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ORTHOGONAL HULL OF A STRONGLY PROJECTABLE LATTICE ORDERED GROUP

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Let G be a lattice ordered group. The underlying lattice will be denoted by $l(G)$. A lattice ordered group is said to be *strongly projectable* if each its polar is a direct factor. We denote by $o(G)$ the orthogonal hull of G. The following result has been proved in [8]:

(A) *Let G_1 and G_2 be complete lattice ordered groups such that $l(G_1)$ is isomorphic with $l(G_2)$. Then $l(o(G_1))$ is isomorphic with $l(o(G_2))$.*

In this note the following theorem will be established:

(A′) *Let G_1 and G_2 be lattice ordered groups such that $l(G_1)$ is isomorphic with $l(G_2)$. Suppose that G_1 is strongly projectable. Then*

(i) *G_2 is strongly projectable;*

(ii) *$l(o(G_1))$ is isomorphic with $l(o(G_2))$.*

1. PRELIMINARIES

We shall use the standard notation for lattice ordered groups (cf. BIRKHOFF [3], FUCHS [6] and CONRAD [4]). Let G be a lattice ordered group, $\emptyset \neq X \subseteq G$. The set

$$X^\delta = \{y \in G : |y| \wedge |x| = 0 \text{ for each } x \in X\}$$

is said to be *a polar of* G. We put $(X^\delta)^\delta = X^{\delta\delta}$. If $X = \{x\}$ is a one-element set, then we denote $\{x\}^{\delta\delta} = [x]$; $[x]$ is called *a principal polar*. Each polar is a convex l-subgroup of G.

A polar Y is said to be *a direct factor of* G if for each $0 \leqq z \in G$ the set $\{z_1 \in Y : : z_1 \leqq z\}$ possesses a greatest element z_0. In such a case we put $z(Y) = z_0$ and for any $t \in G$ we denote $t(Y) = t^+(Y) - t^-(Y)$; the element $t(Y)$ is called *the component of t in Y*. If Y is a direct factor of G, then the mapping $t \to t(Y)$ is a homomorphism of G onto Y and Y^δ is a direct factor of G as well; for each $t \in G$ we have $t = t(Y) +$

$+ t(Y^\delta)$ and $t \geqq 0$ if and only if $t(Y) \geqq 0$, $t(Y^\delta) \geqq 0$. Under the above assumptions we write $G = Y \oplus Y^\delta$.

If X and Y are direct factors of G, then $X \cap Y$ is also a direct factor of G and for each $t \in G$ we have

$$t(X \cap Y) = (t(X))(Y) = (t(Y))(X).$$

If $X = [x]$ is a direct factor, then we write $t[x]$ instead of $t([x])$.

Let I be a nonempty set. A system $\{g_i\}_{i \in I}$ of elements of G will be called *disjoint* if $g_i \geqq 0$ for each $i \in I$ and $g_i \wedge g_j = 0$ whenever i and j are distinct elements of I. The lattice ordered group G is called *orthogonally complete* if each disjoint system in G possesses the least upper bound in G.

An l-subgroup A of G is said *to be dense* if for each $0 < g \in G$ there exists $a \in A$ with $0 < a \leqq g$.

Let G and G' be lattice ordered groups such that

(i) G is a dense l-subgroup of G';

(ii) G' is orthogonally complete;

(iii) if G'' is an l-subgroup of G' with $G \subseteq G''$ and if G'' is orthogonally complete, then $G'' = G'$.

Under these assumptions G' is said to be *an orthogonal hull of* G. Each lattice ordered group possesses an orthogonal hull and this is defined uniquely up to isomorphism (cf. Bernau [1]; for representable lattice ordered groups this was proved by Conrad [5] and for complete lattice ordered groups by Pinsker [10] and Nakano [9]).

If G is archimedean and orthogonally complete, then it is strongly projectable (Bernau [2] and Rotkovič [11]); a non-archimedean orthogonally complete lattice ordered group need not be strongly projectable (cf. Ex. 6.1 below).

2. STRONG PROJECTABILITY

Let G_1 and G_2 be lattice ordered groups such that G_1 is strongly projectable. Assume that φ is an isomorphism of the lattice $l(G_1)$ onto $l(G_2)$. Then the mapping ψ defined by

$$\psi(x) = \varphi(x) - \varphi(0)$$

is an isomorphism of the lattice $l(G_1)$ onto $l(G_2)$ fulfilling $\psi(0) = 0$.

Let us remark that the notion of strong projectability of G_1 is defined by means of properties of polars, and defining polars we used the operation $|x|$ for $x \in G$. When proving the strong projectability of G_2 we could attempt to use the relation

$$\psi(|x|) = |\psi(x)| \tag{1}$$

for elements $x \in G_1$. However, this method is impossible, since (1) fails to be valid in general (cf. Ex. 6.2 below).

Let G be a lattice ordered group. Consider the lattice ordered semigroup $(G^+; +, \leqq)$. Let P, Q be subsets of G^+ with the following properties:

(i) P and Q are subsemigroups and sublattices in G^+;

(ii) for each $g \in G^+$ there are uniquely determined elements $g_1 \in P$ and $g_2 \in Q$ with $g = g_1 + g_2 = g_2 + g_1$;

(iii) if $x, y \in G^+$, $x_1, y_1 \in P$, $x_2, y_2 \in Q$, $x = x_1 + x_2$, $y = y_1 + y_2$, then $x \circ y = (x_1 \circ x_2) + (y_1 \circ y_2)$ for each $\circ \in \{+, \wedge, \vee\}$.

Under these assumptions the lattice ordered semigroup G^+ will be said to be *a direct sum of P and Q*; we write $G^+ = P \oplus Q$.

The following result is well-known (cf. Šimbireva [14]).

2.1. Theorem. *Let $P, Q \subseteq G^+$ with $G^+ = P \oplus Q$. Then there are l-subgroups P' and Q' in G such that $P = (P')^+$, $Q = (Q')^+$ and $G = P' \oplus Q'$.*

2.2. Lemma. *Let P, Q be convex sublattices of the lattice $(G^+; \leqq)$ with $P \cap Q = \{0\}$. Assume that for each $g \in G$ there exist $p \in P$ and $q \in Q$ such that $g = p \vee q$. Then $G^+ = P \oplus Q$.*

Proof. From $P \cap Q = \{0\}$ and from the convexity of the sublattices P, Q we infer that $p \wedge q = 0$ for each $p \in P$ and each $q \in Q$. Let $p_1, p_2 \in P$. Then $p_1 + p_2 \in G^+$, hence there are elements $p \in P$ and $q \in Q$ with $p_1 + p_2 = p \vee q$. Since $p_i \wedge q = 0$ $(i = 1, 2)$, we have $q = q \wedge (p_1 + p_2) = 0$ and therefore $p_1 + p_2 = p \in P$. Thus P is a subsemigroup of G^+. Analogously, Q is a subsemigroup of G^+.

Let $g \in G$, $p, p_1 \in P$, $q, q_1 \in Q$, $g = p \vee q = p_1 \vee q_1$. Then

$$p = p \wedge g = p \wedge (p_1 \vee q_1) = p \wedge p_1 ,$$

and similarly we obtain $p_1 = p \wedge p_1$. Hence $p = p_1$ and analogously $q = q_1$. Therefore in the expression $g = p \vee q$ $(p \in P,\ q \in Q)$ the elements p and q are uniquely determined by g. Moreover, from $p \wedge q = 0$ it follows that

$$g = p \vee q = p + q = q + p . \tag{2}$$

Hence the conditions (i) and (ii) are valid. The condition (iii) is an easy consequence of (2).

