, $0 \leqq b \in G_j$, then $a \wedge b = 0$. Let $0 \leqq g \in G$. Put $g_i = g(G_i)$ for each $i \in I$. We have $g_i \leqq g$ for each $i \in I$. If $h < g$ and $g_i \leqq h$ for each $i \in I$, then the element $0 < g - h$

must be disjoint with each e_i, which is a contradiction. Thus $g = \bigvee_{i \in I} g_i$. Suppose that for each $i \in I$ we have $0 \leqq h_i \in G_i$ and that $g = \bigvee_{i \in I} h_i$. Then $g(G_i) = h_i$ for each $i \in I$. Therefore G is a completely subdirect product of the system $\{G_i\}$ $(i \in I)$. For each $i \in I$, e_i is a weak unit in G_i. According to 3.1, all G_i are totally a-inhomogeneous.

3.4. Lemma. *Let G_i be a complete lattice ordered group. Then G_i can be written as $G_i = A_i \times A_i'$, where A_i is a vector lattice and A_i' is singular. If G_i has a weak unit, then both A_i and A_i' have a weak unit.*

Proof. The first assertion has been proved in [3]. If e_i is a weak unit in G_i, then $e_i(A_i)$ and $e_i(A_i')$ is a weak unit in A_i or A_i', respectively.

For $0 \leqq g \in G$ we denote $C([0, g]) = C(g)$.

3.5. Lemma. *Let G have a weak unit e. Then $P(G)$ is isomorphic with $C(e)$.*

Proof. For each $X_1 \in P(G)$ we put $\varphi(X_1) = e(X_1)$. If $X_1, X_2 \in P(G)$, $X_1 \subseteq X_2$, then $e(X_1) \leqq e(X_2)$. We have $X_1 = \{e_1\}^{\delta\delta}$ where $e_1 = e(X_1)$, since e_1 is a weak unit in X_1. Thus if $e_i = \varphi(X_i)$, $i = 1, 2$, then $e_1 \leqq e_2$ implies $X_1 \subseteq X_2$.

For each $e_0 \in C(e)$, the relation $\varphi(\{e_0\}^{\delta\delta}) = e_0$ is valid. This can be verified as folows. We have $\varphi(\{e_0\}^{\delta\delta}) = e(\{e_0\}^{\delta\delta})$; let us denote this element by e_1. Then e_1 is the least upper bound of the set $\{a \in \{e_0\}^{\delta\delta} : 0 \leqq a \leqq e\}$; the element e_0 belongs to this set and hence $e_0 \leqq e_1$. On the other hand, e_0 possesses a relative complement e_0' in $[0, e]$. Thus e_0 is the greatest of those elements belonging to $[0, e]$ which are disjoint with e_0'; e_1 is one of these elements, since $e_1 \in \{e_0\}^{\delta\delta}$ implies $e_1 \wedge e_0' = 0$. From this we obtain $e_1 \leqq e_0$. Hence we conclude $e_0 = e_1$. Therefore φ is an isomorphism of $P(G)$ onto $C(e)$.

3.6. Corollary. *Let e_1, e_2 be weak units in G. Then $C(e_1)$ is isomorphic with $C(e_2)$.*

For the proofs of the following results cf. [14] (Thm. V. 4.1; Thm. V.3.1; Thms. V.5.1 and V.5.2).

(*) *Let G be a complete vector lattice with a weak unit. Then there exists an isomorphism φ of G into $F_\infty(B)$ with $B = P(G)$ such that $F_b(B) \subseteq \varphi(G)$.*

(**) *Let G be a complete vector lattice with a strong unit. Then there exists an isomorphism of G onto $F_b(B)$ with $B = P(G)$.*

(***) *Let G be a complete vector lattice. Assume that G is orthogonally complete. Then G is isomorphic with $F_\infty(B)$, where $B = P(G)$.*

3.7. Lemma. *Let $A \neq \{0\}$ be a complete vector lattice with a weak unit. Assume that A is totally a-inhomogeneous. Then the Boolean algebra $P(A)$ is totally inhomogeneous.*

Proof. Let e be a weak unit in A. Suppose that $P(A)$ fails to be totally inhomogeneous. According to Lemma 3.5, $C(e)$ is not totally inhomogeneous. Thus there is $0 < e_1 \in C(e)$ such that the set $C_1 = \{e_i \in C(e) : e_i \leqq e_1\}$ is a homogeneous Boolean algebra. Clearly $C_1 = C(e_1)$.

Put $D_1 = [e_1]$. Thus e_1 is a strong unit in D_1. Since A is totally a-inhomogeneous, there exists a principal l-ideal $D_2 \neq \{0\}$ of A with $D_2 \subset D_1$ such that D_2 is not isomorphic with D_1. Let D_2 be generated by an element $0 < e_2$. Put

$$D_3 = \{e_2\}^{\delta\delta}$$

where the symbol δ is taken with respect to the lattice ordered group D_1. Then D_3 is a direct factor of D_1. Thus the element $e^* = e_1(D_3)$ is a strong unit in D_3. This implies that e^* is a weak unit in D_3. From the definition of D_3 it follows immediately that e_2 is a weak unit in D_3. Thus according to Corollary 3.6, $C(e_2)$ is isomorphic with $C(e^*)$. Since the Boolean algebra $C(e_1)$ is homogeneous and since $0 < e^* \in C(e_1)$ we obtain that $C(e^*)$ is isomorphic with $C(e_1)$. Thus $C(e_2)$ is isomorphic with $C(e_1)$. Hence we infer from 3.5 and $(**)$ that D_2 is isomorphic with D_1, which is a contradiction.

