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NOTE ON HOMOLOGY AND COHOMOLOGY WITH $\mathbb{Z}_P$-COEFFCIENTS

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1. INTRODUCTION

The object of this note is to provide elementary proofs of two useful results on the homology and cohomology of topological spaces with coefficients in the P-localized integers $\mathbb{Z}_P$, where P is an arbitrary family of primes. The first result, due to A. K. BOUSFIELD, is stated as Theorem 3.1 and asserts that a map $f : X \to Y$ induces homology isomorphisms with $\mathbb{Z}_P$ coefficients if and only if it induces cohomology isomorphisms with $\mathbb{Z}_P$ coefficients. The second result, due to D. SULLIVAN [3; p. 18] is stated as Theorem 3.2 and asserts that a map $f : X \to Y$ induces homology isomorphisms with $\mathbb{Z}_P$ coefficients if and only if it induces homology isomorphisms with $\mathbb{Q}$ coefficients and with $\mathbb{Z}/p$ coefficients for all $p \in P$. Our proofs are simple and complete and are based on elementary observations on the homological algebra of $\mathbb{Z}_P$-modules, which constitute the content of Section 2. Thus we regard Theorems 3.1 and 3.2 as generalizing the case $P = \Pi$, the family of all primes, when, of course, $\mathbb{Z}_P = \mathbb{Z}$.

In particular, we generalize in Section 2 the result due to STEIN-SERRE [2; p. 108] that if A is an abelian group of countable rank with $\operatorname{Ext}(A, \mathbb{Z}) = 0$ then A is free.*) The generalization presents no difficulty since, in fact, we get the same derived functors Tor and Ext, whether we regard two $\mathbb{Z}_P$-modules A and B as $\mathbb{Z}_P$-modules or as abelian groups.

2. THE HOMOLOGICAL ALGEBRA OF $\mathbb{Z}_P$-MODULES

Throughout this section A will be a $\mathbb{Z}_P$-module, where P is a non-empty family of primes; the family complementery to P will be denoted P'. Since, for any $\mathbb{Z}_P$-module B,

$$(2.1) \qquad \operatorname{Hom}(A. B) = \operatorname{Hom}_{\mathbb{Z}_P}(A, B), \quad \operatorname{Ext}(A, B) = \operatorname{Ext}_{\mathbb{Z}_P}(A, B),$$

*) The celebrated Whitehead conjecture has, in the meantime, been solved by S. SHELAH.

we will feel free to suppress $\mathbb{Z}_P$ from the notation for Hom and Ext; we will likewise suppress $\mathbb{Z}_P$ from the notation for the tensor and torsion products and will in fact write $A * B$ for the torsion product of A and B. We note that Hom (A, B), Ext (A, B), $A \otimes B$ and $A * B$ are all $\mathbb{Z}_P$-modules, since $\mathbb{Z}_P$ is commutative.

Since $\mathbb{Z}_P$ is a principal ideal domain (or, in view of the identifications above) we have, exactly as (V. 4.3)–(V. 4.6) in [2]

Proposition 2.1. *Let A, B, C be $\mathbb{Z}_P$-modules. Then*

(i) $$(A * B) \otimes C \oplus (A \otimes B) * C \cong A * (B \otimes C) \oplus A \oplus (B * C);$$

(ii) $$(A * B) * C \cong A * (B * C),$$

(iii) $$\operatorname{Hom}(A * B, C) \oplus \operatorname{Ext}(A \otimes B, C) \cong \operatorname{Hom}(A, \operatorname{Ext}(B, C)) \oplus \oplus \operatorname{Ext}(A, \operatorname{Hom}(B, C));$$

(iv) $$\operatorname{Ext}(A * B, C) \cong \operatorname{Ext}(A, \operatorname{Ext}(B, C)).$$

Corollary 2.2. *Let* Hom $(B, C) = 0$, Ext $(B, C) = 0$. *Then* Ext $(A \otimes B, C) = 0$ *for all $\mathbb{Z}_P$-modules A.*

Proposition 2.3. *If A is a torsion $\mathbb{Z}_P$-module, then* Ext $(A, \mathbb{Z}) =$ Ext $(A, \mathbb{Z}_P)$.

