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CONDITIONS FOR STRONG MAXIMALITY OF LOCAL DIFFUSIONS IN MULTI-DIMENSIONAL CASE

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Introduction. Let an Itô stochastic differential equation

$$\mathrm{d}x = a(t, x)\,\mathrm{d}t + B(t, x)\,\mathrm{d}w$$

be given in a region Q, $Q = (0, L) \times D$ where D is a region in the n-dimensional Euclidean space, the n-dimensional vector function $a(t, x)$, the matrix function $B(t, x)$ of the type $n \times n$ and the region D fulfil conditions guaranteeing the existence and unicity of solutions, $w(t)$ is an n-dimensional Wiener process.

Denote by $x(t, x_0)$ the solution of the Itô equation fulfilling the initial condition $x(0, x_0) = x_0$ (x_0 being a deterministic value) and by $P(B, a, x_0, Q)$ the probability that the solution $x(t, x_0)$ leaves the region D during the time interval $\langle 0, L\rangle$, i.e.

$$P(B, a, x_0, Q) = P\{\exists\{\tau : x(\tau, x_0) \notin D,\ \tau \in \langle 0, L\rangle\}\} .$$

The matrix function $B(t, x)$ or $A(t, x)$ ($A(t, x) = B(t, x)\,B^{\mathrm{T}}(t, x)$ where B^{T} is the transposed matrix) is called *strongly maximal with respect to $a(t, x)$ and Q* if

$$P(B, a, x_0, Q) \geqq P(B', a, x_0, Q)$$

for all $x_0 \in D$ and for all matrix functions $B'(t, x)$ fulfilling the conditions guaranteeing the existence and unicity and such that $A(t, x) - A'(t, x)$ ($A'(t, x) = B'(t, x)\,B'^{\mathrm{T}}(t, x)$) is a positive semi-definite matrix at every point $[t, x] \in Q$.

This definition was used in the papers [1], [2], [5] with the following conditions guaranteeing existence and unicity:

i) *$a(t, x)$, $B(t, x)$ are Hölder continuous in t;*
ii) *$a(t, x)$, $B(t, x)$ are Lipschitz continuous in x;*
iii) *$A(t, x) = B(t, x)\,B^{\mathrm{T}}(t, x)$ is uniformly positive definite in Q;*
iv) *the region D is bounded and has the outside strong sphere property* [4].

If the matrix functions $B(t, x)$ and $A(t, x)$ are diagonal at every point $[t, x] \in Q$, the matrix function $B(t, x)$ $(A(t, x))$ is called *maximal with respect to* $a(t, x)$ *and* Q if

$$P(B, a, x_0, Q) \geqq P(B', a, x_0, Q)$$

for all $x_0 \in D$ and for all diagonal matrix functions $B'(t, x)$ fulfilling the conditions guaranteeing the existence and unicity (e.g. i) to iii)) and such that $A(t, x) - A'(t, x)$ is a positive semi-definite matrix at every point $[t, x] \in Q$.

The matrix function $A(t, x)$ $(B(t, x))$ is strongly maximal (maximal) with respect to $a(t, x)$ and Q if and only if the bounded solution $u(t, x)$ of the parabolic equation

$$\frac{\partial u}{\partial t} = \frac{1}{2} \sum_{i,j=1}^{n} A_{ij}(L - t, x) \frac{\partial^2 u}{\partial x_i \, \partial x_j} + \sum_{i=1}^{n} a_i(L - t, x) \frac{\partial u}{\partial x_i}$$

fulfilling $u(0, x) = 0$ for $x \in D$ and $u(t, x) = 1$ for $t > 0$, $x \in \dot{D}$ ($\dot{D}$ is the boundary of D) is a convex function of x $(\partial^2 u / \partial x_i^2 \geqq 0,\ i = 1, \ldots, n)$ in Q.

This result was proved in [2]. The problem to find conditions guaranteeing the strong maximality (the maximality) of $A(t, x)$ is thus transformed to the problem to find conditions guaranteeing convexity $(\partial^2 u / \partial x_i^2 \geqq 0,\ i = 1, \ldots, n)$ of the given solution of the parabolic equation.

The paper [5] is devoted to the one-dimensional case $(n = 1)$ and presents explicit conditions ensuring maximality of $B(x)$ (Theorems 3 and 4 [5]).

The problem to find conditions guaranteeing maximality of $A(t, x)$ in the multi-dimensional case is treated in [1] (Theorem 4 [1]). It is assumed in Theorem 4 [1] that the solution $u(t, x)$ is convex on the side-boundary S of Q, i.e. on $S = \langle 0, L \rangle \times \times \dot{D}$. Therefore conditions are given guaranteeing the convexity of u on S provided the coefficients of the Itô equation as well as the region D depend on a small parameter (Theorem 6 [1]).

The method which enables us to prove Theorems 3 and 4 in [5] is based on Theorem 3 [2] which is valid for linear drift coefficients only. Theorem 1 of the present paper is a modification of Theorem 3 [2] to the multi-dimensional case. Examples 1 to 3 [2] and Lemma 1 [1] show that Theorem 3 [2] cannot be generalized to the multidimensional case without additional assumptions on A and D.

The meaning of Hypotheses (A) to (C) (see page 202–203) is evident. Hypothesis (D) is closely related to Theorem 5 [1] and enables us to investigate more precisely the behaviour of $u(t, x)$ near the points $[0, x]$, $x \in \dot{D}$ where the intitial and boundary values differ.

Hypothesis (E) is simplified to (0, 4) in Remark 2 and in Section 13 it is more closely discussed for $n = 2$ (Examples 1 and 2).

In the Sections 7 to 12 of the paper the possibility of extending Theorem 1 to nonlinear drift coefficients is investigated. Formally it is possible to use the method from [5] which was mentioned above and which is described in more detail in the introduction to Theorems 2 and 3. Since the general case would be too complicated the investigation is limited to weakly nonlinear drift coefficients, i.e. to the drift coefficients of the type $-\lambda_i x_i + \varepsilon\, a_i(x)$ where ε is a small parameter.

Some assumptions of Theorem 3 could be weakened, for example the assumption that a, A are real analytic functions and the assumption that zero is not an eigenvalue of (7, 5). However, these generalizations would lead to considerable technical difficulties.

The last section is devoted to spherically symmetric equations with linear drift coefficients. Example 3 gives conditions under which the problem can be solved by virtue of Theorem 1.