3.8. Lemma. *Let $S_1 \neq \{0\}$ be a singular complete lattice ordered group with a weak unit e. Then there exists a singular element s_0 in S_1 such that s_0 is a weak unit in S_1. Moreover, s_0 is the join of all singular elements of S_1.*

Proof. Let S_0 be the set of all singular elements $s \in S_1$ with $s \leqq e$. Denote $s_0 = \sup S_0$. Then s_0 is singular. Assume that there exists a singular element s in S_1 such that s non $\leqq s_0$. Put $s_1 = s_0 \vee s$. Then s_1 is singular, $s_0 < s_1$. Hence $0 < s_2 = s_1 - s_0$ is singular and $s_2 \wedge s_0 = 0$. We have $s_2 \wedge e > 0$, $s_2 \wedge e$ is singular and $(s_2 \wedge e) \wedge s_0 = e \wedge (s_2 \wedge s_0) = 0$. On the other hand, since $s_2 \wedge e \leqq e$, we infer that $s_2 \wedge e \leqq s_0$, hence $0 < s_2 \wedge e = (s_2 \wedge e) \wedge s_0$, which is a contradiction. Thus s_0 is the join of all singular elements. Since S_1 is singular, s_0 is a weak unit in S_1.

Let $S_1 \neq \{0\}$ be a singular complete lattice ordered group with a weak unit s_0 such that s_0 is a singular element of S_1. Put $B = C(s_0) = [0, s_0]$.

Let N be the set of all positive integers.

The following result has been proved in [6] (Thm. 3.2):

(α) *If S_1 is orthogonally complete and $0 < g \in S_1$, then there is a subset $N(g) \subseteq N$ and a disjoint system $\{t_n\}$ $(n \in N(g))$ such that $s_0 \geqq t_n$ for each $n \in N(g)$ and $g = \bigvee nt_n$ $(n \in N(g))$.*

It can be easily verified that the assumption of orthogonal completeness is redundant in (α). If we put $t_n = 0$ for each $n \in N \setminus N(g)$ and $t_0 = s_0 - \bigvee_{n\in N} t_n$, we obtain:

(+) Let $0 < g \in S_1$. Then there exist elements $t_n \in B$ $(n = 0, 1, 2 \ldots)$ with

$$g = \bigvee nt_n \quad (n = 0\ 1, 2, \ldots)$$

such that $\bigvee t_n = s_0$ and $t_n \wedge t_m = 0$ whenever $n, m \in \{0, 1, 2, \ldots\}$, $n \neq m$.

If S_1 is orthogonally complete, then for each disjoint subset $\{t_n\}$ $(n = 1, 2, \ldots)$ of elements of B the join $\bigvee nt_n$ exists in S_1. Hence from 3.3, [6] and from $(+)$ it follows that $(S_1)^+$ is determined up to isomorphism by the Boolean algebra B. Since S_1 is uniquely determined by $(S_1)^+$, we obtain:

$(++)$ Let S_1 be orthogonally complete. Then S_1 is determined up to isomorphism by $C(s_0)$.

Suppose that s_0 is a strong unit in S_1 and let g, t_n be as in $(+)$. There is a positive integer n_1 with $n_1 s_0 \geqq g$. Let $n > n_1$. Suppose that $t_n > 0$. Thus $nt_n > n_1 t_n$. Denote $\{0, 1, 2, \ldots\} = N_0$. We have

$$n_1 s_0 = n_1(\bigvee_{m \in N_0} t_m) = \bigvee_{m \in N_0} n_1 t_m ,$$
$$nt_n \leqq g \leqq \bigvee_{m \in N_0} n_1 t_m$$

and hence

$$nt_n = nt_n \wedge (\bigvee_{m \in N_0} n_1 t_m) = \bigvee_{m \in N_0} (nt_n \wedge n_1 t_m) =$$
$$= nt_n \wedge n_1 t_n = (n \wedge n_1)\, t_n = n_1 t ,$$

which is a contradiction. Thus $t_n = 0$ for each $n > n_1$ and thus

$$g = t_1 \vee 2t_2 \vee \ldots \vee n_1 t_{n_1} .$$

Hence S_1^+ is determined up to isomorphism by B and therefore we have:

$(+++)$ Let S_1 have a strong unit that is a singular element in S_1. Then S_1 is determined up to isomorphism by $C(s_0)$.