Proof. Since $\mathbb{Z}_P/\mathbb{Z}$ is divisible we have an exact sequence Hom $(A, \mathbb{Z}_P/\mathbb{Z}) \to$ $\to$ Ext $(A, \mathbb{Z}) \twoheadrightarrow$ Ext $(A, \mathbb{Z}_P)$. But $\mathbb{Z}_P/\mathbb{Z}$ is a P'-torsiongroup, so Hom $(A, \mathbb{Z}_P/\mathbb{Z}) = 0$.

Corollary 2.4. *If A is a finite $\mathbb{Z}_P$-module, then* Ext $(A, \mathbb{Z}_P) \cong A$.

Proof. Certainly Ext $(A, \mathbb{Z}) \cong A$ if A is finite.

Proposition 2.5. *If* Ext $(A, \mathbb{Z}_P) = 0$ *then A is torsionfree.*

Proof. If A has torsion then $\mathbb{Z}/p \subseteq A$ for some $p \in P$ and Ext $(A, \mathbb{Z}_P) \twoheadrightarrow$ $\twoheadrightarrow$ Ext $(\mathbb{Z}/p, \mathbb{Z}_P)$. But, by Corollary 2.4, Ext $(\mathbb{Z}/p, \mathbb{Z}_P) \cong \mathbb{Z}/p$, so that Ext $(A, \mathbb{Z}_P) \neq 0$.

Theorem 2.6. *If A is a $\mathbb{Z}_P$-module of countable rank with* Ext $(A, \mathbb{Z}_P) = 0$, *then A is free.*

Proof. The proof proceeds exactly as in the case $P = \Pi$, the collection of all primes. First A is torsionfree by Proposition 2.5. Next we observe that a $\mathbb{Z}_P$-module of countable rank, all of whose submodules of finite rank are free, is itself free. Since if Ext $(A, \mathbb{Z}_P) = 0$, then Ext $(A_0, \mathbb{Z}_P) = 0$ for all submodules A_0 of A, it now follows that it suffices to prove that A is free if Ext $(A, \mathbb{Z}_P) = 0$ and A is of *finite* rank. Thus we prove the assertion that, if A is a torsionfree, non-free $\mathbb{Z}_P$-module of finite rank, then Ext $(A, \mathbb{Z}_P) \neq 0$.

Now

$$\operatorname{Ext}(A, \mathbb{Z}_P) = \operatorname{Hom}(A, \mathbb{Q}/\mathbb{Z}_P)/\operatorname{Im} \operatorname{Hom}(A, \mathbb{Q}).$$

We prove that, under the given hypotheses on A,

$$\left|\operatorname{Hom}(A, \mathbb{Q})\right| = \aleph_0, \quad \left|\operatorname{Hom}(A, \mathbb{Q}/\mathbb{Z}_P)\right| = c; \tag{2.2}$$

this will certainly establish the assertion.

Let $S = (a_1, \ldots, a_k)$ be a maximal independent set in A and let A_0 be the submodule generated by S. Then a homomorphism from A to $\mathbb{Q}$ is essentially just a function $S \to \mathbb{Q}$ and plainly there are $\aleph_0$ such functions. Thus the first part of (2.2) is established.

To establish the second part we first observe that $\left|\operatorname{Hom}(A_0, \mathbb{Q}/\mathbb{Z}_P)\right| = \aleph_0$. Since A is not free, $A_0 \neq A$. Let $b_1 \in A - A_0$ and let n be the smallest positive integer m such that $mb_1 \in A_0$. Such an m obviously exists since b_1 depends on S. Then $n \in P$; for if $n = n'n''$ with $n' \in P'$, $n'' \in P$, then $n''b_1 = (1/n')(nb_1) \in A_0$. It is then plain that, given $\phi : A_0 \to \mathbb{Q}/\mathbb{Z}_P$ we have n distinct extensions of ϕ to $A_1 = (A_0, b_1)$. For if ψ is such an extension, then ψb_1 is an element x of $\mathbb{Q}/\mathbb{Z}_P$ such that $nx = \phi(nb_1)$. We may identify $\mathbb{Q}/\mathbb{Z}_P$ with $\mathbb{Z}_{P'}/\mathbb{Z} \subseteq \mathbb{Q}/\mathbb{Z}$. Then the equation $nx = \phi(nb_1)$ has n solutions in $\mathbb{Q}/\mathbb{Z}$ and, since $\phi(nb_1) \in \mathbb{Z}_{P'}/\mathbb{Z}$ and $n \in P$, every solution in fact lies in $\mathbb{Z}_{P'}/\mathbb{Z}$. Thus ϕ has n extensions to A_1 with $n > 1$. Call one such extension ϕ_1, and write $n = n_1$.