As in the previous papers we shall use the following notation. Let a function $f(x_1, \ldots, x_n)$ be given in a neighbourhood of a point $[x_1^0, \ldots, x_n^0]$. The function is convex or strictly convex at the point $[x_1^0, \ldots, x_n^0]$ if the matrix $\partial^2 f/\partial x_i\, \partial x_j\, (x_1^0, \ldots, x_n^0)$ is positive semi-definite or positive-definite, respectively.

Let l be a nonzero real n-dimensional vector. The derivative of the function f in the direction l at the point $x^0 = [x_1^0, \ldots, x_n^0]$ is denoted by $\mathrm{d}f/\mathrm{d}l(x^0)$ and defined as usual by

$$\mathrm{d}f/\mathrm{d}l(x^0) = \lim_{h \to 0} (f(x^0 + lh) - f(x^0))/h\,.$$

The column vector of the first derivatives of f : $[\partial f/\partial x_1, \ldots, \partial f/\partial x_n]$ will be denoted by $\mathrm{d}f/\mathrm{d}x$ and the matrix of the second derivatives $\partial^2 f/\partial x_i\, \partial x_j$ will be denoted by $\mathrm{d}^2 f/\mathrm{d}x^2$.

Analogously if f is a vector valued function then $\mathrm{d}f/\mathrm{d}x$ denotes the matrix $\{\partial f_i/\partial x_j\}_{i,j}$.

We shall study the partial differential equation of parabolic type

$$\frac{\partial u}{\partial t} = \frac{1}{2} \sum_{i,j=1}^{n} A_{ij}(t, x) \frac{\partial^2 u}{\partial x_i\, \partial x_j} + \sum_{i=1}^{n} a_i(t, x) \frac{\partial u}{\partial x_i} \tag{0,1}$$

in a cylindric region $Q = (0, L) \times D$. Consider a bounded solution fulfilling the initial condition

$$u(0, x) = 0 \quad \text{for} \quad x \in D \tag{0,2}$$

and the boundary condition

$$u(t, x) = 1 \quad \text{for} \quad t > 0\,, \quad x \in \dot{D} \tag{0,3}$$

($\dot{D}$ is the boundary of the region D).

Throughout the paper the following assumptions will be used.

Hypotheses. (A) *The coefficients $a_i(t, x)$ are Hölder continuous in $\bar{Q}$ ($\bar{Q}$ is the closure of Q) and $A_{ij}(t, x)$ have bounded and continuous derivatives $\partial A_{ij}/\partial x_k$, $\partial^2 A_{ij}/\partial x_k\, \partial x_l$, $\partial A_{ij}/\partial t$, $\partial^3 A_{ij}/\partial x_k\, \partial x_l\, \partial x_m$ and $\partial^2 A_{ij}/\partial t^2$. The matrix function $A(t, x)$ is uniformly positive definite in Q, i.e., there exists a constant K_1 such that $\sum_{i,j=1}^{n} A_{ij}(t, x)\, \lambda_i \lambda_j \geqq K_1 \sum_{i=1}^{n} \lambda_i^2$ for all real numbers λ_i.*

(B) *The region D is bounded and to every point $P \in \dot{D}$ there exists a ball K with its centre at P and a system of orthogonal coordinates $\hat{x}_1, \ldots, \hat{x}_n$ where $\hat{x}_n$ has the direction of the inward normal to $\dot{D}$ with respect to D at the point P such that the boundary $\dot{D}$ can be expressed in the ball as a function $\hat{x}_n = h(\hat{x}_1, \ldots, \hat{x}_{n-1})$ for $[\hat{x}_1, \ldots, \hat{x}_{n-1}] \in K^+ \subset K^*$ with Hölder continuous second derivatives. The set K^* is defined by $K^* = \{[\hat{x}_1, \ldots, \hat{x}_{n-1}] : [\hat{x}_1, \ldots, \hat{x}_{n-1}, 0] \in K\}$ and K^+ is an open subset of K^* containing the origin of the $\hat{x}_1, \ldots, \hat{x}_{n-1}$ – coordinate system.*

(C) *The region D is strictly convex, i.e., the matrices $\partial^2 h/\partial\hat{x}_i\, \partial\hat{x}_j(0)$ are positive definite for every $P \in \dot{D}$.*

Denote by $\bar{x}_1, \ldots, \bar{x}_n$ a local coordinate system corresponding to P, $P \in \dot{D}$. We do not require the local coordinate system $\bar{x}_1, \ldots, \bar{x}_n$ to be orthogonal at the moment but we suppose that the boundary of D in a neighbourhood of P can be expressed by $\bar{x}_n = \bar{h}(\bar{x}_1, \ldots, \bar{x}_{n-1})$ where $\bar{h}(0, \ldots, 0) = 0$, $\partial\bar{h}/\partial\bar{x}_k(0, \ldots, 0) = 0$ for $k = 1, \ldots, n-1$. Denote by $\bar{A}_{ij}$, $\bar{a}_i$ the coefficients of (0,1) in the $\bar{x}_1, \ldots, \bar{x}_n$ – coordinate system, i.e.

$$\frac{\partial\bar{u}}{\partial t} = \frac{1}{2}\sum_{i,j=1}^{n} \bar{A}_{ij}(t, \bar{x}) \frac{\partial^2 \bar{u}}{\partial\bar{x}_i\, \partial\bar{x}_j} + \sum_{i=1}^{n} \bar{a}_i(t, \bar{x}) \frac{\partial\bar{u}}{\partial\bar{x}_i}, \quad \bar{u}(t, \bar{x}) = u(t, x).$$

Let the coordinate system $\bar{x}_1, \ldots, \bar{x}_n$ – be chosen so that $\bar{A}(0, \ldots, 0) = I$ (I is the unit matrix). Define the matrix Γ: $\Gamma_{ij} = \partial^2\bar{h}/\partial\bar{x}_i\, \partial\bar{x}_j(0, \ldots, 0)$ for $i, j = 1, \ldots, n-1$, $\Gamma_{in} = \Gamma_{ni} = \frac{1}{2}\,\partial\bar{A}_{nn}/\partial\bar{x}_i(0, \ldots, 0)$, $i < n$, $\Gamma_{nn} = 2\bar{a}_n(0, \ldots, 0) - \sum_{i=1}^{n-1} \Gamma_{ii}$.

(D) *The determinant of Γ is uniformly positive, i.e.,* $\inf_P \det \Gamma > 0$.

The value of $\det \Gamma$ is independent of the choice of the coordinate system $\bar{x}_1, \ldots, \bar{x}_n$ if only the above formulated conditions are fulfilled.