2.3. Theorem. *Let G_1 and G_2 be lattice ordered groups such that the lattices $l(G_1)$ and $l(G_2)$ are isomorphic. Suppose that G_1 is strongly projectable. Then G_2 is strongly projectable.*

Proof. As we have already remarked above, there exists an isomorphism ψ of $l(G_1)$ onto $l(G_2)$ such that $\psi(0) = 0$. Let Y be a polar in G_2, $Z = Y^\delta$. Put $P = Y^+$, $Q = Z^+$, $P_1 = \psi^{-1}(P)$, $Q_1 = \psi^{-1}(Q)$. From the definition of P and Q and from the isomorphism ψ^{-1} we obtain that

$$P_1 = \{g \in G_1^+ : g \wedge q_1 = 0 \text{ for each } q_1 \in Q_1\},$$

$$Q_1 = \{g \in G_1^+ : g \wedge p_1 = 0 \text{ for each } p_1 \in P_1\}.$$

Hence there are polars Y_1 and Z_1 in G_1 such that $Z_1 = Y_1^\delta$, $P_1 = Y_1^+$ and $Q_1 = Z_1^+$.

Since G_1 is strongly projectable, we have $G_1 = Y_1 \oplus Z_1$. From this we obtain immediately that $G_1^+ = P_1 \oplus Q_1$. Thus if $g_1 \in G_1^+$, then there are elements $p_1 \in P_1$ and $q_1 \in Q_1$ with $g_1 = p_1 + q_1$. Moreover, $P_1 \cap Q_1 = \{0\}$ and hence $g_1 = p_1 \vee g_1$. Clearly $P \cap Q = \{0\}$ and from the isomorphism ψ it follows that for each $g \in G_2^+$ there are $p \in P$ and $q \in Q$ with $g = p \vee q$. The sets P and Q are convex sublattices of the lattice $(G_2^+; \leqq)$. Thus according to 2.2, $G_2^+ = P \oplus Q$.

Now according to 2.1 there are l-subgroups P' and Q' of G_2 such that $P = (P')^+$, $Q = (Q')^+$ and $G_2 = P' \oplus Q'$. Since each l-subgroup of G_2 is uniquely determined by its positive cone, we obtain $P' = Y$, $Q' = Z$. Therefore $G_2 = Y \oplus Z$. Thus G_2 is strongly projectable.

Let $x. y \in G$ $x \geqq 0$ $y \leqq 0$. We put $x \,\delta\, y$ if there exists $z \in G$ such that $z \wedge 0 = y$, $z \vee 0 = x$.

2.4. Lemma. *Let $a, b \in G$. The following conditions are equivalent:*

(i) $|a| \wedge |b| = 0$.

(ii) $(a \vee 0) \wedge (b \vee 0) = 0$, $(a \wedge 0) \vee (b \wedge 0) = 0$, $(a \vee 0)\, \delta\, (b \wedge 0)$, $(b \vee 0)\, \delta$ $\delta\, (a \wedge 0)$.

Proof. Let (i) be valid. We have $a \in [a]$, $b \in [a]^\delta$. Hence $a \wedge 0$, $a \vee 0 \in [a]$ and $b \wedge 0$, $b \vee 0 \in [a]^\delta$. Thus $(a \vee 0) \wedge (b \vee 0) = 0$ and $(a \wedge 0) \vee (b \wedge 0) = 0$. Put $z = (a \vee 0) + (b \wedge 0)$. It is a routine to verify that $z \wedge 0 = b \wedge 0$, $z \vee 0 = a \vee 0$. Therefore $(a \vee 0)\, \delta\, (b \wedge 0)$. Analogously, $(b \vee 0)\, \delta\, (a \wedge 0)$.

Conversely, assume that (ii) holds. Hence there are elements z_1, z_2 in G such that z_1 is the relative complement of 0 in the interval $[b \wedge 0, a \vee 0]$ and z_2 is the relative complement of 0 in the interval $[a \wedge 0, b \vee 0]$. Thus $z_1^+ = a \vee 0$, $-z_1^- = b \wedge 0$, $z_2^+ = b \vee 0$, $-z_2^- = a \wedge 0$. Because $z_1^+ \wedge z_1^- = 0 = z_2^+ \wedge z_2^-$ and $|a| = (a \wedge 0) \vee (-(a \wedge 0))$, $|b| = (b \wedge 0) \vee (-(b \wedge 0))$, we easily obtain that $|a| \wedge |b| = 0$.

2.5. Lemma. *Let G_1 and G_2 be lattice ordered groups and let ψ be an isomorphism of $l(G_1)$ onto $l(G_2)$ such that $\psi(0) = 0$. Let A be a polar in G_1. Then $\psi(A)$ is a polar in G_2 and $\psi(A^\delta) = (\psi(A))^\delta$.*

Proof. From 2.4 it follows that for each set $\emptyset \neq M \subseteq G_1$ the polar M^δ can be constructed by using merely the set M, the element 0 and the lattice operations in $l(G_1)$. Hence $\psi(M^\delta) = (\psi(M))^\delta$ holds. This implies $(\psi(A))^{\delta\delta} = \psi(A^{\delta\delta}) = \psi(A)$, thus $\psi(A)$ is a polar in G_2.

Let A, B be lattices. Their direct product will be denoted by $A \times B$ (cf. [3]). Let L be a lattice and let φ be an isomorphism of L onto $A \times B$. Let $x_0 \in L$, $\varphi(x_0) = (a_0, b_0)$. Put

$$A^0 = \varphi^{-1}(\{(a, b_0) : a \in A\}), \quad B^0 = \varphi^{-1}(\{(a_0, b) : b \in B\}).$$

Then we shall write $L = A^0 \otimes B^0$. Clearly $A^0 \cap B^0 = \{x_0\}$.

We need the following result:

2.6. Theorem. (Cf. [7], Thm. 3) *Let G be a lattice ordered group, $A^0 \subseteq G$, $B^0 \subseteq G$, $A^0 \cap B^0 = \{0\}$. Assume that $l(G) = A^0 \times B^0$. Then A^0 and B^0 are l-subgroups of G and $G = A^0 \oplus B^0$.*

2.7. By using 2.6 we obtain an alternative proof of 2.3:

Let G_1 and G_2 be lattice ordered groups and suppose that $l(G_1)$ is isomorphic with $l(G_2)$. Then there is an isomorphism ψ of $l(G_1)$ onto $l(G_2)$ such that $\psi(0) = 0$. Let P be a polar in G_2. Put $Q = P^\delta$,

$$A = \psi^{-1}(P), \quad B = \psi^{-1}(Q).$$

According to 2.5, A and B are polars in G_1 and $B = A^\delta$. Assume that G_1 is strongly projectable. Hence $G_1 = A \oplus B$. This yields

$$l(G_1) = A \otimes B. \tag{$*$}$$

The relation $(*)$ and the isomorphism ψ implies that

$$l(G_2) = P \otimes Q$$

is valid. Since $P \cap Q = \{0\}$, from 2.6 we infer that $G_2 = P \oplus Q$ holds. Therefore G_2 is strongly projectable.

3. THE LATTICE H

In this section we assume that G is a strongly projectable lattice ordered group. The general idea of the method to be used for constructing the orthogonal hull of G is analogous to that used in [8] for complete lattice ordered groups.

We denote by H_1 the system of all disjoint subsets of G. For $h_1 = \{x_i\}_{i \in I} \in H_1$ and $h_2 = \{y_j\}_{j \in J} \in H_1$ we put $h_1 \leqq h_2$ if for each $i \in I$ the relation

$$x_i = \bigvee_{j \in J}(x_i \wedge y_j) \tag{3}$$

is valid. Obviously $h_1 \leqq h_1$ for each $h_1 \in H_1$.

3.1. Lemma. *$(H_1, \leqq)$ is a quasiordered set.*

Proof. We have to verify that the relation $\leqq$ on H_1 is transitive. Let h_1, h_2 be as above and let $h_3 = \{z_k\}_{k \in K} \in H_1$. Suppose that $h_1 \leqq h_2$ and $h_2 \leqq h_3$ is valid. Hence for each $i \in I$ we have

$$x_i = \bigvee_{j \in J}(x_i \wedge y_j) = \bigvee_{j \in J}(x_i \wedge \bigvee_{k \in K}(y_j \wedge z_k)) = \\ = \bigvee_{j \in J}\bigvee_{k \in K}(x_i \wedge y_j \wedge z_k) .$$

From this and from the obvious inequality

$$x_i \wedge y_j \wedge z_k \leqq x_i \wedge z_k \leqq x_i$$

we infer that

$$x_i = \bigvee_{k \in K}(x_i \wedge z_k)$$

holds for each $i \in I$. Thus $h_1 \leqq h_3$.