We now proceed exactly as in the classical case. Since A_1 is free we know that $A_1 \neq A$. Choose $b_2 \in A - A_1$ and let $n_2 \in P$ be the smallest positive integer with $n_2 b_2 \in A_1$. Then ϕ_1 has n_2 extensions to $A_2 = (A_1, b_2)$. We thus find a sequence of elements $b_1, b_2, \ldots$, and modules $A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots$ with $b_i \in A_i - A_{i-1}$ and positive integers $n_i \in P$, $n_i > 1$, which are minimal for the property that $n_i b_i \in A_{i-1}$, $i = 1, 2, \ldots$. If $\tilde{A} = \bigcup A_i$ then there are plainly $\aleph_0 \cdot n_1 n_2 \ldots = c$ homomorphisms from $\tilde{A}$ to $\mathbb{Q}/\mathbb{Z}_P$ and each extends to A since $\mathbb{Q}/\mathbb{Z}_P$ is divisible. Thus

$$\left|\operatorname{Hom}(A, \mathbb{Q}/\mathbb{Z}_P)\right| \geqq c. \tag{2.3}$$

However if A is a torsionfree $\mathbb{Z}_P$-module of countable rank it is certainly countable, so that, trivially,

$$\left|\operatorname{Hom}(A, \mathbb{Q}/\mathbb{Z}_P)\right| \leqq c, \tag{2.4}$$

and so (2.2), and hence Theorem 2.6 follows. Note that it follows from (2.2) that if A is torsionfree, non-free of finite rank, then

$$\left|\operatorname{Ext}(A, \mathbb{Z}_P)\right| = c \tag{2.5}$$

We proceed to make a special study of $\operatorname{Ext}(\mathbb{Q}, \mathbb{Z}_P)$. To this end we first state proposition whose proof exactly parallels that of the special case $P = \Pi$.

Proposition 2.7. *Let $p \in P$. Then*

(i) *A is p-torsionfree if and only if* $\operatorname{Ext}(A, \mathbb{Z}_P)$ *is p-divisible;*

(ii) *If A is p-divisible,* $\operatorname{Ext}(A, \mathbb{Z}_P)$ *is p-torsionfree. Conversely, if* $\operatorname{Ext}(A, \mathbb{Z}_P)$ *is p-torsionfree and* $\operatorname{Hom}(A, \mathbb{Z}_P) = 0$, *$A$ is p-divisible.*

Corollary 2.8. $\operatorname{Ext}(\mathbb{Q}, \mathbb{Z}_P) \cong \mathbb{R}$.

Proof. By Proposition 2.7, $\operatorname{Ext}(\mathbb{Q}, \mathbb{Z}_P)$ is p-torsionfree, p-divisible for all $p \in P$. But $\operatorname{Ext}(\mathbb{Q}, \mathbb{Z}_P)$ is a $\mathbb{Z}_P$-module, so it is torsionfree, divisible, and hence a vector space over $\mathbb{Q}$. By (2.5), $|\operatorname{Ext}(\mathbb{Q}, \mathbb{Z}_P)| = \boldsymbol{c}$, so that a basis for $\operatorname{Ext}(\mathbb{Q}, \mathbb{Z}_P)$ as a vector space has cardinality $\boldsymbol{c}$. But $\mathbb{R}$ is a vector space over $\mathbb{Q}$ with a basis of cardinality $\boldsymbol{c}$, so that $\operatorname{Ext}(\mathbb{Q}, \mathbb{Z}_P) \cong \mathbb{R}$ as $\mathbb{Q}$-vector spaces.