(E) *The coefficients $a_i(t, x)$ are linear in x and*

$$\operatorname{tr}\left[\frac{1}{2} AA^{\mathrm{T}} \frac{\mathrm{d}^2 A}{\mathrm{d}l^2}(t, x) + ABA^{\mathrm{T}} \frac{\mathrm{d}A}{\mathrm{d}l}(t, x) + AB^2A^{\mathrm{T}} A(t, x)\right] \geqq 0$$

for every point $[t, x] \in Q$, every real matrix A of the type $n \times n$, with orthogonal columns, every real symmetric matrix B of the type $n \times n$ which has the same zero columns as A (i.e., if the k-th column of A is the zero vector then also the k-th column of B is the zero vector) and for every unit vector l fulfilling $A^{\mathrm{T}} l = 0$.

The next theorem is very similar to Theorem 4 [1]. The assumptions on the diffusion coefficients in Theorem 4 [1] are very restrictive and they do not permit any application similar to Theorem 3 of the present paper. Theorem 1 is more applicable than Theorem 4 [1] in spite of its more complicated structure.

Theorem 1. *Let $u(t, x)$ be the bounded solution of* (0,1) *fulfilling* (0,2) *and* (0,3). *Suppose that $u(t, x)$ is convex at the points $[t, x]$, $x \in \dot{D}$, $0 < t < L$. If the assumptions* (A), (B), (C), (D) *and* (E) *are fulfilled, then $u(t, x)$ is convex with respect to the spatial variables $x_1, \ldots, x_n$ in Q.*

Remark 1. Hypothesis (D) can be formulated in another way. Choose $P \in \dot{D}$. In virtue of (B) there exists a system of orthogonal coordinates $\hat{x}_1, \ldots, \hat{x}_n$. Let the coordinate systems $x_1, \ldots, x_n$, $\hat{x}_1, \ldots, \hat{x}_n$ be related by $x = P + T\hat{x}$ where T is a unitary matrix. Let $\hat{A}_{ij}(t, \hat{x})$, $\hat{a}_i(t, \hat{x})$ be the coefficients of (0,1) in the $\hat{x}_1, \ldots, \hat{x}_n$ coordinate system. Denote $\hat{\Gamma}$ the $(n \times n)$ – matrix defined as follows: $\hat{\Gamma}_{ij}$ are the elements of the matrix $((L^{(1)})^{-1})^{\mathrm{T}} H (L^{(1)})^{-1}$ for $i, j = 1, \ldots, n-1$, $\hat{\Gamma}_{in} = \hat{\Gamma}_{ni} = \frac{1}{2} \sum_{s=1}^{n} \partial \hat{A}_{nn}(0,0)/\partial \hat{x}_s (L^{(1)})^{-1}_{si} (\hat{A}_{nn}(0,0))^{-1/2}$ for $i < n$ and $\hat{\Gamma}_{nn} = 2\hat{a}_n(0,0) - \sum_{i=1}^{n-1} \hat{\Gamma}_{ii}$ where H is the matrix of the type $(n-1) \times (n-1)$ whose elements are: $H_{ij} = \partial^2 h/\partial \hat{x}_i \, \partial \hat{x}_j(0)$ while $L^{(1)}$ is the matrix of the type $(n-1) \times (n-1)$ which is constructed in Lemma 2 [1]. Hypothesis (D) is equivalent to the assumption $\inf_P \det \hat{\Gamma} > 0$. The value of $\det \hat{\Gamma}$ is independent of the choice of $L^{(1)}$ (under the condition that the assumptions of Lemma 2 [1] are fulfilled). Remark 1 together with Lemma 2 [1] provide a method for evaluation of the determinant of $\hat{\Gamma}$.

Remark 2. *Let $\mathrm{d}^2 A/\mathrm{d}l^2(t, x)$ be positive semi-definite in Q for all unit vectors l. Obviously there exist matrix functions $M(t, x)$, $N_l(t, x)$, $P_l(t, x)$ defined in Q all of the type $n \times n$ fulfilling*

$$M^{\mathrm{T}}(t, x)\, M(t, x) = A(t, x)\,, \quad N_l^{\mathrm{T}}(t, x)\, N_l(t, x)\, n^2 = \mathrm{d}^2 A/\mathrm{d}l^2(t, x)\,,$$
$$P_l(t, x) = \mathrm{d}A/\mathrm{d}l(t, x)\,.$$

If

$$M^{\mathrm{T}}(t, x)\, N_l(t, x) + N_l^{\mathrm{T}}(t, x)\, M(t, x) + \sqrt{(2)}\, P_l(t, x) \quad \text{and} \tag{0,4}$$
$$M^{\mathrm{T}}(t, x)\, N_l(t, x) + N_l^{\mathrm{T}}(t, x)\, M(t, x) - \sqrt{(2)}\, P_l(t, x)$$

are positive semi-definite matrices for every $[t, x] \in \bar{Q}$ and for every unit vector l, then

$$\sum_{i=1}^{q-1} \frac{\partial^2 \tilde{A}_{ii}}{\partial y_q^2} \alpha_i^2 + 2 \sum_{i,j=1}^{q-1} \frac{\partial \tilde{A}_{ij}}{\partial y_q} \alpha_i \alpha_j \beta_{i,j} + 2 \sum_{i,j=1}^{q-1} \tilde{A}_{ij} \alpha_i \alpha_j \sum_{p=1}^{q-1} \beta_{ip} \beta_{jp} \geqq 0 \tag{0,5}$$

for all real numbers α_i, $\beta_{ij} (\beta_{ij} = \beta_{ji})$ any index q, $q > 1$ and an arbitrary unitary matrix T where $x = Ty$, $\tilde{A}(t, y) = T^{\mathrm{T}} A(t, x)\, T$.

Inequality (0,5) *implies* (4,4) *and is equivalent with Hypothesis* (E) *on* A.

It means that the assumptions of the remark together with the linearity of a_i can substitute Hypothesis (E). Even though the assumptions of the Remark are stronger than Hypothesis (E), they are simpler and more explicit so that they can be used in the next theorem.

The equivalence of the Hypothesis (E) on A with (0,5) is obtained easily by putting $A_{ij} = T_{ij}\alpha_j$, $B_{ij} = \beta_{ij}$ where $x = Tz$ is the transformation from Remark 2, and applying the well-known relations $\sum C_{ij} F_{ij} = \operatorname{tr} F^{\mathrm{T}} C$, $\operatorname{tr} CF = \operatorname{tr} FC$.