For $h_1, h_2 \in H_1$ we put $h_1 = h_2$ if $h_1 \leqq h_2$ and $h_2 \leqq h_1$. Let H be the corresponding set of equivalence classes in H_1; then $(H; \leqq)$ is a partially ordered set. The equivalence class containing $\{x_i\}_{i \in I} \in H_1$ will be denoted by $S_{i \in I}\{x_i\}$. Let $x, y \in H$,

$$x = S_{i \in I}\{x_i\} , \quad y = S_{j \in J}\{y_j\} .$$

3.2. Lemma. *Suppose that for each $i \in I$ there exists $j(i) \in J$ with $x_i \leqq y_{j(i)}$. Then $x \leqq y$.*

Proof. Let $i \in I$. Since $x_i = x_i \wedge y_{j(i)}$, the relation (3) obviously holds.

Now let us denote

$$\begin{aligned} X' &= \{x_i : i \in I\}^\delta , & Y' &= \{y_j : j \in J\}^\delta , \\ X &= (X')^\delta , & Y &= (Y')^\delta , \\ x_{ij} &= x_i[y_j] , & y_{ji} &= y_j[x_i] , \\ x_i' &= x_i(Y') , & y_j' &= y_j(X') , \\ x_i^0 &= x_i(Y) , & y_j^0 &= y_j(X) \end{aligned}$$

for each $i \in I$ and each $j \in J$.

An element $0 < e$ of a lattice ordered group G is said to be *a weak unit in* G if $0 < e \wedge g$ for each $0 < g \in G$. For each $0 < e \in G$, e is a weak unit in $[e]$. If e is a weak unit in G, then $[e] = G$. If e is a weak unit in G and A is a direct factor in G, then $e(A)$ is a weak unit in A.

3.3. Lemma. $[x_{ij}] = [y_{ji}]$ *for each $i \in I$ and each $j \in J$.*

Proof. If $x_{ij} = 0$, then $x_i \wedge y_j = 0$, hence $y_{ji} = 0$, and conversely. Let $x_{ij} > 0$. Then $x_i > 0$ and x_i is a weak unit in $[x_i]$. We have $x_{ij} \in [x_i] \cap [y_j]$ and

$$x_{ij} = x_i[y_j] = (x_i[x_i])\,[y_j] = x_i([x_i] \cap [y_j]),$$

hence x_{ij} is a weak unit in $[x_i] \cap [y_j]$. Thus $[x_{ij}] = [x_i] \cap [y_j]$. Analogously we obtain $[y_{ij}] = [x_i] \cap [y_j]$ and hence $[x_{ij}] = [y_{ji}]$.

3.4. Lemma. *For each $i \in I$ we have*

$$x_i^0 = \bigvee_{j \in J} x_{ij} \,. \tag{4}$$

Proof. Let $i \in I$. For each $j \in I$ the relation $[y_j] \subseteq Y$ is valid and hence we obtain

$$x_{ij} = x_i[y_j] \leqq x_i(Y) = x_i^0 \,.$$

Let $t \in G$ such that $t \leqq x_i^0$ and $x_{ij} \leqq t$ for each $j \in J$. Put $z = -t + x_i^0$. Thus $z \geqq 0$.

Suppose that $z[y_j] > 0$ for some $j \in J$. Hence

$$x_i[y_j] = x_{ij} < x_{ij} + z[y_j] \leqq t + z = x_i^0 \leqq x_i$$

and $x_{ij} + z[y_j] \in [y_j]$. This is a contradiction. Thus $z[y_j] = 0$ for each $j \in J$. Therefore $z \in Y'$. At the same time, from $0 \leqq z \leqq x_i^0 \in Y$ we get $z \in Y$. Thus $z = 0$ and hence (4) is valid.

3.5. Corollary. *For each $i \in I$ and each $j \in J$ we have*

$$x_i = (\bigvee_{j \in J} x_{ij}) \vee x_i' \,, \tag{5}$$

$$y_j = (\bigvee_{i \in I} y_{ji}) \vee y_j' \,. \tag{5'}$$

It is easy to verify that the sets

$$\{x_{ij}, x_i'\}_{i \in I, j \in J} \,, \quad \{y_{ji}, y_j'\}_{i \in I, j \in J}$$

belong to H_1, hence $x_0 = S_{i \in I, j \in J}\{x_{ij}, x_i'\}$ and $y_0 = S_{i \in I, j \in J}\{y_{ji}, y_j'\}$ belong to H.

3.6. Lemma. $x = x_0$ *and* $y = y_0$.

Proof. For each $i \in I$ and each $j \in J$ we have $x_{ij} \leqq x_i$ and $x_i' \leqq x_i$. Hence from 3.2 we obtain $x_0 \leqq x$. From 3.5 and from the definition of the relation $\leqq$ in H it follows immediately that $x \leqq x_0$. Hence $x = x_0$. Analogously we can verify that $y = y_0$ is valid.

3.7. Lemma. $x \leqq y$ *if and only if* $x_i' = 0$ *and* $x_{ij} \leqq y_{ji}$ *for each* $i \in I$ *and each* $j \in J$.

Proof. Let $x \leqq y$ and let $i \in I, j \in J$. From (3) we obtain $x_i \in Y$ (since Y is a closed l-subgroup of G) and thus $x_i' = x_i(Y') = 0$. For each $k \in J$, $x_{ij} \wedge y_k' = 0$. If $k \in J$ and $s \in I$ such that $s \neq i$ or $j \neq k$, then $x_{ij} \wedge y_{ks} = 0$. Thus from 3.6 we get $x_{ij} \leqq y_{ji}$.

Conversely, assume that $x'_i = 0$ and $x_{ij} \leqq y_{ji}$ for each $i \in I$ and each $j \in J$. Then from 3.2 and 3.6 we infer that $x \leqq y$.

The system $\{x_{ij} \wedge y_{ji}\}_{i\in I, j\in J}$ obviously belongs to H_1. Denote

$$z = S_{i\in I, j\in J}\{x_{ij} \wedge y_{ji}\} .$$

3.8. Lemma. $x \wedge y = z$.

Proof. According to 3.2 and 3.6 we have $z \leqq x$ and $z \leqq y$. Let $u \in H$, $u \leqq x$, $u \leqq y$, $u = S_{k\in K}\{u_k\}$. Let $k \in K$. From the definition of the partial order $\leqq$ on H we obtain

$$u_k = \bigvee_l (u_k \wedge t_l) \tag{6}$$

with t_l running over the set $\{x_{ij}, x'_i\}_{i\in I, j\in J}$, and

$$u_k = \bigvee_m (u_k \wedge s_m) \tag{7}$$

with s_m running over the set $\{y_{ji}, y'_j\}_{i\in I, j\in J}$.

Since each t_l belongs to X, (6) implies that $u_k \in X$ and hence $u_k \wedge y'_j = 0$ for each $j \in J$. Analogously, from (7) it follows that $u_k \wedge x'_i = 0$ for each $i \in I$. Thus (6) and (7) can be reduced to

$$u_k = \bigvee_{i\in I, j\in J}(u_k \wedge x_{ij}) ,$$

$$u_k = \bigvee_{i\in I, j\in J}(u_k \wedge y_{ji}) .$$

Hence

$$u_k = u_k \wedge u_k = (\bigvee_{i\in I, j\in J}(u_k \wedge x_{ij})) \wedge (\bigvee_{i_1\in I, j\in J}(u_k \wedge y_{j_1, i_1})) =$$

$$= \bigvee_{i\in I, j\in J}(u_k \wedge x_{ij} \wedge y_{ji}) .$$

Therefore $u \leqq z$ and so $z = x \wedge y$.

Let us consider the system $\{x_{ij} \vee y_{ji}, x'_i, y'_j\}$. This system is disjoint and thus

$$v = S_{i\in I, j\in J}\{x_{ij} \vee y_{ji}, x'_i\ y'_j\}$$

belongs to H.