3.9. Lemma. *Let $S_1 \neq \{0\}$ be a complete singular lattice ordered group with a weak unit. Suppose that S_1 is totally a-inhomogeneous. Then the Boolean algebra $P(S_1)$ is totally inhomogeneous.*

Proof. According to Lemma 3.8, the join s_0 of all singular elements of S_1 is a weak unit in S_1. Assume that $P(S_1)$ is not totally inhomogeneous. Thus by Lemma 3.5, $C(s_0)$ is not totally inhomogeneous. Since s_0 is singular, we have $C(s_0) = [0, s_0]$. Hence there is $0 < s_2 \leqq s_0$ such that the Boolean algebra $[0, s_2]$ is homogeneous. Put $S_2 = [s_2]$. Because S_1 is totally a-inhomogeneous, there exists a principal l-ideal $S_3 \neq \{0\}$ of S_1 with $S_3 \subset S_2$ such that S_3 is not isomorphic with S_2. There is $0 < e \in S_1$ with $S_3 = [e]$. Moreover, S_3 is singular and complete. Since s_2 is a singular strong unit in $S_2 = [s_2]$ and $e \in S_2$, we obtain by an analogous reasoning as above that there are elements $t_i \in [0, s_2]$ $(i = 1, 2, \ldots, n_1)$ with $e = t_1 \vee 2t_2 \vee \ldots \vee n_1 t_1$, $t_n \wedge t_m = 0$ whenever $n, m \in \{1, 2, \ldots, n_1\}$, $n \neq m$. All t_i are singular and belong to S_3, hence the element $s_3 = t_1 \vee \ldots \vee t_{n_1}$ is singular as well and $s_3 \in S_3$. We have $n_1 s_3 = n_1 t_1 \vee \ldots \vee n_1 t_n \geqq e$. Because e is a strong unit in $S_3 = [e]$, the element s_3 is a strong unit in S_3. Obviously $0 < s_3 \leqq s_2$ and hence $[0, s_3]$ is isomorphic with $[0, s_2]$. Thus according to $(+++)$, S_3 is isomorphic with S_2, which is a contradiction.

3.10. Proposition. *Let $G \neq \{0\}$ be a complete lattice ordered group that is totally a-inhomogeneous. Then G fulfils the following condition*:

(c) *G is a completely subdirect product of lattice ordered groups $H_j \neq \{0\}$ $(j \in J)$ such that for each $j \in J$, H_j has a weak unit, the Boolean algebra $P(H_j) = B_j$ is totally inhomogeneous and either*

(i) *there exists an isomorphism φ_j of H_j into $F_\infty(B_j)$ such that $F_b(B_j) \subseteq \varphi_j(H_j)$, or*

(ii) *H_j is singular.*

Proof. Let $\{G_i\}$ $(i \in I)$ be as in Lemma 3.3. Further, let A_i and A_i' be as in 3.4. Let $\{H_j\}$ $(j \in J)$ be the system of all l-subgroups A_i and A_i' that are distinct from $\{0\}$. According to 3.2, all H_j are totally a-inhomogeneous. Because G is a completely subdirect product of the system $\{G_i\}$ $(i \in I)$, it is also a completely subdirect product of the system $\{H_j\}$ $(j \in J)$. Let $j \in J$. If H_j is a vector lattice, then by $(*)$ and 3.7 the condition (i) holds and $P(H_j)$ is totally inhomogeneous. If H_j is singular, then according to 3.9, $P(H_j)$ is totally inhomogeneous.

3.11. Theorem. *Let $G \neq \{0\}$ be a complete lattice ordered group such that the Boolean algebra $P(G)$ is totally inhomogeneous. Then G is both totally a-inhomogeneous and totally b-inhomogeneous.*

Proof. Let $[x_1]$ be a principal l-ideal of G with $x_1 > 0$. Put $X_1 = \{x_1\}^{\delta\delta}$. Then $X_1 \in P(G)$ and the interval $[\{0\}, X_1]$ of $P(G)$ coincides with $P(X_1)$. Hence $P(X_1)$ is totally inhomogeneous. Thus according to 3.5, $C(x_1)$ is totally inhomogeneous. Hence there is $0 < x_2 \in C(x_1)$ such that $C(x_2)$ is not isomorphic with $C(x_1)$. Therefore $[x_2] \subseteq [x_1]$, $P([x_1])$ is isomorphic with $C(x_1)$, $P([x_2])$ is isomorphic with $C(x_2)$ and thus $[x_2]$ is not isomorphic with $[x_1]$. Hence G is totally a-inhomogeneous.

Let $X \neq \{0\}$ be a direct factor of G. Choose $0 < x_1 \in X$ and let x_2 be as above. Put $X_1 = \{x_1\}^{\delta\delta}$, $X_2 = \{x_2\}^{\delta\delta}$. Then x_i is a weak unit in X_i $(i = 1, 2)$ and similarly as we did above for $[x_1]$, $[x_2]$ we can verify that X_1 is not isomorphic with X_2. Both X_1 and X_2 are direct factors of G, $X_i \subseteq X$ $(i = 1, 2)$ and either X_1 or X_2 fails to be isomorphic with X. Thus G is totally b-inhomogeneous.

3.12. Proposition. *Let $G \neq \{0\}$ be a complete lattice ordered group fulfilling the condition* (c) *from* 3.10. *Then G is totally a-inhomogeneous.*

Proof. According to 3.11 all lattice ordered groups H_j are totally a-inhomogeneous and hence by 3.1, G is totally a-inhomogeneous as well.

From 3.10 and 3.11 it follows that the assertion (A) in § 2 is valid.