Remark. Since P is non-empty, it follows that $\operatorname{Hom}(\mathbb{Q}, \mathbb{Z}_P) = 0$. We thus obtain the commutative diagram, with exact rows, and columns

$$\begin{tikzcd}
\operatorname{Hom}(\mathbb{Q}, \mathbb{Z}_P/\mathbb{Z}) \arrow[r, tail] & \operatorname{Ext}(\mathbb{Q}, \mathbb{Z}) \arrow[r, two heads] & \operatorname{Ext}(\mathbb{Q}, \mathbb{Z}_P) \\
\operatorname{Hom}(\mathbb{Q}, \mathbb{Z}_P/\mathbb{Z}) \arrow[r, tail] \arrow[u, equal] & \operatorname{Hom}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) \arrow[r, two heads] \arrow[u, two heads] & \operatorname{Hom}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}_P) \arrow[u, two heads] \\
 & \operatorname{Hom}(\mathbb{Q}, \mathbb{Q}) \arrow[r, equal] \arrow[u, tail] & \operatorname{Hom}(\mathbb{Q}, \mathbb{Q}) \arrow[u, tail]
\end{tikzcd}$$

Since $\operatorname{Hom}(\mathbb{Q}, \mathbb{Z}_P/\mathbb{Z}) \neq 0$, it follows that the homomorphism $\operatorname{Ext}(\mathbb{Q}, \mathbb{Z}) \to$ $\to \operatorname{Ext}(\mathbb{Q}, \mathbb{Z}_P)$, induced by the embedding $\mathbb{Z} \subseteq \mathbb{Z}_P$, is surjective but not injective (unless $P = \Pi$, of course). Notice that, although the middle row splits, the splitting will not be compatible with the embedding of $\operatorname{Hom}(\mathbb{Q}, \mathbb{Q}) \cong \mathbb{Q}$.

3. HOMOLOGY WITH $\mathbb{Z}_P$-COEFFICIENTS

Theorem 3.1. *Let P be an arbitrary family of primes, and let $f : X \to Y$ be a map of topological spaces. Then*

$$f_* : H_*(X; \mathbb{Z}_P) \to H_*(Y; \mathbb{Z}_P)$$

is an isomorphism if and only if

$$f^* : H^*(Y; \mathbb{Z}_P) \to H^*(X; \mathbb{Z}_P)$$

is an isomorphism.

Proof. By passing to the mapping cone of f, it suffices to show that, if W is a topological space and H_*, H^* refer to reduced homology, then

$$H_*(W; \mathbb{Z}_P) = 0 \quad \text{if and only if} \quad H^*(W; \mathbb{Z}_P) = 0\,. \tag{3.1}$$

This is obvious if $P = \emptyset$ since $H^*(W; \mathbb{Q}) = \operatorname{Hom}(H_*(W; \mathbb{Q}), \mathbb{Q})$ so we assume $P \neq \emptyset$. Now $H^n(W; \mathbb{Z}_P) = \operatorname{Hom}(H_n(W; \mathbb{Z}_P), \mathbb{Z}_P) \oplus \operatorname{Ext}(H_{n-1}(W; \mathbb{Z}_P), \mathbb{Z}_P)$, so that certainly $H^*(W; \mathbb{Z}_P) = 0$ if $H_*(W; \mathbb{Z}_P) = 0$. Conversely, suppose that $H^*(W; \mathbb{Z}_P) = 0$. Then, for each n,

$$\operatorname{Hom}(H_n(W; \mathbb{Z}_P), \mathbb{Z}_P) = 0\,, \quad \operatorname{Ext}(H_n(W; \mathbb{Z}_P), \mathbb{Z}_P) = 0\,.$$

Thus, by Corollary 2.2, $\operatorname{Ext}(\mathbb{Q} \otimes H_n(W; \mathbb{Z}_P), \mathbb{Z}_P) = 0$, and, by Proposition 2.5, $H_n(W; \mathbb{Z}_P)$ ist torsionfree. Thus if $H_n(W; \mathbb{Z}_P) \neq 0$, $\mathbb{Q} \otimes H_n(W; \mathbb{Z}_P)$ is a non-trivial vector space over $\mathbb{Q}$ and we are in contradiction with Corollary 2.8. This proves that $H_n(W; \mathbb{Z}_P) = 0$, establishing (3.1) and hence the theorem.

Theorem 3.2. *Let P be an arbitrary family of primes and let $f : X \to Y$ be a map of topological spaces. Then*

$$f_* : H_*(X; \mathbb{Z}_P) \to H_*(Y; \mathbb{Z}_P)$$

is an isomorphism if and only if

$$f_* : H_*(X; \mathbb{Q}) \to H_*(Y; \mathbb{Q})$$

is an isomorphism and

$$f_* : H_*(X; \mathbb{Z}/p) \to H_*(Y, \mathbb{Z}/p)$$

is an isomorphism for all $p \in P$.