3.9. Lemma. $v = x \vee y$.

Proof. 3.2 and 3.6 imply $x \leqq v$ and $y \leqq v$. Let $u = S_{k\in K}\{u_k\} \in H$ and assume that $x \leqq u$, $y \leqq u$. Hence from 3.6 we obtain

$$x_{ij} = \bigvee_{k\in K}(x_{ij} \wedge u_k) , \tag{8}$$

$$x'_i = \bigvee_{k\in K}(x'_i \wedge u_k) , \tag{9}$$

$$y_{ji} = \bigvee_{k\in K}(y_{ji} \wedge u_k) , \tag{10}$$

$$y'_j = \bigvee_{k\in K}(y'_j \wedge u_k) . \tag{11}$$

The relations (8) and (10) imply

$$x_{ij} \vee y_{ji} = \bigvee_{k\in K}((x_{ij} \vee y_{ji}) \wedge u_k). \tag{12}$$

From (9), (11) and (12) we obtain $v \leqq u$. Hence $v = x \vee y$.

We have verified that H is a lattice. If $x = S_{i\in I}\{x_i\} \in H$ and $I = \{i\}$ is a one-element set, then x can be identified with x_i and hence we can consider G^+ as a subset of H; then G^+ is obviously a sublattice of H and 0 is the least element of H.

4. THE SEMIGROUP OPERATION IN H

Let $\{x_i\}_{i\in I}$ and $\{y_j\}_{j\in J}$ be elements of H_1. If for each $i \in I$ there is $j(i) \in J$ with $x_i \leqq y_{j(i)}$ then we shall write $\{x_i\}_{i\in I} \prec \{y_j\}_{j\in J}$. Let x_{ij}, x'_i, y_{ji} and y'_j be as in § 3. The set

$$\{x_{ij} + y_{ji},\ x'_i,\ y'_j\}_{i\in I, j\in J} \tag{13}$$

is disjoint in G^+ and hence it belongs to H_1. We define a binary operation $+$ on H_1 by putting

$$\{x_i\}_{i\in I} + \{y_j\}_{j\in J} = \{x_{ij} + y_{ji},\ x'_i,\ y'_j\}_{i\in I, j\in J}.$$

The element $S_{i\in I, j\in J}\{x_{ij} + y_{ji}, x'_i, y'_j\}$ will be denoted also by $S(\{x_i\}_{i\in I} + \{y_j\}_{j\in J})$.

4.1. Lemma. *Let $\{x_i\}_{i\in I}$, $\{x^*_k\}_{k\in K}$, $\{y_j\}_{j\in J} \in H_1$. Assume that $\{x^*_k\}_{k\in K} \prec \{x_i\}_{i\in I}$ and $S_{i\in I}\{x_i\} = S_{k\in K}\{x^*_k\}$. Then*

$$S(\{x_i\}_{i\in I} + \{y_j\}_{j\in J}) = S(\{x^*_k\}_{k\in K} + \{y_j\}_{j\in J}), \tag{14}$$

$$\{x^*_k\}_{k\in K} + \{y_j\}_{j\in J} \prec \{x_i\}_{i\in I} + \{y_j\}_{j\in J}. \tag{14'}$$

Proof. Without loss of generality we can assume that the sets I, J and K are mutually disjoint. Let us consider the elements

$$x = S_{k\in K}\{x^*_k\}, \quad y = S_{j\in J}\{y_j\}.$$

According to 3.6 we can write

$$x = S_{k\in K, j\in J}\{x^*_{kj}, x^{*\prime}_k\},$$

$$y = S_{j\in J, k\in K}\{y_{jk}, y''_j\},$$

where the symbols x^*_{kj}, $x^{*\prime}_k$, y_{jk} and y''_j have analogous meanings as x_{ij}, x'_i, y_{ji} and y'_j in 3.6. Hence

$$\{x^*_k\}_{k\in K} + \{y_j\}_{j\in J} = \{x^*_{kj} + y_{jk},\ x^{*\prime}_k,\ y''_j\}. \tag{15}$$

Now let us compare the elements of (13) and (15). Let $j \in J$, $k \in K$. There exists $i(k) \in I$ with $x_k^* \leqq x_{i(k)}$. Hence

(16) $$x_{kj}^* = x_k^*[y_j] \leqq x_{i(k)}[y_j] = x_{i(k),j} .$$

Moreover,

$$[x_k^*] \subseteq [x_{i(k)}] ,$$

thus

(17) $$y_{jk} = y_j[x_k^*] \leqq y_j[x_{i(k)}] = y_{j,i(k)} .$$

From (16) and (17) we obtain

(18) $$x_{kj}^* + y_{jk} \leqq x_{i(k),j} + y_{j,i(k)} .$$

Let X' and Y' have the same meaning as in § 3. Then

(19) $$x_k^{*\prime} = x_k^*(Y') \leqq x_{i(k)}(Y') = x'_{i(k)} .$$

From $S_{i\in I}\{x_i\} = S_{k\in K}\{x_k^*\}$ we obtain $X' = \{x_i\}_{i\in I}^\delta = \{x_k^*\}_{k\in K}^\delta$, hence

(20) $$y_j'' = y_j' .$$

The relations (18), (19) and (20) imply that (14′) is valid.

Now let $i \in I$ and $j \in J$. Denote

$$K_i = \{k \in K : i(k) = i\} .$$

We have

(21) $$x_i = \bigvee_{k\in K}(x_i \wedge x_k^*) .$$

If $i \neq i(k)$, then $x_i \wedge x_k^* = 0$. Thus it follows from (21) that $K_i \neq \emptyset$ and

(22) $$x_i = \bigvee_{k\in K_i} x_k^* .$$

Hence

(23) $$x_{ij} = x_i[y_j] = (\bigvee_{k\in K_i} x_k^*)\,[y_j] = \bigvee_{k\in K_i}(x_k^*[y_j]) = \bigvee_{k\in K_i} x_{kj}^* .$$

Next we shall verify that the relation

(24) $$y_{ji} = \bigvee_{k\in K_i} y_{jk}$$

is valid.

For each $k \in K_i$ we have (cf. (17))

(25) $$y_{jk} \leqq y_{ji} .$$

Let $t \in G$, $y_{jk} \leqq t \leqq y_{ji}$ for each $k \in K_i$. Denote $-t + y_{ji} = q$. Then $0 \leqq q \leqq y_{ji}$, hence $q \in [y_{ji}] = [x_{ij}]$ (cf. Lemma 3.3). Since $[x_{ij}] \subseteq [x_i]$, we get $q \in [x_i]$. Assume that $q > 0$. Then $x_i \wedge q > 0$, thus (22) yields that $x_k^* \wedge q = q_1 > 0$ for some $k \in K_i$. Therefore $q_1 \in [x_k^*]$ and $y_{jk} + q_1 \in [x_k^*]$; since $[x_k^*] \subseteq [x_i]$, we have

$$y_{jk} = y_j[x_k^*] = y_j([x_i] \cap [x_k^*]) = (y_j[x_i])\,[x_k^*] =$$
$$= y_{ji}[x_k^*] = (t + q)\,[x_k^*] \geqq (y_{jk} + q_1)\,[x_k^*] = y_{jk} + q_1 > y_{jk},$$

which is impossible. Thus (24) is valid.

From (23) and (24) we obtain

$$x_{ij} + y_{ji} = (\bigvee_{k \in K_i} x_{kj}^*) + (\bigvee_{k' \in K_i} y_{jk'}) = \bigvee_{k \in K_i, k' \in K_i} (x_{kj}^* + y_{jk'}) .$$

If $k \neq k'$, then $x_{kj}^* \wedge y_{jk'} = 0$ and hence

$$x_{kj}^* + y_{jk'} = x_{kj}^* \vee y_{jk'} \leqq (x_{kj}^* + y_{jk}) \vee (x_{k'j}^* + y_{jk'}) ;$$

therefore

$$x_{ij} + y_{ji} = \bigvee_{k \in K_i} (x_{kj}^* + y_{jk}) . \tag{26}$$

Further, we have according to (22)

$$x_i' = x_i(Y') = (\bigvee_{k \in K_i} x_k^*)\,(Y') = \bigvee_{k \in K_i} (x_k^*(Y')) = \bigvee_{k \in K_i} x_k^{*\prime} . \tag{27}$$

From (26), (27) and (20) it follows that

$$S_{i \in I, j \in J}\{x_{ij} + y_{ji}, x_i', y_j'\} \geqq S_{k,j}\{x_{kj}^* + y_{jk}, x_k^{*\prime}, y_j''\} . \tag{28}$$

By (14′) and (28), the relation (14) is valid.