3.13. Theorem. *Let $G \neq \{0\}$ be a complete lattice ordered group that is totally a-inhomogeneous. Then the Boolean algebra $P(G)$ is totally inhomogeneous.*

Proof. Assume that $P(G)$ is not totally inhomogeneous. Then there exists $\{0\} \neq X \in P(G)$ such that the interval $[\{0\}, X]$ of $P(G)$ is homogeneous. Choose $0 < x \in X$ and put $\{x\}^{\delta\delta} = X_1$. Then the interval $[\{0\}, X_1]$ of $P(G)$ coincides with $P(X_1)$. Hence $P(X_1)$ is homogeneous. Since x is a weak unit in X_1, we infer from 3.5 that $C(x)$ is homogeneous.

Let $\{H_j\}$ $(j \in J)$ be as in 3.10. The lattice ordered group X_1 is a completely subdirect product of lattice ordered groups $X_1 \cap H_j$ $(j \in J)$ and for each $j \in J$, $x_j = x(H_j)$ is a weak unit in $X_1 \cap H_j$. There exists $j \in J$ with $X_1 \cap H_j \neq \{0\}$; then $x_j > 0$. We have $x_j \in C(x)$ and hence $C(x_j) \subseteq C(x)$. Thus $C(x_j)$ is homogeneous. Hence with respect to 3.5, $P(X_1 \cap H_j)$ is homogeneous.

$X_1 \cap H_j$ is complete and it follows from 3.10 that $X_1 \cap H_j$ is either a vector lattice or a singular lattice ordered group. Since G is totally a-inhomogeneous, $X_1 \cap H_j$ is totally a-inhomogeneous as well. Hence we obtain from 3.7 and 3.9 that $P(X_1 \cap H_j)$ is totally inhomogeneous, which is a contradiction.

From 3.11 and 3.13 we obtain:

3.14. Corollary. *Let G be a complete lattice ordered group that is totally a-inhomogeneous. Then G is totally b-inhomogeneous.*

3.15. Theorem. *Let $G \neq \{0\}$ be a complete lattice ordered group that is totally a-inhomogeneous. Then the center of each nontrivial interval of G is a totally inhomogeneous Boolean algebra.*

Proof. Let $a, b \in G$, $a < b$. Assume that $C([a, b])$ is not totally inhomogeneous. Put $y = b - a$. Since the intervals $[a, b]$ and $[0, y]$ are isomorphic, the Boolean algebra $C(y)$ is not totally inhomogeneous. Hence there is $0 < x \in C(y)$ such that $C(x)$ is homogeneous. Put $X_1 = \{x\}^{\delta\delta}$. Now by the same method as in the proof of 3.12 we arrive at a contradiction.

3.16. Theorem. *Let $G \neq \{0\}$ be a complete lattice ordered group. Suppose that for each $0 < g \in G$ there exists $0 < g_1 \in G$ with $g_1 \leqq g$ such that the center of the interval $[0, g_1]$ is a totally inhomogeneous Boolean algebra. Then G is totally a-inhomogeneous.*

Proof. According to 3.11 it suffices to show that $P(G)$ is totally inhomogeneous. Assume that $P(G)$ fails to be totally inhomogeneous. Then there is $X \in P(G)$, $X \neq \{0\}$ such that the interval $[\{0\}, X]$ of $P(G)$ is homogeneous. Choose $0 < x \in X$. According to the assumption there is $0 < x_1 \in G$ with $x_1 \leqq x$ such that $C(x_1)$ is totally inhomogeneous. We have $\{x_1\}^{\delta\delta} \subseteq X$ and $P(\{x_1\}^{\delta\delta})$ coincides with the interval $[\{0\}, \{x_1\}^{\delta\delta}]$ of $P(G)$. Hence $P(\{x_1\}^{\delta\delta})$ is homogeneous; by using 3.5 we get that $C(x_1)$ is homogeneous, which is a contradiction.

From 3.11, 3.12, 3.15 and 3.16 it follows that the conditions (a)–(c) mentioned in § 2 are equivalent.

Let G be a lattice ordered group that need not be abelian. For $g \in G$ let $c(g)$ be the convex l-subgroup of G generated by g. Suppose that for each $0 < g_1 \in G$ there is $0 < g_2 \in c(g_1)$ such that $c(g_2)$ is not isomorphic with $c(g_1)$. Then G is called totally inhomogeneous [7]. For abelian lattice ordered groups the notions of the total inhomogeneity and that of the total a-inhomogeneity coincide. In [7] (Thm. 5.5) it has been shown that in each lattice ordered group G there is a largest convex totally inhomogeneous l-subgroup, but the existence of totally inhomogeneous complete lattice ordered groups distinct from $\{0\}$ was not examined in [7].

4. TOTAL b-INHOMOGENEITY

4.1. Lemma. *Let $A \neq \{0\}$ be a complete vector lattice with a strong unit. Suppose that A is totally b-inhomogeneous. Then the Boolean algebra $P(A)$ is totally inhomogeneous.*

Proof. Let e be a strong unit in A. According to 3.5 it suffices to verify that $C(e)$ is totally inhomogeneous. Let $0 < e_1 \in C(e)$, $A_1 = \{e_1\}^{\delta\delta}$. Then A_1 is a direct factor of A with a strong unit e_1. According to the assumption there is a direct factor $A_2 \neq \{0\}$ of A such that $A_2 \subset A_1$ and A_2 is not isomorphic with A_1. The element $e_2 = e_1(A_2) = e(A_2)$ is a strong unit in A_2. According to $(**)$ and 3.5, A_i is isomorphic with $F_b(C(e_i))$ $(i = 1, 2)$. Thus $C(e_1)$ is not isomorphic with $C(e_2)$. Hence $C(e)$ is totally inhomogeneous.