Proof of Remark 2. Inequality (0,5) can be rewritten in the form

$$2\sum_{i=1}^{n}(Bz_i, \beta^2 z_i) + 2\sum_{i=1}^{n}(B'z_i, \beta z_i) + \sum_{i=1}^{n}\left(\frac{\partial^2 \tilde{A}_{ii}}{\partial y_q^2}\alpha_i^2 z_i, z_i\right) \geqq 0$$

choosing

$$\alpha_i = \beta_{ij} = 0 \quad \text{for} \quad i, j \geqq q .$$

where z_i is the system of orthonormal vectors which is given by the columns of the unit matrix. The elements of the matrix B are $\tilde{A}_{ij}\alpha_i\alpha_j$ while the elements of the matrix B' are $(\partial\tilde{A}_{ij}/\partial y_q)\,\alpha_i\alpha_j$. The elements of β are β_{ij}. As in the proof of Lemma 3 [2] we can substitute the vectors z_i by the eigenvectors of the matrix β (the matrix β is symmetric). Nonetheless, we preserve the notation of z_i without a change while the eigenvalues of β will be denoted by λ_i. Then the last inequality is equivalent to

$$2\sum_i \lambda_i^2(Bz_i, z_i) + 2\sum_i \lambda_i(B'z_i, z_i) + \sum_i\left(\frac{\partial^2\tilde{A}_{ii}}{\partial y_q^2}\alpha_i^2 z_i, z_i\right) \geqq 0 .$$

Inequality (0,5) will be obviously valid if

$$2\lambda^2 \sum_{i,j}\tilde{A}_{ij}v_iv_j + 2\lambda\mu\sum_{i,j}\frac{\partial\tilde{A}_{ij}}{\partial y_q}v_iv_j + \frac{\mu^2}{n}\sum_i\frac{\partial^2\tilde{A}_{ii}}{\partial y_q^2}v_i^2 \geqq 0$$

where v_i, λ, μ are arbitrary real numbers. The relation between A and $\tilde{A}$ implies that the last inequality is equivalent to

$$(0,6) \qquad 2\lambda^2\sum_{p,q}A_{pq}\sum_{i,j}T_{pi}T_{qj}v_iv_j + 2\lambda\mu\sum_{p,q}\frac{\mathrm{d}A_{pq}}{\mathrm{d}l}\sum_{i,j}T_{pi}T_{qj}v_iv_j +$$
$$+\frac{\mu^2}{n}\sum_{p,q}\frac{\mathrm{d}^2A_{pq}}{\mathrm{d}l^2}\sum_i T_{pi}T_{qi}v_i^2 \geqq 0\,, \quad l = [T_{1,q}\ldots T_{n,q}]$$

where T_{ij} are the elements of the unitary matrix T. Suppose $T = T_1T_2$ where T_1, T_2 are unitary matrices, T_1 transforming $\mathrm{d}^2A/\mathrm{d}l^2$ into a diagonal matrix, i.e. $T_1^{\mathrm{T}}\,\mathrm{d}^2A/\mathrm{d}l^2\,T_1 = S$ where S is a diagonal matrix. The last term in (0,6) can be rewritten as

$$\frac{\mu^2}{n}\sum_i(T^{\mathrm{T}}\,\mathrm{d}^2A/\mathrm{d}l^2\,T)_{ii}\,v_i^2 = \frac{\mu^2}{n}\sum_i(T_2^{\mathrm{T}}ST_2)_{ii}\,v_i^2 = \frac{\mu^2}{n}\sum_p S_{pp}\sum_i(T_2)_{pi}^2v_i^2 \geqq$$

$$\geqq \frac{\mu^2}{n^2}\sum_p S_{pp}(\sum_i(T_2)_{pi}\,v_i)^2 = \frac{\mu^2}{n^2}\sum_{i,j}(T_2^{\mathrm{T}}ST_2)_{ij}\,v_iv_j = \frac{\mu^2}{n^2}\sum_{i,j}(T^{\mathrm{T}}\,\mathrm{d}^2A/\mathrm{d}l^2T)_{ij}\,v_iv_j\,.$$

Using this estimate we see that (0,5) is valid if

$$2\lambda^2 \sum_{p,q} A_{pq} w_p w_q + 2\lambda\mu \sum_{p,q} \frac{dA_{pq}}{dl} w_p w_q + \frac{\mu^2}{n^2} \sum_{p,q} \frac{d^2 A_{pq}}{dl^2} w_p w_q \geqq 0$$

where w_p are arbitrary real numbers. Recalling the definitions of M, P_l, N_l we conclude

$$(0,7) \quad 2\lambda^2 M^T M + 2\lambda\mu P_l + \mu^2 N_l^T N_l = (\lambda \sqrt{(2)}\, M^T + \mu N_l^T)(\lambda \sqrt{(2)}\, M + \mu N_l) + \\ + \lambda\mu \sqrt{(2)} (\sqrt{(2)}\, P_l - M^T N_l - N_l^T M)$$

and

$$(0,8) \quad 2\lambda^2 M^T M + 2\lambda\mu P_l + \mu^2 N_l^T N_l = (\lambda \sqrt{(2)}\, M^T - \mu N_l^T)(\lambda \sqrt{(2)}\, M - \mu N_l) + \\ + \lambda\mu \sqrt{(2)} (\sqrt{(2)}\, P_l + M^T N_l + N_l^T M)\,.$$

If $0 < \lambda\mu$ we use the first assumption of (0,4) and the relation (0,8). If $0 > \lambda\mu$ we use the second assumption of (0,4) and the relation (0,7). In both cases we obtain that the left hand sides of (0,7) or (0,8) are positive definite which proves Remark 2. The proof of Theorem 1 is divided into several lemmas.

1

Lemma 1. *Let D be a region in a p-dimensional Euclidean space. Assume that a positive semi-definite symmetric matrix function $M(x_1, \ldots, x_p)$ is defined on D. Let the matrix $M(x_1, \ldots, x_p)$ be of the type $n \times n$ and let the matrix function be continuous in D. If there exists a point $[x_1^0, \ldots, x_p^0] \in D$ such that $M(x_1^0, \ldots, x_p^0)$ is positive definite than there exists a region D_1, $D_1 \subset D$, $[x_1^0, \ldots, x_p^0] \in D_1$ such that $M(x_1, \ldots, x_p)$ is positive definite in D_1, it is not positive definite on $\dot{D}_1 \cap D$ and* det $M(x_1, \ldots, x_p) = 0$ *on $\dot{D}_1 \cap D$.*