4.2. Lemma. *Let $\{x_i\}_{i \in I}, \{x_m^{\sim}\}_{m \in M}, \{y_j\}_{j \in J} \in H_1$. Assume that $S_{i \in I}\{x_i\} = S_{m \in M}\{x_m^{\sim}\}$. Then*

$$S(\{x_i\}_{i \in I} + \{y_j\}_{j \in J}) = S(\{x_m^{\sim}\}_{m \in M} + \{y_j\}_{j \in J}) .$$

Proof. Without loss of generality we may assume that $I \cap M = \emptyset$. Let us construct elements x_{im} and $x_{mi}^{\sim}$ analogously as we did for x_{ij} and y_{ji} in § 2. According to Lemma 3.7 we have $[x_{im}] = [x_{mi}^{\sim}]$ for each $i \in I$ and each $m \in M$; moreover, if we put $\{x_{im}\}_{i \in I, m \in M} = \{x_k^*\}_{k \in K}$, then

$$S_{i \in I}\{x_i\} = S_{k \in K}\{x_k^*\} = S_{m \in M}\{x_m^{\sim}\} , \quad \{x_k^*\}_{k \in K} \prec \{x_i\}_{i \in I} , \quad \{x_k^*\} \prec \{x_m^{\sim}\}_{m \in M} .$$

Now the assertion of the lemma follows immediately from 4.1.

Analogously we obtain:

4.3. Lemma. *Let $\{x_i\}_{i \in I}, \{y_j\}_{j \in J}, \{y_k^{\sim}\}_{k \in K} \in H$. Assume that $S_{j \in J}\{y_j\} = S_{k \in K}\{y_k^{\sim}\}$.* Then

$$S(\{x_i\}_{i \in I} + \{y_j\}_{j \in J}) = S(\{x_i\}_{i \in I} + \{y_j^{\sim}\}_{j \in J}) .$$

Now we define a binary operation $+$ on H as follows. Let $x, y \in H$. There are $\{x_i\}_{i\in I}, \{y_j\}_{j\in J} \in H_1$ with $x = S_{i\in I}\{x_i\}$, $y = S_{j\in J}\{y_j\}$. Put $x + y = S(\{x_i\}_{i\in I} + \{y_j\}_{j\in J})$. From 4.2 and 43 it follows that the operation $+$ in H is correctly defined.

4.4. Lemma. *Let $x, y, z \in H$, $x \leqq y$. Then $x + z \leqq y + z$ and $z + x \leqq z + y$.*

Proof. Let $z = S_{k\in K}\{z_k\}$. From 3.6 and 3.7 it follows that there are $\{x_i\}_{i\in I}$, $\{y_j\}_{j\in J} \in H_1$ with $x = S_{i\in I}\{x_i\}$, $y = S_{j\in J}\{y_j\}$, $\{x_i\}_{i\in I} \prec \{y_j\}_{j\in J}$. Hence we have

$$\{x_i\}_{i\in I} + \{z_k\}_{k\in K} \prec \{y_j\}_{j\in J} + \{z_k\}_{k\in K} .$$

Therefore $x + z \leqq y + z$. Analogously we can verify that $z + x \leqq z + y$.

4.5. Remark. From the definition of the operation $+$ in H it follows that

(i) if $x, y \in G^+$, then $x + y$ in G coincides with $x + y$ in H;
(ii) if $x, y \in H$, then $x \leqq x + y$ and $y \leqq x + y$;
(iii) if $x, y \in H$ and $x + y = 0$, then $x = y = 0$.

The assertions (ii) and (iii) are obvious. Let us verify that (i) is valid. Let $x, y \in G^+$. Then by 3.6 we have

$$S\{x\} = S\{x[y], x[y]^\delta\}, \quad S\{y\} = S\{y[x], y[x]^\delta\} .$$

From the definition of the operation $+$ in H we obtain

$$(*) \qquad S\{x\} + S\{y\} = S\{x[y] + y[x], x[y]^\delta, y[x]^\delta\} .$$

Since $x[y], x[y]^\delta \leqq x$ and $y[x], y[x]^\delta \leqq y$, we get

$$S\{x\} + S\{y\} \leqq S\{x + y\} .$$

Further we have

$$(x + y)([x] \cap [y]) = x([x] \cap [y]) + y([x] \cap [y]) =$$
$$= (x[x])[y] + (y[y])[x] = x[y] + y[x]$$

and by similar arguments,

$$(x + y)([x] \cap [y]^\delta) = x[y]^\delta, \quad (x + y)([x]^\delta \cap [y]) = y[x]^\delta,$$
$$(x + y)([x]^\delta \cap [y]^\delta) = 0 .$$

From $G = [x] \oplus [x]^\delta = [y] \oplus [y]^\delta$ it follows that

$$G = ([x] \cap [y]) \oplus ([x] \cap [y]^\delta) \oplus ([x]^\delta \cap [y]) \oplus ([x]^\delta \cap [y]^\delta) .$$

Thus

$$x + y = (x + y)([x] \cap [y]) \vee (x + y)([x] \cap [y]^\delta) \vee$$
$$\vee (x + y)([x]^\delta \cap [y]) \vee (x + y)([x]^\delta \cap [y]^\delta) =$$
$$= (x[y] + y[x]) \vee (x[y]^\delta) \vee (y[x]^\delta).$$

Hence according to (*) we have $S\{x + y\} \leqq S\{x\} + S\{y\}$ and therefore $S\{x + y\} = S\{x\} + S\{y\}$.

4.6. Lemma. *Let $x, y \in H$, $x = S_{i \in I}\{x_i\}$, $y = S_{j \in J}\{y_j\}$. Suppose that $I = I_1 \cup I_2$, $J = J_1 \cup J_2$ with $I_1 \cap I_2 = \emptyset = J_1 \cap J_2$ and that there is a one-to-one mapping φ of I_1 onto J_1 such that the following conditions are fulfilled:*

(i) *if $i \in I_1$, $j \in J$, $j \neq \varphi(i)$, then $x_i \wedge y_j = 0$;*

(ii) *if $i \in I_2$, $j \in J$, then $x_i \wedge y_j = 0$.*

Then $x + y = S_{i_1 \in I_1, i_2 \in I_2, j_2 \in J_2}\{x_{i_1} + y_{\varphi(i_1)}, x_{i_2}, y_{j_2}\}$.

Proof. From (i) and (ii) it follows that the system $\{x_{i_1} + y_{\varphi(i_1)}, x_{i_2}, y_{j_2}\}$ $(i_1 \in I_1, i_2 \in I_2. j_2 \in J_2)$ is disjoint, hence there is $u \in H$ with

$$u = S_{i_1 \in I_1, i_2 \in I_2, j_2 \in J_2}\{x_{i_1} + y_{\varphi(i_1)}, x_{i_2}, y_{j_2}\}.$$

Let $i_1 \in I_1$. If $j \in J$, $j \neq \varphi(i_1)$, then $x_{i_1 j} = 0$ and $y_{j i_1} = 0$. Suppose that $j = \varphi(i_1)$. Then $x_{i_1 j} + y_{j i_1} \leqq x_{i_1} + y_{\varphi(i_1)}$.

Let $i_2 \in I_2$. Then $x_{i_2 j} = 0$ and $y_{j i_2} = 0$ for each $j \in J$. Moreover, if Y and Y' are as in § 3, then $x_{i_2}(Y) = 0$, hence

$$x'_{i_2} = x_{i_2}(Y') = x_{i_2}$$

and analogously $y'_{j_2} = y_{j_2}$. Hence according to 3.7 we have $x + y \leqq u$.