4.2. Lemma. *Let S be a complete singular lattice ordered group that is totally b-inhomogeneous. Suppose that S possesses a strong unit s_0 such that s_0 is a singular element of S. Then the Boolean algebra $P(S)$ is totally inhomogeneous.*

The proof is analogous to that of 4.1 with the distinction that we use $(+++)$ instead of $(**)$.

For a lattice ordered group G we denote by $S_0(G)$ the set of all singular elements of G and we put $S(G) = (S_0(G))^{\delta\delta}$. Then $S(G)$ is the largest convex singular l-subgroup of G. If G is complete, then $S(G)$ is a direct factor of G.

4.3. Theorem. *Let G be a complete lattice ordered group with a strong unit e such that $e(S(G))$ is a singular element in G. Suppose that G is totally b-inhomogeneous. Then $P(G)$ is totally inhomogeneous.*

Proof. According to 3.4 we have $G = A \times S(G)$, where A is a vector lattice. The element $e(A)$ is a strong unit in A. Hence by 4.1, $P(A)$ is totally inhomogeneous. Moreover, by 4.2 $P(S(G))$ is totally inhomogeneous. Since G is a direct product $A \times S(G)$, $P(G)$ is isomorphic with the direct product $P(A) \times P(S(B))$. From this it easily follows that $P(G)$ is totally inhomogeneous.

From 4.3 and 3.13 we obtain:

4.4. Corollary. *Let G be a complete lattice ordered group with a strong unit e such that $e(S(G))$ is singular. Suppose that G is totally b-inhomogeneous. Then G is totally a-inhomogeneous.*

4.5. Theorem. *Let $G \neq \{0\}$ be a complete lattice ordered group that is orthogonally complete. Suppose that G is totally b-inhomogeneous. Then $P(G)$ is totally inhomogeneous.*

Proof. We have $G = A \times S(G)$, where A is a vector lattice. Both A and $S(G)$ are orthogonally complete and totally b-inhomogeneous. By the same method as in the proof of 4.1 (with the distinction that we use (***) instead of (**)) we obtain that $P(A)$ is totally inhomogeneous. Similarly, by using $(++)$, we get that $P(S(G))$ is totally inhomogeneous. Hence $P(G)$ is totally inhomogeneous.

4.6. Corollary. *Let G be a complete lattice ordered group that is orthogonally complete. If G is totally b-inhomogeneous. the it is totally a-inhomogeneous.*

From 4.6, (***), 3.14 and the assertion (A) from § 2 it follows that the assertion (B) in § 2 is valid.

In the next section it will be shown that in general the total b-inhomogeneity does not imply the total a-inhomogeneity for complete lattice ordered groups.

5. TOTALLY b-INHOMOGENEOUS LATTICE ORDERED GROUPS WITH HOMOGENEOUS BOOLEAN ALGEBRAS OF POLARS

Let N be the set of all positive integers. Let α, β, α_n $(n \in N)$ be infinite cardinals with $\alpha < \alpha_1 < \alpha_2 < \ldots < \beta$. Let Q be a set with card $Q = \beta$. The system of all sequences $\{q_n\}$ $(n = 1, 2, \ldots)$ of elements of Q will be denoted by S. Let m be a positive integer and let $q_1, q_2, \ldots, q_m$ be fixed elements of Q. We denote by $S(q_1, \ldots, q_m)$ the set of all sequences $\{p_n\} \in S$ such that $p_i = q_i$ for $i = 1, \ldots, m$. The system of all $S(q_1, \ldots, q_m)$ (with $q_1, \ldots, q_m$ running over Q) will be denoted by S_m.

Let H be the set of all integer valued functions defined on the set S. The set H is a group under addition. For $h_1, h_2 \in H$ we put $h_1 \leqq h_2$ if $h_1(x) \leqq h_2(x)$ for each $x \in S$. Then H is a lattice ordered group.

Let $h \in H$ and let $m, n \in N$. We put

$$s(h, m, n) = \{Y \in S_m : |h(y)| > n \text{ for some } y \in Y\},$$
$$s_0(h, m) = \inf\{\text{card } s(h, m, n)\}_{n \in N}.$$

Let $Y \in S_m$. We denote by $F(Y)$ the set of all $h \in H$ such that $h(x) = 0$ for each $x \in S \setminus Y$. Further, let $F_c(Y)$ be the set of all $h \in F(Y)$ that are constant on Y (i.e., $h(x_1) = h(x_2)$ for each pair $x_1, x_2 \in Y$).

Let us denote by H^1 the set of all $h_1 \in H^+$ such that either $h_1 = 0$ or h_1 can be written as

$$h_1 = \bigvee_{i \in I} h_i$$

where $\{h_i\}$ $(i \in I)$ is a disjoint subset of H such that for each $i \in I$ there exist $m(i) \in N$ and $Y_i \in S_{m(i)}$ with $h_i \in F_c(Y_i)$. We denote by H^2 the set of all $h \in H$ such that both h^+ and h^- belong to H^1. Then H^2 is an l-subgroup of H. Let $m \in N$, $Y \in S_m$; we put

$$H^2(Y) = F(Y) \cap H^2 .$$

Then $H^2(Y)$ is a direct factor of H^2. Moreover, from the construction of H^2 it follows that H^2 is orthogonally complete and thus for each polar $X \neq \{0\}$ in H^2 there is $0 < h_1 \in X$ such that $X = \{h_1\}^{\delta\delta}$. Hence each polar of H^2 is principal.