Proof. Evidently, the set of points at which the matrix M is positive definite is open. Denote by D_1 the maximal region containing $[x_1^0, \ldots, x_n^0]$ in which M is positive definite. Certainly M is not positive definite on $\dot{D}_1 \cap D$. It remains to prove the statement about det M. Let $[x_1', \ldots, x_p'] \in \dot{D}_1 \cap D$, then there exists a nonzero n-dimensional vector y' such that

$$(1,1) \qquad (M(x_1', \ldots, x_p')\, y', y') = 0$$

and

$$(1,2) \qquad (M(x_1', \ldots, x_p')\, y, y) \geqq 0 \quad \text{for every vector } y\,.$$

Since $M(x_1', \ldots, x_p')$ is a symmetric matrix there exists such a real unitary matrix T that $M(x_1', \ldots, x_p') = T^T S T$ where S is a diagonal matrix. Denote $z = Ty$ and

$z' = Ty'$. Using (1,1) we obtain

$$\sum_i S_{ii}(z_i')^2 = 0 \tag{1,3}$$

and using (1,2) we obtain $\sum_i S_{ii}(z_i)^2 \geqq 0$. The last inequality implies $S_{ii} \geqq 0$ and then (1,3) implies $S_{ii} = 0$ for some i. It means that the determinant of S and also that of $M(x_1', \ldots, x_p')$ must be zero. Lemma 1 is proved.

In the sequel we shall need the following remark which is presented without proof.

Remark 3. *Let M be a positive semi-definite symmetric matrix of a type $n \times n$. If $M_{ii} = 0$ for some i, then $M_{ij} = M_{ji} = 0$ for $j = 1, \ldots, n$.*

In order to prove the convexity of $u(t, x)$ we need some approximations of $u(t, x)$. In the next section an approximation of $u(t, x)$ will be used which involves four terms. The reason is that the second derivatives which are crucial for the convexity are unbounded near the points $[0, x], x \in \dot{D}$. Due to a specific behaviour of $u(t, x)$ near such points the convexity of $u(t, x)$ can be proved only in a part of their neighbourhoods. These parts of neighbourhoods are denoted by Q_p. The corresponding statement is given in Lemma 2. The second part of the proof consists in the study of convexity in the whole region Q. Since the original equation is transformed in the course of the proof the last part of the proof consists in the reformulation of results.

First we carry out transformations which simplify the necessary estimates. The coefficients $a_i(t, x)$, $A_{ij}(t, x)$ can be extended to the whole strip $(0, L) \times R^n$ so that assumption (A) is fulfilled. Let a point $P \in \dot{D}$ be given. Due to Hypothesis (A) there exists a linear transformation which maps $u(t, x)$ to $\bar{u}(t, \bar{x})$ and equation (0,1) to a parabolic equation with the coefficients $\bar{A}_{ij}(t, \bar{x})$, $\bar{a}_i(t, \bar{x})$ such that $\bar{A}(0, 0) = I$ (the unit matrix). The second part of Lemma 2 [1] states that there exist a number $\delta > 0$ and a function $h^0(\bar{x}_1, \ldots, \bar{x}_n)$ such that $h^0(\bar{x}_1, \ldots, \bar{x}_n) = \bar{h}(\bar{x}_1, \ldots, \bar{x}_n)$ for $|\bar{x}| < \delta$, $h^0(\bar{x}_1, \ldots, \bar{x}_n) = 0$ for $|x| > 2\delta$ and the function has all continuous derivatives. The function $\bar{h}$ is the same as in the definition of Γ – see Hypothesis (D). Denote

$$\bar{x}_i = y_i\,, \quad i = 1, \ldots, n-1\,, \quad \bar{x}_n = y_n + h^0(y_1, \ldots, y_{n-1})\,, \tag{1,4}$$
$$\bar{u}(t, \bar{x}) = v(t, y)\,.$$

Equation (0,1) is transformed to

$$\frac{\partial v}{\partial t} = \frac{1}{2} \sum_{i,j} A_{ij}^0(t, y) \frac{\partial^2 v}{\partial y_i \partial y_j} + \sum_i a_i^0(t, y) \frac{\partial v}{\partial y_i} \tag{1,5}$$

where a_i^0, A_{ij}^0 are determined in (3,10) [1]. The image of the region D will be denoted by $D(P)$. Evidently the boundary of $D(P)$ is described in the δ-neighbourhood of P by $y_n = 0$. Further, $\delta > 0$ is chosen such that assumption (A) is valid for (1,5). Certain approximations of $v(t, y)$ will be constructed in the next section.

2

We shall submit the equation to still another transformation

$$t = \lambda\tau\,, \quad y = \xi\sqrt{\lambda}\,, \quad v(t, y) = v_\lambda(\tau, \xi) \quad \text{where} \quad \lambda > 0\,. \tag{2,0}$$

Equation (0,1) is transformed to

$$\frac{\partial v}{\partial \tau} = \frac{1}{2}\sum_{i,j} A^0_{ij}(\tau\lambda, \xi\sqrt{\lambda})\frac{\partial^2 v}{\partial \xi_i\,\partial \xi_j} + \sum_i a^0_i(\tau\lambda, \xi\sqrt{\lambda})\sqrt{\lambda}\,\frac{\partial v}{\partial \xi_i} \tag{2,1}$$

and the conditions $v_\lambda(0, \xi) = 0$ for $\xi \in D(P, \lambda)$, $v_\lambda(\tau, \xi) = 1$ for $\tau > 0$, $\xi \in \dot{D}(P, \lambda)$ where $D(P, \lambda)$ is the image of $D(P)$.

Let $v_0(\tau, \xi)$ be the solution of $\partial v_0/\partial\tau = \frac{1}{2}\sum_i \partial^2 v_0/\partial\xi_i^2$ fulfilling $v_0(0, \xi) = 0$ for $\xi \in D(P, 0)$ and $v_0(\tau, \xi) = 1$ for $\tau > 0$, $\xi \in \dot{D}(P, 0)$ where $D(P, 0) = \{\xi : \xi_n > 0\}$. We have

$$v_0(\tau, \xi) = 1 - \sqrt{\left(\frac{2}{\pi}\right)}\int_0^{\xi_n/\sqrt{\tau}} e^{-\mu^2/2}\,\mathrm{d}\mu\,. \tag{2,2}$$