Let $i \in I_1$, $j = \varphi(i)$. From $y_j \in Y$ it follows that $[y_j]^\delta \supseteq Y'$, hence $x'_i = x_i(Y') \leqq x_i[y_j]^\delta \leqq x_i$. Because $x_i \wedge y_{j_3} = 0$ for each $j_3 \in J$ with $j_3 \neq j$, we infer that $x_i[y_j]^\delta \wedge y_m = 0$ for each $m \in J$, thus $x_i[y_j]^\delta \in Y'$ and so

$$x_i(Y') = x_i[y_j]^\delta.$$

Hence

$$x_i = x_i[y_j] + x_i[y_j]^\delta = x_{ij} + x'_i$$

and analogously

$$y_j = y_{ji} + y'_j.$$

Therefore

$$x_i + y_j = x_{ij} + x'_i + y_{ji} + y'_j = (x_{ij} + y_{ji}) + x'_i + y'_j =$$
$$= (x_{ij} + y_{ji}) \vee x'_i \vee y'_j.$$

Thus $u \leqq x + y$ and by combining both inequalities, $u = x + y$.

4.7. Lemma. *Let $x, y, z \in H$, $x + y = x + z$. Then $y = z$.*

Proof. As above, we can write

$$x = S_{i\in I, j\in J}\{x_{ij}, x'_i\}\,, \quad y = S_{i\in I, j\in J}\{y_{ji}, y'_j\}\,,$$

$$x + y = S_{i\in I, j\in J}\{x_{ij} + y_{ji}, x'_i, y'_j\}\,. \tag{29}$$

Let $z = S_{k\in K}\{z_k\}$. According to 4.5 (ii) we have $z \leqq x + y$. From 3.3 we obtain $[x_{ij} + y_{ji}] = [x_{ij}] = [y_{ji}]$. Hence from 3.6 we infer

$$z = S_{i\in I, j\in J, k\in K}\{z_k[x_{ij}], z_k[x'_i], z_k[y'_j]\}\,.$$

Denote $x_{ij}[z_k[x_{ij}]] = x_{ijk}$, $x'_i[z_k[x'_i]] = x'_{ik}$. Since

$$x_{ij}[z_k[x'_i]] = 0 = x_{ij}[z_k[y'_j]]\,,$$

$$x'_i[z_k[x_{ij}]] = 0 = x'_i[z_k[y'_j]]\,,$$

we have according to 3.6

$$x = S_{i\in I, j\in J, k\in K}\{x_{ijk}, x'_{ik}\}\,.$$

Under the analogous notation, the relation

$$y = S_{i\in I, j\in J, k\in K}\{y_{jik}, y'_{jk}\}$$

is valid. Thus according to Lemma 4.6,

$$x + y = S_{i\in I, j\in J, k\in K}\{x_{ijk} + y_{jik}, x'_{ik}, y'_{jk}\}\,, \tag{29'}$$

$$x + z = S_{i\in I, j\in J, k\in K}\{x_{ijk} + z_k[x_{ij}], x'_{ik} + z_k[x'_i], z_k[y'_j]\}\,. \tag{30}$$

Let $i \in I$, $j \in J$, $k \in K$. Put $x_{ijk} + z_k[x_{ij}] = t$, $x_{ijk} + y_{jik} = t'$. Since $x + y = x + z$ and

$$t \wedge x'_{ik} = t \wedge y'_{jk} = 0\,, \quad t' \wedge (x'_{ik} + z_k[x'_i]) = t' \wedge (z_k[y'_j]) = 0\,,$$

we infer from (29') and (30) that

$$x_{ijk} + y_{jik} = x_{ijk} + z_k[x_{ij}]\,,$$

thus $y_{jik} = z_k[x_{ij}]$. Similarly we get

$$x'_{ik} = x'_{ik} + z_k[x'_i]\,, \quad y'_{jk} = z_k[y'_j]\,.$$

Hence $y = z$.

4.7'. Lemma. *Let $x, y, z \in H$, $x + y = z + y$. Then $x = z$.*

The proof is analogous to that of 4.7.

4.8. Lemma. *The operation + on H is associative.*

Proof. Let $x, y, z \in H$. Under the same notation as above we can write

$$x = S_{i\in I, j\in J}\{x_{ij}, x'_i\}, \quad y = S_{i\in I, j\in J}\{y_{ji}, y'_j\},$$
$$z = S_{k\in K}\{z_k\}.$$

We can assume that the sets I, J and K are mutually disjoint. Denote

$$M' = \{x_{ij}, x'_i, y_{ji}, y'_j\}^{\delta}_{i\in I, j\in J}, \quad M = (M')^{\delta},$$
$$Z' = \{z_k\}^{\delta}_{k\in K}, \quad Z = (Z')^{\delta}.$$

Further we put

$$x_{ijk} = x_{ij}[z_k], \quad x'_{ik} = x'_i[z_k], \quad y_{jik} = y_{ji}[z_k], \quad y'_{jk} = y'_j[z_k],$$
$$x'_{ij} = x_{ij}(Z'), \quad x''_j = x'_i(Z'), \quad y'_{ji} = y_{ji}(Z'), \quad y''_j = y'_j(Z'),$$
$$z_{kij} = z_k[x_{ij}] = z_k[y_{ji}], \quad z_{ki} = z_k[x'_i], \quad z_{kj} = z_k[y'_j],$$
$$z'_k = z_k(M').$$

Then we have (cf. 3.6)

$$x = S_{i\in I, j\in J, k\in K}\{x_{ijk}, x'_{ij}, x'_{ik}, x''_i\},$$
$$y = S_{i\in I, j\in J, k\in K}\{y_{jik}, y'_{ji}, y'_{jk}, y''_j\},$$
$$z = S_{i\in I, j\in J, k\in K}\{z_{kij}, z_{ki}, z_{kj}, z'_k\}.$$

From this and from Lemma 4.6 it follows that

$$(x + y) + z = S_{i\in I, j\in J, k\in K}\{x_{ijk} + y_{jki} + z_{kij},$$
$$x''_{ij} + y''_{ji}, x'_{ik} + z_{ki}, y'_{jk} + z_{kj}, x''_i, y''_j, z'_k\}$$

and the same results is obtained for $x + (y + z)$. Hence $(x + y) + z = x + + (y + z)$.

4.9. Lemma. *Let $x = S_{i\in I}\{x_i\} \in H$. Then $x = \bigvee_{i\in I} x_i$ holds in H.*

Proof. From 3.2 it follows that $x_i \leqq x$ for each $i \in I$. Let $y = S_{j\in J}\{y_j\} \in H$, $x_i \leqq y$ for each $i \in I$. Hence $x_i = \bigvee_{j\in J}(x_i \wedge y_j)$ is valid for each $i \in I$, thus $x \leqq y$. Therefore, $x = \bigvee_{i\in I} x_i$.

Let G and H be as above.

From 4.4, 4.5 (iii), 4.7, 4.7 , 4.8 and Thm. 3, Chap. XIV, [3] we obtain:

5.1. Lemma. *There exists a lattice ordered group G' such that $((G')^+; +, \leqq) = (H; +, \leqq)$.*

5.2. Remark. *Since G^+ is a subsemigroup and a sublattice of H, G is an l-subgroup of G'.*

5.3. Lemma. *G' is orthogonally complete.*

Proof. Let $\{x^k\}_{k\in K}$ be a disjoint subset of G'. Each x^k belongs to H, hence it can be expressed as

$$x^k = S_{i\in I_k}\{x_{ki}\}$$

and without loss of generality we can assume that the sets I_k $(k \in K)$ are mutually disjoint. Then $\{x_{ki}\}$ $(k \in K,\ i \in I_k)$ is a disjoint subset of G and hence there exists

$$y = S_{k\in K, i\in I_k}\{x_{ki}\}$$

in H. According to 3.2, $x^k \leqq y$ for each $k \in K$ and by the definition of the relation $\leqq$ in H we have obviously $y \leqq z$ whenever z is an elemnent of H such that $x^k \leqq z$ for each $k \in K$. Hence $y = \bigvee_{k\in K} x^k$ holds in G .