Proof. Again it suffices to show that if W is a topological space then

$$H_*(W; \mathbb{Z}_P) = 0 \quad \text{if and only if} \quad H_*(W; \mathbb{Q}) = 0 \quad \text{and} \quad H_*(W, \mathbb{Z}/p) = 0 \quad \text{for all} \quad p \in P\,. \tag{3.2}$$

Since $H_*(W; \mathbb{Z}_P)$ is the P-localization of H_*W it follows immediately (see also Lemma 4.3 of [1]) that

$$H_*(W; \mathbb{Z}_P) = 0 \quad \text{if and only if } H_*W \text{ is } P'\text{-torsion}\,. \tag{3.3}$$

Now $H_*(W; \mathbb{Q})$ is the rationalization of $H_*(W; \mathbb{Z}_P)$. Moreover, since, for $p \in P$, $\mathbb{Z}/p$ is P-local,

$$H_n(W; \mathbb{Z}/p) = H_n(W; \mathbb{Z}_P) \otimes \mathbb{Z}/p \oplus \operatorname{Tor}(H_{n-1}(W; \mathbb{Z}_P), \mathbb{Z}/p)\,.$$

It is thus plain that if $H_*(W; \mathbb{Z}_P) = 0$, then $H_*(W; \mathbb{Q}) = 0$ and $H_*(W; \mathbb{Z}/p) = 0$ for all $p \in P$. Since, by (3.3), $H_*(W; \mathbb{Q}) = 0$ if and only if H_*W is a torsiongroup, it is obvious that the converse implication of (3.2) follows from (3.3) and

$$H_*(W; \mathbb{Z}/p) = 0 \quad \text{implies that } H_*W \text{ has no } p\text{-torsion}\,. \tag{3.4}$$

In fact, we prove a stronger result than (3.4), namely

Lemma 3.3. *Let A be an abelian group. Then*

(i) Tor $(A, \mathbb{Z}/p) = 0 \Leftrightarrow A$ *has no p-torsion*

(ii) Tor $(A, \mathbb{Z}/p) = 0,\ A \otimes \mathbb{Z}/p = 0 \Leftrightarrow A$ *is p'-local.*

Proof. The exact sequence $\mathbb{Z} \rightarrowtail^{p} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/p$ yields the exact sequence

$$0 \to \mathrm{Tor}\,(A, \mathbb{Z}/p) \to A \xrightarrow{p} A \to A \otimes \mathbb{Z}/p \to 0\,.$$

Thus Tor $(A, \mathbb{Z}/p) = 0 \Leftrightarrow p : A \to A$ is injective $\Leftrightarrow A$ has no p-torsion, establishing (i); and Tor $(A, \mathbb{Z}/p) = 0,\ A \otimes \mathbb{Z}/p = 0 \Leftrightarrow p : A \to A$ is bijective $\Leftrightarrow A$ is p'-local, establishing (ii).

It is plain that Lemma 3.3 establishes (3.4) and hence completes the proof of (3.2).

Remarks. (a) We may obviously replace homology by cohomology in Theorem 3.2.

(b) It was observed in [4] that, if the homology of X, Y is of finite type then $f_* : H_*(X; \mathbb{Z}_P) \cong H_*(Y; \mathbb{Z}_P)$ if and only if $f_* : H_*(X; \mathbb{Z}/p) \cong H_*(Y; \mathbb{Z}/p)$ for all $p \in P$, provided that P is not vacuous. This may be regarded as a corollary of Lemma 3.3, since if A is finitely generated and p'-local then A is finite. Thus, we infer from $H_*(W; \mathbb{Z}/p) = 0$ for all $p \in P$, with H_*W of finite type, that H_*W is P'-torsion, so that $H_*(W; \mathbb{Z}_P) = 0$.

References

[1] *A. Deleanu* and *P. Hilton:* Localization, homology and a construction of Adams, Trans. Am. Math. Soc., *179* (1973), 349—362.

[2] *P. Hilton* and *U. Stammbach:* A course in homological algebra, Springer 1971.

[3] *D. Sullivan:* Genetics of homotopy theory and the Adams conjecture, Ann. of Math., *100* (1974), 1—79.

[4] *P. Hilton, G. Mislin* and *J. Roitberg:* Localization of nilpotent groups and spaces, North-Holland/American Elsevier, 1975.

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