Let $\Delta_0(\tau, \xi)$ be the solution of

$$\frac{\partial \Delta}{\partial \tau} = \frac{1}{2}\sum_i \frac{\partial^2 \Delta}{\partial \xi_i^2} + \frac{1}{2}\frac{\partial^2 v_0}{\partial \xi_n^2}\sum_i \frac{\partial A^0_{nn}}{\partial y_i}(0, 0)\,\xi_i + a^0_n(0, 0)\,\frac{\partial v_0}{\partial \xi_n}$$

fulfilling $\Delta_0(0, \xi) = 0$ for $\xi \in D(P, 0)$ and $\Delta_0(\tau, \xi) = 0$ for $\tau > 0$, $\xi \in \dot{D}(P, 0)$. The solution $\Delta_0(\tau, \xi)$ can be written in the form

$$\Delta_0(\tau, \xi) = \sum_{i=1}^{n-1} \xi_i\,\Delta_i(\tau, \xi_n) + \Delta_n(\tau, \xi_n) \tag{2,3}$$

where

$$\Delta_i(\tau, \xi_n) = \frac{1}{\sqrt{(2\pi)}}\,\frac{\partial A^0_{nn}}{\partial y_i}(0, 0)\,\frac{\xi_n}{\sqrt{\tau}}\,e^{-\xi_n{}^2/2\tau}\,, \quad i = 1, \dots, n-1$$

and

$$\Delta_n(\tau, \xi_n) = -2a^0_n(0, 0)\,\frac{1}{\pi}\,e^{-\xi_n{}^2/2\tau}\sqrt{\tau}\int_0^1\int_0^{(\xi_n/\sqrt{\tau})\sqrt{(\lambda/(1-\lambda))}} e^{-\mu^2/2}\,\mathrm{d}\mu\,\mathrm{d}\lambda +$$

$$+ \frac{\partial A^0_{nn}}{\partial y_n}(0, 0)\,\frac{1}{\pi}\,e^{-\xi_n{}^2/2\tau}\sqrt{\tau}\int_0^1\left[\frac{\xi_n}{\sqrt{\tau}}\sqrt{(\lambda(1-\lambda))}\,e^{-(\xi_n{}^2/2\tau)(\lambda/(1-\lambda))} +\right.$$

$$\left. + \left(1 - \lambda + \frac{\xi_n^2}{\tau}\lambda\right)\int_0^{(\xi_n/\sqrt{\tau})\sqrt{(\lambda/(1-\lambda))}} e^{-\mu^2/2}\,\mathrm{d}\mu\right]\mathrm{d}\lambda\,.$$

Further, let $\varrho_0(\tau, \xi)$ be the solution of

$$(2,4)\qquad \frac{\partial \varrho_0}{\partial \tau} = \frac{1}{2} \sum_i \frac{\partial^2 \varrho_0}{\partial \xi_i^2} + \frac{1}{2} \sum_{i,j} \frac{\partial^2 \Delta_0}{\partial \xi_i \, \partial \xi_j} \sum_k \frac{\partial A_{ij}^0}{\partial y_k}(0, 0)\, \xi_k +$$

$$+ \sum_i a_i^0(0, 0) \frac{\partial \Delta_0}{\partial \xi_i} + \frac{1}{2} \frac{\partial^2 v_0}{\partial \xi_n^2} \left[\frac{\partial A_{nn}^0}{\partial t}(0, 0)\, \tau + \right.$$

$$\left. + \frac{1}{2} \sum_{k,l} \frac{\partial^2 A_{nn}^0}{\partial y_k \, \partial y_l}(0, 0)\, \xi_k \xi_l \right] + \frac{\partial v_0}{\partial \xi_n} \sum_k \frac{\partial a_n^0}{\partial y_k}(0, 0)\, \xi_k$$

fulfilling $\varrho_0(0, \xi) = 0$ for $\xi \in D(P, 0)$ and $\varrho_0(\tau, \xi) = 0$ for $\tau > 0$, $\xi \in \dot{D}(P, 0)$. Put

$$R_\lambda(\tau, \xi) = \frac{v_\lambda(\tau, \xi) - v_0(\tau, \xi)}{\lambda \sqrt{\lambda}} - \frac{\Delta_0(\tau, \xi)}{\lambda} - \frac{\varrho_0(\tau, \xi)}{\sqrt{\lambda}}.$$

The function $R_\lambda(\tau, \xi)$ is the solution of

$$(2,5)\qquad \frac{\partial R_\lambda}{\partial \tau} = \frac{1}{2} \sum_{i,j} A_i^0(\tau\lambda, \xi \sqrt{\lambda}) \frac{\partial^2 R_\lambda}{\partial \xi_i \, \partial \xi_j} + \sum_i a_i^0(\tau\lambda, \xi \sqrt{\lambda}) \sqrt{\lambda} \frac{\partial R_\lambda}{\partial \xi_i} +$$

$$+ \frac{1}{2\sqrt{\lambda}} \sum_{i,j} \left(A_{ij}^0(\tau\lambda, \xi \sqrt{\lambda}) - A_{ij}^0(0, 0) \right) \frac{\partial^2 \varrho_0}{\partial \xi_i \, \partial \xi_j} + \sum_i a_i^0(\tau\lambda, \xi \sqrt{\lambda}) \frac{\partial \varrho_0}{\partial \xi_i} +$$

$$+ \frac{1}{2\lambda} \sum_{i,j} \frac{\partial^2 \Delta_0}{\partial \xi_i \, \partial \xi_j} \left(A_{ij}^0(\tau\lambda, \xi \sqrt{\lambda}) - A_{ij}^0(0, 0) - \sqrt{\lambda} \sum_k \frac{\partial A_{ij}^0}{\partial y_k}(0, 0)\, \xi_k \right) +$$

$$+ \frac{1}{\sqrt{\lambda}} \sum_i \left(a_i^0(\tau\lambda, \xi \sqrt{\lambda}) - a_i^0(0, 0) \right) \frac{\partial \Delta_0}{\partial \xi_i} + \frac{1}{2\lambda \sqrt{\lambda}} \left(A_{nn}^0(\tau\lambda, \xi \sqrt{\lambda}) - \right.$$

$$- A_{nn}^0(0, 0) - \sqrt{\lambda} \sum_i \frac{\partial A_{nn}^0}{\partial y_i}(0, 0)\, \xi_i - \lambda \frac{\partial A_{nn}^0}{\partial t}(0, 0)\, \tau -$$

$$\left. - \frac{\lambda}{2} \sum_{k,l} \frac{\partial^2 A_{nn}^0}{\partial y_k \, \partial y_l}(0, 0)\, \xi_k \xi_l \right) \frac{\partial^2 v_0}{\partial \xi_n^2} + \frac{1}{\lambda} \left(a_n^0(\tau\lambda, \xi \sqrt{\lambda}) - a_n^0(0, 0) - \right.$$

$$\left. - \sqrt{\lambda} \sum_k \frac{\partial a_n^0}{\partial y_k}(0, 0)\, \xi_k \right) \frac{\partial v_0}{\partial \xi_n}$$

fulfilling $R_\lambda(0, \xi) = 0$ for $\xi \in D(P, \lambda)$ and

$$R_\lambda(\tau, \xi) = \frac{1 - v_0(\tau, \xi)}{\lambda \sqrt{\lambda}} - \frac{\Delta_0(\tau, \xi)}{\lambda} - \frac{\varrho_0(\tau, \xi)}{\sqrt{\lambda}} \quad \text{for} \quad \tau > 0, \quad \xi \in \dot{D}(P, \lambda).$$