Let h_1, h_i, X be as above. Let $0 < h \in H^2$. There are subsets Y_j $(j \in J)$ of S such that $I \cap J = \emptyset$, each Y_j belongs to some $S_{m(j)}$ and there are elements $h_j \in F_c(Y_j)$ such that $\{h_j\}$ is a disjoint subset of H^2 and

$$h = \bigvee_{j \in J} h_j .$$

If $i \in I, j \in J$, then either $Y_i \cap Y_j = \emptyset$ or there is $m(i, j) \in N$ such that $Y_i \cap Y_j \in S_{m(i,j)}$.

We define $h' \in H$ as follows. Let $x \in S$. If there are $i \in I$ and $j \in J$ such that $x \in$ $\in Y_i \cap Y_j$, then we put

$$h'(x) = h(x) ;$$

otherwise we set $h'(x) = 0$.

It is not hard to verify that h' belongs to X and that h' is the greatest element of the set $\{h'' \in X : h'' \leqq h\}$. Hence X is a direct factor of H^2. Thus we have the following

5.1. Lemma. *The lattice ordered group H^2 is strongly projectable.*

Let H^0 be the set of all elements $h \in H^2$ such that

$$s_0(h, m) \leqq \alpha_m$$

is valid for each $m \in N$. Then H^0 is a convex l-subgroup of H^2. Since each convex l-subgroup of a strongly projectable lattice ordered group is again strongly projectable, 5.1 yields:

5.2. Lemma. *The lattice ordered group H^0 is strongly projectable.*

For $m \in N$, $Y \in S_m$ put $F(Y) \cap H^0 = H^0(Y)$. Clearly $H^0(Y)$ is a direct factor of H^0. Let us denote by G the Dedekind completion of H^0. For $m \in N$, $Y \in S_m$ we set

$$G(Y) = (H^0(Y))^{\delta\delta} .$$

5.3. Lemma. *For each $m \in N$ and each $Y \in S_m$ we have*

(a) *$G(Y)$ is direct factor of G;*

(b) *$G(Y)$ is a Dedekind completion of $H^0(Y)$.*

Proof. In [8] (Thm. 2.6) it has been shown that if B is the Dedekind completion of an archimedean lattice ordered group A and if $A = A_1 \times A_2$, then $B = B_1 \times B_2$, where B_i is the convex l-subgroup of B generated by A_i, and B_i is the Dedekind completion of A_i $(i = 1, 2)$. Thus to prove the assertion of the lemma it suffices to verify that (under the above notation) we have $B_1 = (A_1)^{\delta\delta}$ (the symbol δ being taken with respect to B).

Let $0 < x \in B_1$. Since B_1 is the Dedekind completion of A_1, there is $a \in A_1$ with $x \leqq a$. Because $(A_1)^{\delta\delta}$ is a convex l-subgroup of B and $a \in (A_1)^{\delta\delta}$, we get $x \in (A_1)^{\delta\delta}$. Hence $B_1 \subseteq (A_1)^{\delta\delta}$. Conversely, suppose that $0 < x \in (A_1)^{\delta\delta}$. Then there are elements $0 \leqq b_i \in B_i$ $(i = 1, 2)$ with $x = b_1 + b_2$. Further, there is $a_2 \in A_2$ with $b_2 \leqq a_2$. From $A = A_1 \times A_2$ it follows that $a_2 \in A_1^\delta$, hence $b_2 \in A_1^\delta$ and thus $x \wedge b_2 = 0$. Since $x \wedge b_2 = b_2$, we obtain $b_2 = 0$, therefore $x = b_1 \in B_1$. Hence $(A_1)^{\delta\delta} \subseteq B_1$.

5.4. Lemma. *Let $m \in N$, $Y \in S_m$. Let γ be a cardinal with $\alpha_{m+1} < \gamma \leqq \beta$. Then there is a disjoint subset $\{h_{i,n}\}$ $(i \in I,\ n \in N)$ in $G(Y)$ such that*

(i) card $I = \gamma$;

(ii) *the set $\{h_{i,n}\}$ $(i \in I,\ n \in N)$ is not upper bounded in $G(Y)$.*

Proof. There exists a system $\{Z_{i,n}\}$ $(i \in I,\ n \in N)$ of mutually disjoint nonempty sets $\{Z_{i,n}\}$ such that card $I = \gamma$, each $Z_{i,n}$ belongs to S_{m+1} and is a subset of Y. For $i \in I$ and $n \in N$ we define $h_{i,n} \in H$ as follows:

$h_{i,n}(x) = n$ if $x \in Z_{i,n}$ and $h_{i,n}(x) = 0$ otherwise. Then each $h_{i,n}$ belongs to $G(Y)$ and obviously $\{h_{i,n}\}$ is a disjoint system.