5.4. Lemma. *G' is an orthogonal hull of G.*

Proof. From 4.9, 5.1 and 5.2 it follows that G is a dense l-subgroup of G'. Since G' is orthogonally complete, it suffices to verify that $G' = A$ whenever A is an orthogonally complete l-subgroup of G' such that $G \subseteq A$.

Let A be an l-subgroup of G'. Suppose that A is orthogonally complete and $G \subseteq A$. Then A is a dense l-subgroup of G'. Let $0 < x \in G'$. By 5.1 we have $x \in H$ and thus according to 4.9 there exists a disjoint subset $\{x_i\}_{i\in I}$ of G such that $x = \bigvee_{i\in I} x_i$ holds in G'. Since A is orthogonally complete, there is $y \in A$ such that $y = \bigvee_{i\in I} x_i$ is valid in A. From this and from Lemma 2.3 in [5] it follows that $y = \bigvee_{i\in I} x_i$ is valid in G' as well. Thus $x = y$ and therefore $(G')^+ = H \subseteq A$. Hence $A = G'$. This completes the proof.

5.5. Lemma. *G' is strongly projectable.*

Proof. Since G' is orthogonally complete, each polar of G' is principal. Let $[x]$ be a principal polar of G'. Without loss of generality we can suppose that $x \geqq 0$. Let $0 \leqq y \in G'$. There are $\{x_i\}_{i\in I}, \{y_j\}_{j\in J} \in H_1$ with $x = S_{i\in I}\{x_i\}$, $y = S_{j\in J}\{y_j\}$. Under the above notation $x = S_{i\in I, j\in J}\{x_{ij}, x'_i\}$, $y = S_{i\in I, j\in J}\{y_{ji}, y'_j\}$. There is $t \in G'$ with $t = S_{i\in I, j\in J}\{y_{ji}\}$. Clearly $t \in [x]$ and $t \leqq y$. Let $z \in [x]$, $0 \leqq z \leqq y$. We can use the same notation for x, y, z as in the proof of Lemma 4.8. From $z \in [x]$ we obtain

$$z_{kj} = z'_k = 0 .$$

Next from $z \leqq y$ we infer that $z_{ki} = 0$. Thus

$$z = S_{i \in I, j \in J, k \in K}\{z_{kij}\} .$$

Because $z \leqq y$, we get

$$z_{kij} \leqq y_{jik} \quad \text{for each} \quad i \in I , \quad j \in J , \quad k \in K .$$

From 3.6 it follows that

$$t = S_{i \in I, j \in J, k \in K}\{y_{jik}, y'_{ji}\} .$$

Then by 3.2 we have

$$z \leqq t \leqq y .$$

Hence

$$t = \max \{u \in [x] : 0 \leqq u \leqq y\} .$$

Therefore G' is strongly projectable.

5.6. Lemma. *Let G_1 and G_2 be lattice ordered groups such that the lattice $(G_1^+; \leqq)$ is isomorphic with the lattice $(G_2^+; \leqq)$. Then the lattices $l(G_1)$ and $l(G_2)$ are isomorphic.*

Proof. Let φ be an isomorphism of the lattice $(G_1^+; \leqq)$ onto the lattice $(G_2^+; \leqq)$. Then clearly $\varphi(0) = 0$. For each $g \in G_1$ we put

$$\psi(g) = \varphi(g^+) - \varphi(g^-) .$$

If $g \in G_1^+$, then $\psi(g) = \varphi(g)$. Since $g^+ \wedge g^- = 0$, we have

$$\varphi(g^+) \wedge \varphi(g^-) = 0$$

and hence we obtain

$$(\psi(g))^+ = \varphi(g^+) , \quad (\psi(g))^- = -\varphi(g^-) .$$

Now it is not difficult to verify that ψ is onto and isotone. Hence ψ is an isomorphism of $l(G_1)$ onto $l(G_2)$.

5.7. Theorem. *Let G_1 and G_2 be lattice ordered groups such that the lattices $l(G_1)$ and $l(G_2)$ are isomorphic. Assume that G_1 is strongly projectable. Then*

(i) *each element of $o(G_i)$ is a join of a disjoint subset of G_i $(i = 1, 2)$;*

(ii) *the lattices $l(o(G_1))$ and $l(o(G_2))$ are isomorphic.*

Proof. According to Thm. 2.3, G_2 is strongly projectable. Thus we can construct lattices $H(G_i)$ for G_i $(i = 1, 2)$ analogously as we constructed the lattice H for the lattice ordered group G in § 3. According to the assumption there exists an isomorphism of $l(G_1)$ onto $l(G_2)$ and hence there exists an isomorphism φ_1 of the lattice

$(G_1^+; \leqq)$ onto the lattice $(G_2^+; \leqq)$. Since in the construction of $H(G_i)$ merely the lattice properties of $(G_i^+; \leqq)$ are used, we infer that the isomorphism φ_1 can be extended to an isomorphism φ of the lattice $H(G_1)$ onto the lattice $H(G_2)$. Let G_i' be the orthogonal hull of G_i $(i = 1, 2)$; according to 5.1 and 5.4 we can assume that $(G_i')^+ = H(G_i)$. From this and from 5.6 it follows that there exists an isomorphism of $l(G_1')$ onto $l(G_2')$. Thus (ii) is valid. The assertion (i) is a consequence of 5.1 and 4.9.

5.8. Remarks. (a) The assertion (i) need not hold if G_i fails to be strongly projectable (cf. Example 6.3 below). (b) If G_1, G_2 are lattice ordered groups such that G_1 is strongly projectable and the lattice $l(G_1)$ is isomorphic with $l(G_2)$, then G_1 need not be isomorphic with G_2 (Cf. Example 6.4 below.)

A lattice ordered group G is said to be representable if there exists a system $\{A_i\}_{i \in I}$ of linearly ordered groups A_i and an isomorphism φ of G into the direct product $\prod_{i \in I} A_i$ such that for each $i \in I$ and each $a^i \in A_i$ there exists $g \in G$ with $(\varphi(g)(i) = a^i$. It is well-known (cf. Šik [13]) that a lattice ordered group is representable if and only if each of its polars is a normal subgroup. From this it follows that each strongly projectable lattice ordered group is representable. Under the above notation, the isomorphism

$$\varphi : G \to \prod_{i \in I} A_i$$

is called a representation of G.

5.9. Proposition. Let G_1 and G_2 be lattice ordered groups such that the lattices $l(G_1)$ and $l(G_2)$ are isomorphic. Assume that G_1 is strongly projectable. Then (a) the lattice ordered group G_2 is representable, and (b) there exist representations $\varphi_1 : G_1 \to \prod_{i \in I} A_i$ and $\varphi_2 : G_2 \to \prod_{i \in I} B_i$ such that, for each $i \in I$, the lattices $l(A_i)$ and $l(B_i)$ are isomorphic.

We need some auxiliary notation and results.

Let $G \neq \{0\}$ be a strongly projectable lattice ordered group and let $\mathscr{P}(G)$ be the set of all polars of G. The set $\mathscr{P}(G)$ is partially ordered by inclusion. Then $\mathscr{P}(G)$ is a Boolean algebra and for each $A \in \mathscr{P}$, A^δ is the complement of A in $\mathscr{P}(G)$ (cf. Šik [12]).

Let $A \in \mathscr{P}(G)$. We have $G = A \oplus A^\delta$. For $g_1, g_2 \in G$ we put $g_1 \equiv g_2(R(A))$ if $g_1(A^\delta) = g_2(A^\delta)$. Then $R(A)$ is a congruence relation on the lattice ordered group G. Clearly $g_1 \equiv g_2(R(A))$ if and only if $g_1 \wedge g_2 \equiv g_1 \vee g_2(R(A))$.

For the notion of projectivity of intervals in a lattice cf. [3].

Under the above notation we have:

5.10. Lemma. *Let $g_1, g_2 \in G$, $g_1 \leqq g_2$. Then the following conditions are equivalent:*

(a) $g_1 \equiv g_2(R(A))$.