We need estimates for the solution R_λ. To this aim we write $R_\lambda(\tau, \xi) = \sigma_\lambda(\tau, \xi) + R_\lambda^*(\tau, \xi)$ where $\sigma_\lambda(\tau, \xi)$ is the solution of (2,5) in $(0, L) \times D(P, 0)$ fulfilling the zero initial and boundary conditions. The function R_λ^* is then the solution of (2,1) in $(0, L) \times D(P, \lambda)$ fulfilling the zero initial condition and the boundary condition

$$(2,6) \qquad R_\lambda^*(\tau, \xi) = \frac{1 - v_0(\tau, \xi)}{\lambda \sqrt{\lambda}} - \frac{\Delta_0(\tau, \xi)}{\lambda} - \frac{\varrho_0(\tau, \xi)}{\sqrt{\lambda}} - \sigma_\lambda(\tau, \xi)$$

$$\text{for} \quad \tau > 0\,, \quad \xi \in \dot{D}(P, \lambda)\,.$$

We shall need the solutions $v_0(\tau, \xi)$, $\Delta_0(\tau, \xi)$ in an explicit form – see (2,2) and (2,3). The explicit form of ϱ_0 would be too complicated but it is not necessary since the following estimates are sufficient

$$(2,7) \qquad |\varrho_0(\tau, \xi)| \leqq c_1 e^{-\xi_n^2/2\tau} \left(1 + \frac{1}{\tau} \sum_{k \neq n} \xi_k^2\right) \tau\,,$$

$$(2,8) \qquad \left|\frac{\partial \varrho_0(\tau, \xi)}{\partial \xi_i}\right| \leqq c_1 e^{-\xi_n^2/2\tau} \left(1 + \frac{|\xi_n|^6}{\tau^3}\right) \left(1 + \frac{1}{\tau} \sum_{k \neq n} \xi_k^2\right) \sqrt{\tau}\,,$$

$$\left|\frac{\partial^2 \varrho_0(\tau, \xi)}{\partial \xi_i\, \partial \xi_j}\right| \leqq c_1 e^{-(\xi_n^2/2\tau)} \left(1 + \frac{|\xi_n|^7}{\tau^3 \sqrt{\tau}}\right) \left(1 + \frac{1}{\tau} \sum_{k \neq n} \xi_k^2\right).$$

These estimates can be derived by means of the formula

$$\varrho_0(\tau, \xi) = (2\pi)^{-n/2}\, \tau e^{-(\xi_n^2/2\tau)} \Bigg[\int_0^1 \int_{-\infty}^{\infty} \dots$$

$$\dots \int_{-\infty}^{\infty} \int_{-(\xi_n/\sqrt{(\tau)})\sqrt{(v/(1-v))}}^{\infty} \sqrt{(v)}\, e^{-(1/2)\sum_i \mu_i^2} \varphi(\tau v, \xi_1 + \mu_1 \sqrt{(\tau(1 - v))}, \dots$$

$$\dots \xi_{n-1} + \mu_{n-1} \sqrt{(\tau(1 - v))}, \quad \xi_n v + \mu_n \sqrt{(\tau v(1 - v))})\, d\mu_n\, d\mu_{n-1} \dots d\mu_1\, dv -$$

$$- \int_0^1 \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \int_{(\xi_n/\sqrt{(\tau)})\sqrt{(v/(1-v))}}^{\infty} \sqrt{(v)}\, e^{-(1/2)\sum_i \mu_i^2} \varphi(\tau v, \xi_1 + \mu_1 \sqrt{(\tau(1 - v))}, \dots$$

$$\dots \xi_{n-1} + \mu_{n-1} \sqrt{(\tau(1 - v))}, \quad \xi_n v - \mu_n \sqrt{(\tau v(1 - v))})\, d\mu_n\, d\mu_{n-1} \dots d\mu_1\, dv \Bigg],$$

where $\exp\{-\xi_n^2/2\tau\}\, \varphi(\tau, \xi)$ is the nonhomogeneous term of (2,4). The coefficient c_1 is independent of P. Estimates for σ_λ can be derived analogously. We have

$$(2,9) \qquad \sigma_\lambda(\tau, \xi) = \lambda^{n/2} \int_0^\tau \int_{-\infty}^{\infty} \dots$$

$$\dots \int_{-\infty}^{\infty} \int_0^{\infty} \Gamma(\tau\lambda, \xi \sqrt{\lambda};\; v\lambda, \eta \sqrt{\lambda})\, e^{-\eta_n^2/2v}\, \varphi(v, \eta)\, d\eta_n \dots d\eta_1\, dv\,,$$

where $\Gamma(t, x; \xi, y)$ is the Green function of (1,5) with respect to the region $D(P, 0)$ and $e^{-\xi_n^2/2\tau}\, \varphi(\tau, \xi)$ is the nonhomogeneous term of (2,5).

Owing to Theorem 16.3 [3],

$$|\Gamma(t, x; \xi, y)| \leqq c_1|t - \xi|^{-n/2} \exp\{-c_2 \sum_i |x_i - y_i|^2/(2(t - \xi))\}$$

and thus (2,9) implies

$$|\sigma_\lambda| \leqq c_1 \tau e^{-c_2(\xi_n^2/2\tau)} \int_0^1 \int_{-\infty}^{\infty} \dots$$

$$\dots \int_{-\infty}^{\infty} \int_{(\xi_n/\sqrt{\tau})\sqrt{(v/(1-v))}}^{\infty} \sqrt{(v)}\, e^{-(c_2/2)\sum_i \mu_i^2} |\varphi(\tau v, \xi_1 + \mu_1 \sqrt{(\tau(1 - v))}, \dots$$

$$\dots \xi_{n-1} + \mu_{n-1}\sqrt{(\tau(1 - v))}, \quad \xi_n v + \mu_n \sqrt{(\tau v(1 - v))})| \, d\mu_n \dots d\mu_1 \, dv .$$