Suppose that the system $\{h_{i,n}\}$ has an upper bound g in $G(Y)$. From the assertion (b) of 5.3 it follows that there exists $h_0 \in H^0(Y)$ with $g \leqq h_0$. Let h be the join of the system $\{h_{i,n}\}$ in H (this join exists since H is orthogonally complete). We have $h \leqq h_0$. In view of the construction of the system $\{h_{i,n}\}$ we get

$$s_0(h, m + 1) = \gamma > \alpha_{m+1}$$

and thus

$$s_0(h_0, m + 1) > \alpha_{m+1}\,,$$

hence h_0 does not belong to $H^0(Y) \subseteq H^0$, which is a contradiction.

For $\emptyset \neq Z \subseteq H^0$ we denote

$$Z^\beta = \{h^0 \in H^0 : |h^0| \wedge |z| = 0 \quad \text{for each} \quad z \in Z\}\,.$$

5.5. Lemma. *Let $m \in N$, $Y \in S_m$. Let γ be an infinite cardinal with $\gamma \leqq \alpha_{m+1}$. Let $\{h_{i,n}\}$ $(i \in I,\ n \in N)$ be a disjoint subset of $G(Y)$ such that* card $I = \gamma$. *Then the set $\{h_{i,n}\}$ is upper bounded in $G(Y)$.*

Proof. For each $i \in I$ and each $n \in N$ let

$$X_{i,n} = \{h_0 \in H^0 : 0 \leqq h_0 \leqq h_{i,n}\}, \quad Y_{i,n} = (X_{i,n})^{\beta\beta}.$$

According to 5.2, each $Y_{i,n}$ is a direct factor of H^0. By the same argument as in the proof of 5.3 we obtain that $(Y_{i,n})^{\delta\delta}$ is a direct factor of G and the Dedekind completion of $Y_{i,n}$.

We have $h_{i,n} = \sup X_{i,n}$ in G, $X_{i,n} \subset (Y_{i,n})^{\delta\delta}$, and since $(Y_{i,n})^{\delta\delta}$ is a closed l-subgroup of G, we get $h_{i,n} \in (Y_{i,n})^{\delta\delta}$. This together with the fact that $(Y_{i,n})^{\delta\delta}$ is the Dedekind completion of $Y_{i,n}$ implies that there is $h'_{i,n} \in Y_{i,n}$ with

$$h_{i,n} \leqq h'_{i,n}.$$

The disjointness of the system $\{h_{i,n}\}$ implies that the system $\{h'_{i,n}\}$ is a disjoint subset of H^0 and clearly $\{h'_{i,n}\} \subset H^0(Y)$. Since the cardinality of the set $\{h'_{i,n}\}$ is $\gamma \leqq \alpha_{m+1}$, the least upper bound of the set $\{h'_{i,n}\}$ in $H^0(Y)$ exists. Thus the set $\{h_{i,n}\}$ is upper bounded in $G(Y)$.

5.6. Lemma. *Let $m, k \in N$, $m < k$, $Y_1 \in S_m$, $Y_2 \in S_k$. Then $G(Y_1)$ is not isomorphic with $G(Y_2)$.*

The proof follows from 5.4 and 5.5.

5.7. Lemma. *The lattice ordered group G is totally b-inhomogeneous and* card $G > \alpha$.

Proof. As we have shown above, for each $m \in N$ and each cardinal γ with $\alpha_m < \gamma \leqq \beta$ there exists a disjoint set of elements of G such that the cardinality of this set is γ. Since $\alpha < \alpha_m$, we get card $G > \alpha$.

Let $A \neq \{0\}$ be a direct factor of G. Thus there is $0 < a \in A$. Hence there exists $h \in H^0$ such that $0 < h \leqq a$. From the definition of H^0 it follows that there are $m \in N$ and $Y_1 \in S_m$ such that $h(y) = h(y') \neq 0$ for each pair of elements $y, y' \in Y_1$. From this we obtain that $H^0(Y_1) \subseteq A$ and therefore $G(Y_1) \subseteq A$.

Let $k > m$. Choose $Y_2 \in S_k$ such that $Y_2 \subset Y_1$. Then $H^0(Y_2) \subset H^0(Y_1)$ and thus $G(Y_2) \subseteq G(Y_1) \subseteq A$. Both $G(Y_1)$ and $G(Y_2)$ are direct factors of G and according to 5.6, $G(Y_1)$ is not isomorphic with $G(Y_2)$. Hence A fails to be isomorphic either with $G(Y_1)$ or with $G(Y_2)$. Therefore G is totally b-inhomogeneous.

Let $e \in H$ such that $e(x) = 1$ for each $x \in S$. The principal l-ideal of H generated by e will be denoted by H_b. Let $m \in N$, $Y \in S_m$. We put

$$H_b^0 = H^0 \cap H_b, \quad H^0(Y)_b = H^0(Y) \cap H_b.$$

5.8. Lemma. *The Boolean algebras $P(H^0)$ and $P(H_b^0)$ are isomorphic.*

Proof. The element e is a weak unit in both lattice ordered groups H^0 and H_b^0.

Since the interval $[0, e]$ of H^0 is a subset of H_b^0, it follows from 3.5 that $P(H^0)$ and $P(H_b^0)$ are isomorphic.