(b) *There are elements $t \in G$, $x_1, x_2, y_1, y_2 \in A$ with $g_1 \leqq t \leqq g_2$, $x_1 \leqq x_2$, $y_1 \leqq y_2$ such that $[g_1, t]$ is projective to $[x_1, x_2]$ and $[t, g_2]$ is projective to $[y_1, y_2]$.*

Proof. If (b) is valid then the regularity of the relation $R(A)$ with respect to the lattice operations $\vee$ and $\wedge$ implies that (a) holds. Conversely, suppose that (a) is valid. Put

$$t = (g_1 \vee 0) \wedge g_2 ,$$

$$x_i = (g_i \wedge 0)(A) , \quad y_i = (g_i \vee 0)(A) \quad (i = 1, 2) .$$

Then $[g_1, t]$ is transposed to $[g_1 \wedge 0, g_2 \wedge 0]$, and $[g_1 \wedge 0, g_2 \wedge 0]$ is transposed to $[x_1, x_2]$; hence $[g_1, t]$ is projective with $[x_1, x_2]$. Similarly, $[t, g_2]$ is projective with $[y_1, y_2]$.

5.11. Lemma. *Let G_1 and G_2 be lattice ordered groups and suppose that G_1 is strongly projectable. Let ψ be an isomorphism of $l(G_1)$ onto $l(G_2)$ with $\psi(0) = 0$. Let $A \in \mathscr{P}(G_1)$, $g_1, g_2 \in G_1$. Then $g_1 \equiv g_2(R(A))$ if and only if $\psi(g_1) \equiv \psi(g_2)$ $(R(\psi(A)))$.*

This is an immediate consequence of 2.3 and 5.10 (recall that $\psi(A) \in \mathscr{P}(G_2)$ by 2.5).

Let G_1 and G_2 be as in 5.11. Let $\{M_i\}$ $(i \in I)$ be the system of all maximal ideals of the Boolean algebra $\mathscr{P}(G_1)$. For $M_i = \{A_{ik}\}$ $(k \in K_i)$ denote $\psi(M_i) = \{\psi(A_{ik})\}$ $(k \in K_i)$. Then it follows from Lemma 2.5 that $\{\psi(M_i)\}$ $(i \in I)$ is the system of all maximal ideals of $\mathscr{P}(G_2)$.

Let $i \in I$. We define a binary relation R_i^1 on G_1 by putting

$$R_i^1 = \bigvee R(A) \ (A \in M_i) .$$

Analogously we put

$$R_i^2 = \bigvee R(\psi(A)) \ (A \in M_i) .$$

R_i^2 and R_i^1 are congruence relations on G_1 or G_2, respectively. From 5.11 it follows:

5.12. Lemma. *Let $g_1, g_2 \in G_1$, $i \in I$. Then $g_1 \equiv g_2(R_i^1)$ if and only if $\psi(g_1) \equiv$ $\equiv \psi(g_2)(R_i^2)$. Hence the lattices $l(G_1/R_i^1)$ and $l(G_2/R_i^2)$ are isomorphic.*

Consider the mappings

$$\varphi_1 : G_1 \to \prod_{i \in I}(G_1/R_i^1) , \quad \varphi_2 : G_2 \to \prod_{i \in I}(G_2/R_i^2)$$

defined by

$$(\varphi_j(g))(i) = g(R_i^j)$$

for each $g \in G_j$ $(j \in \{1, 2\})$ and each $i \in I$, where $g(R_i^j)$ is the class of the congruence R_i^j on G^j containing the element g.

The following result is a consequence of Hilfssatz 1 and Satz 1 of [13].

5.13. Proposition. *Each lattice ordered group G_j/R_i^j $(j \in \{1, 2\}, i \in I)$ is linearly ordered. φ_1 and φ_2 is a representation of G_1 or G_2, respectively.*

If we denote $G_1/R_i^1 = A_i$, $G_2/R_i^2 = B_i$, then 5.9 follows from 5.12 and 5.13.

The following problem remains open: *Does the assertion of* 5.9 *remain valid if the assumption of strong projectability of G_1 is replaced by the weaker assumption of representability of both G_1 and G_2?*

6. EXAMPLES

A non-archimedean orthogonally complete lattice ordered group need not be projectable.

6.1. Example. Let A be the additive lattice ordered group of all integers with the natural linear order and let $B \neq \{0\}$ be an orthogonally complete lattice ordered group. Put $G = A \circ B$ (the symbol $\circ$ denotes the operation of the lexicographic product, cf. [6]). Then G is orthogonally complete, but it fails to be projectable.

Let G_1 and G_2 be lattice ordered groups and suppose that ψ is an isomorphism of $l(G_1)$ onto $l(G_2)$ such that $\psi(0) = 0$. Then for $x \in G_1$ the relation $\psi(|x|) = |\psi(x)|$ need not hold.

6.2. Example. Let R be the set of all reals with the usual linear order and consider the cartesian product $A = R \times R$ with the partial order that is defined component-wise. If the operation $+$ on R has the usual meaning and if we define $+$ on A component-wise, then $G_1 = (A, \leqq, +)$ is a lattice ordered group.

For each $t \in R$ we put $\varphi(t) = t$ if $t \geqq 0$ and $\varphi(t) = 2t$ if $t < 0$. Now we define a binary operation $+_2$ on R by putting

$$t_1 +_2 t_2 = \varphi(t_1) + \varphi(t_2)$$

for each $t_1, t_2 \in R$. Further, let $+_2$ on A be defined component-wise. Then $G_2 = (A; \leqq, +_2)$ is a lattice ordered group and the identical mapping ψ is an isomorphism of $l(G_1)$ onto $l(G_2)$, $\psi(0) = 0$. For $x \in A$ we denote by $|x|_1$ and $|x|_2$ the corresponding absolute value in G_1 or in G_2, respectively. If $x = (1, -1)$, then $\psi(|x|_1) = |x|_1 = (1, 1) \neq (1, 2) = |x|_2 = |\psi(x)|_2$.

Let G be a lattice ordered group, $0 < x \in o(G)$. The element x need not be a join of a disjoint subset of G.

6.3. Example. Let G_1 be the set of all real functions defined on R where R is as in 6.2. The operation $+$ on G_1 has the usual meaning and for $f_1, f_2 \in G_1$ we put $f_1 \leqq f_2$ if $f_1(t) \leqq f_2(t)$ for each $t \in R$. Then G_1 is a lattice ordered group. Let G_2 be the set of all $f \in G_1$ with finite support; G_2 is an l-subgroup of G_1. Let A be as in 6.1; put

$$G_0 = A \circ G_1\,, \quad G = A \circ G_2\,.$$

The lattice ordered group G is a dense l-subgroup of G_0 and G_0 is orthogonally complete.

The elements of G_0 can be written as pairs (a, g) with $a \in A$ and $g \in G_0$. Let B be an l-subgroup of G_1 with $G \subseteq B$ such that B is orthogonally complete. Then B is a dense l-subgroup of G_0. By a method analogous to that used in the proof of 5.5 we can verify that each element $(0, g)$ with $0 < g \in G_1$ belongs to B. Hence $B = G_0$ and this shows that G_0 is the orthogonal hull of G. There exists $g \in G_1$ such that $g_1 > 0$, $g_1 \notin G_2$. Then the element $(1, g)$ belongs to G_0 and it cannot be expressed as a join of a disjoint system of elements of G.

If G_1 and G_2 are strongly projectable lattice ordered groups such that $l(G_1)$ is isomorphic with $l(G_2)$, then G_1 need not be isomorphic with G_2.

6.4. Example. Let A be as in 6.1 and let R_1 be the set of all rationals with the natural linear order and the usual operation $+$. Let I be a nonempty set. Put

$$G_1 = \prod_{i \in I} A_i\,, \quad G_2 = \prod_{i \in I} B_i\,,$$

where $A_i = R_1$ and $B_i = A \circ R_1$ for each $i \in I$. The lattice $l(A_i)$ is isomorphic with $l(B_i)$, hence $l(G_1)$ is isomorphic with $l(G_2)$. Both G_1 and G_2 are orthogonally complete and G_l fails to be isomorphic with G_2.

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