5.9. Lemma. *The lattice ordered groups H_b^0 and $H^0(Y)_b$ are isomorphic.*

Proof. According to the definition of S_m there are elements $q_1, \ldots, q_m \in Q$ such that Y is the set of all sequences s of the form

$$q_1, q_2, \ldots, q_m, p_1, p_2, p_3, \ldots$$

with p_i running over Q for each $i \in N$. Let $p = \{p_n\}_{n \in N} \in S$. Consider the mapping $\varphi(p) = s$, where s is as above. Then $\varphi : S \to Y$ is a bijection. Let $h \in H_b^0$. Let us define a function h' on the set S as follows. For each $x \in S \setminus Y$ let $h'(x) = 0$. For each $y \in Y$ we put

$$h'(y) = h(\varphi^{-1}(y)) .$$

Then $h' \in H^0(Y)_b$. From the construction of H^0 and H_b^0 it follows that the mapping $\psi : H_b^0 \to H^0(Y)_b$ defined by

$$\psi(h) = h' \quad \text{for each} \quad h \in H_b^0$$

is an isomorphism of H_b^0 into $H^0(Y)_b$.

We need the following result of Sikorski [11]:

(S) *Let B_1, B_2 be σ-complete Boolean algebras. Suppose that there exists an isomorphism of B_1 onto an ideal of B_2 and an isomorphism of B_2 onto an ideal of B_1. Then B_1 and B_2 are isomorphic.*

5.10. Lemma. *The Boolean algebra $P(H_b^0)$ is homogeneous.*

Proof. Since e is a weak unit in H_b^0, according to 3.5 it suffices to verify that the center $C(e)$ of the interval $[0, e]$ in H_b^0 is homogeneous. Let $0 < e_0 \in C(e)$. There exist $m \in N$ and $Y \in S_m$ such that $e_0(y) = e_0(y') \neq 0$ for each pair $y, y' \in Y$. Put

$$e_1 = e_0(H^0(Y)) .$$

Then $e_1 \in H^0(Y)_b$ and, moreover, $0 < e_1 \in C(e)$, $e_1 \leqq e_0$. Let ψ be as in 5.9. We have obviously

$$\psi(e) = e_1 ,$$

hence ψ is an isomorphism of $C(e)$ onto $C(e_1)$. Because $C(e_1)$ is a sublattice of $C(e_0)$, ψ is an isomorphism of $C(e)$ into $C(e_0)$. Let ψ_1 be the identical mapping on $C(e_0)$; hence ψ_1 is an isomorphism of $C(e_0)$ into $C(e)$. Clearly $C(e_0)$ is an ideal in $C(e)$ and $C(e_1)$ is an ideal in $C(e_0)$. Because $C(e)$ is complete (being isomorphic with $P(H_0^b)$), we obtain from (S) that $C(e)$ is isomorphic with $C(e_0)$. Hence $C(e)$ is homogeneous.

5.11. Lemma. *The Boolean algebra $P(G)$ is homogeneous.*

Proof. From 5.8 and 5.10 it follows that $P(H_0)$ is homogeneous. For each archimedean lattice ordered group G_1 the Boolean algebras $P(G_1)$ and $P(D(G_1))$ are isomorphic, where $D(G_1)$ is the Dedekind completion of G_1. Hence $P(G)$ is isomorphic with $P(H^0)$. Thus $P(G)$ is homogeneous.

5.12. Theorem. *For each cardinal α there exists a complete lattice ordered group G such that*

(i) card $G > \alpha$,

(ii) *G is totally b-inhomogeneous,*

(iii) *the Boolean algebra $P(G)$ is homogeneous.*

The proof follows from 5.7 and 5.11.

References

[1] *G. Birkhoff:* Lattice theory, third edition, Providence 1967.

[2] *L. Bukovský:* Characterization of generic extensions of models of set theory, Fund. Math. *83* (1973), 35—46.

[3] *P. Conrad, D. McAlister:* The completion of a lattice ordered group. J. Austral. Math. Soc. *9* (1969), 182—208.

[4] *Л. Фукс:* Частично упорядоченные алгебраические системы, Москва 1965.

[5] *J. Jakubík:* Center of a complete lattice. Czech. Math. J. *23* (1973), 125—138.

[6] *J. Jakubík:* Cantor-Bernstein theorem for lattice ordered groups. Czech. Math. J. *22* (1972), 159—175.

[7] *J. Jakubík:* Homogeneous lattice ordered groups. Czech. Math. J. *22* (1972), 325—337.

[8] *J. Jakubík:* Generalized Dedekind completion of a lattice ordered group. Czech. Math. J. *28* (1978), 294—311.

[9] *K. MacAloon:* Consistency results about ordinal definability, Ann. Math. Logic *2* (1971), 449—467.

[10] *R. S. Pierce:* Some questions on Boolean algebras. Proc. Symp. Pure Math. Vol. 2, Lattice theory, Amer. Math. Soc., 1961, 129—140.

[11] *R. Sikorski:* A generalization of theorem of Banach and Cantor-Bernstein. Coll. Math. *1* (1948), 140—144.

[12] *F. Šik:* Über subdirekte Summen geordneter Gruppen. Czech. M. J. *10* (1960), 400—424

[13] *Ф. Шик:* К теории структурно упорядоченных групп, Чех. мат. ж. *6* (1956), 1—25.

[14] *Б. З. Вулих:* Введение в теорию полуупорядоченных пространств, Москва 1961.

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