The last inequality yields

$$|\sigma_\lambda(\tau, \xi)| \leqq c_1 \tau^{3/2} e^{-(c_2/2)(\xi_n^2/2\tau)} \left(1 + \frac{1}{\tau^{3/2}} \sum_{k \neq n} |\xi_k|^3\right). \tag{2,10}$$

We still have to estimate the function $R_\lambda^*(\tau, \xi)$. Put $\hat{R}_\lambda(t, y) = R_\lambda^*(\tau, \xi)$, using the inverse transformation to (2,0). The function $\hat{R}_\lambda(t, y)$ is a solution of (1,5) fulfilling zero initial condition. Using (2,6), estimates (2,10), (2,7) and formulae (2,3), (2,2) we obtain $|\hat{R}_\lambda(t, y)| \leqq c\lambda^{-3/2}$ for $t > 0$, $y \in D(P)$ where c is a constant and $\hat{R}_\lambda(t, y) = 0$ for $t > 0$, $y = [y_1, \dots, y_{n-1}, 0]$, $\|y\| < \delta$. Suppose $r > 0$ is such a number that the ball K_r with radius r and with its centre at the origin of the $y_1 \dots y_n$ – coordinate system fulfils $K_r \cap D(P, 0) \subset D(P)$. Denote by $\bar{R}(t, y)$ the solution of (1,5) fulfilling $\bar{R}(0, y) = 0$ for $y \in K_r$, $\bar{R}(t, y) = 1$ for $t > 0$, $y \in \dot{K}_r$. Certainly $|\hat{R}_\lambda(t, y)| \leqq c\lambda^{-3/2} \bar{R}(t, y)$ for $y \in K_r \cap D(P)$. The Green formula for $\bar{R}(t, y)$ has the form (see for example (7,5) from [1])

$$\bar{R}(t, y) = \int_{K_r} Z(t, y; 0, \eta) \, d\eta + \int_0^t d\tau \int_{\dot{K}_r} Z(t, y; \tau, \eta) \sum_{i,j} A_{ij}^0 \cos(v, \eta_i) \cos(v, \eta_j) \times$$

$$\times \frac{\partial \bar{R}}{\partial v}(\tau, \eta) \, d\sigma_\eta$$

where Z is the fundamental solution of (1,5). Due to the estimate $|Z(t, y; \tau, \xi)| \leqq c_1|t - \tau|^{-n/2} \exp\{-c_2|x - \xi|^2/|t - \tau|\}$ (see (13,1), Chap. IV [3]) and due to $\lim_{t \to 0} \sqrt{(t)} (\partial \bar{R}/\partial \xi_n)(t, P) = \text{const}$ (see Remark 8 [1]) we obtain

$$|\bar{R}(t, y)| \leqq \bar{c} \exp\left\{-\frac{c_2}{2t}\left(\frac{r}{2}\right)^2\right\} \quad \text{for} \quad |y| \leqq \frac{r}{2}.$$

Further,

$$|\hat{R}_\lambda(t, y)| \leqq \frac{\hat{c}}{\lambda\sqrt{\lambda}} e^{-(c_2/2t)(r/2)^2}$$

and

$$|R_\lambda^*(\tau, \xi)| \leqq \frac{\hat{c}}{\lambda\sqrt{\lambda}}\, e^{-(c_2/2\tau\lambda)(r/2)^2} \leqq \frac{\hat{c}\tau\sqrt{\tau}}{r^3}\, e^{-(\bar{c}_2/2\tau\lambda)(r/2)^2} \leqq$$

$$\leqq \frac{\hat{c}\tau\sqrt{\tau}}{r^3}\, e^{-(\bar{c}_2/2)(\xi_n^2/2)} \quad \text{for} \quad \|\xi\| < \frac{r}{2\sqrt{\lambda}}.$$

This inequality together with (2,10) yields

$$|R_\lambda(\tau, \xi)| \leqq c_1\tau^{3/2} e^{-(c_2/2)(\xi_n^2/2)} \left(1 + \frac{1}{\tau^{3/2}} \sum_{k \neq n} |\xi_k|^3\right), \quad \text{for} \quad \|\xi\| < \frac{r}{2\sqrt{\lambda}}. \tag{2,11}$$

The constants c_1, c_2 are positive and independent of P. Let p be a number $0 < p < 1$. By Theorem 4, IV [4] we obtain estimates

$$\left|\frac{\partial R_\lambda}{\partial \xi_i}(\tau, \xi)\right| \leqq K(p)\, c_1\tau^{3/2} e^{-(c_2/2)(\xi_n^2/2\tau)} \left(1 + \frac{1}{\tau^{3/2}} \sum_{k \neq n} |\xi_k|^3\right), \quad 0 < c_2 < 1, \tag{2,12}$$

$$\left|\frac{\partial^2 R_\lambda}{\partial \xi_i\, \partial \xi_j}(\tau, \xi)\right| \leqq K(p)\, c_1\tau^{3/2} e^{-(c_2/2)(\xi_n^2/2\tau)} \left(1 + \frac{1}{\tau^{3/2}} \sum_{k \neq n} |\xi_k|^3\right) \tag{2,13}$$

which are valid for $[\tau, \xi]$ fulfilling $|\xi_i| < p^2$, $i = 1, \ldots, n-1$, $|\xi_n + \tau - p| < p^2$, $\xi_n > 0$, $0 < \tau < p$. The constant $K(p)$ may depend on p.

These estimates of the approximations will be sufficient for the following considerations on convexity of the solution $u(t, x)$.

For brevity we shall denote the point $[0, \ldots, 0, \bar{x}_n]$ in the local coordinate system $\bar{x}_1, \ldots, \bar{x}_n$ by x^*.

The point $[t, x^*]$ is assigned a point $[\tau, 0, \ldots, 0, \xi_n]$ and a number λ by this prescription: Let the number τ be the solution of

$$\frac{\bar{x}_n^2}{t} = \frac{(p-\tau)^2}{\tau}. \tag{2,14}$$

This equation has always two positive roots. The number τ is the less of them, i.e. the one fulfilling $\tau \leqq p$. Put

$$\xi_i = 0, \quad i = 1, \ldots, n-1 \quad \text{and} \quad \xi_n = p - \tau. \tag{2,15}$$

The point $[\tau, \xi_n]$ lies on the straight line $\tau + \xi_n = p$. Put

$$\lambda = \frac{t}{\tau}. \tag{2,16}$$

Conversely, if a point $[\tau, 0, \ldots, 0, \xi_n]$, $\tau + \xi_n = p$ and a number $\lambda > 0$ are given we put $t = \lambda\tau$, $y_i = 0$, $i = 1, \ldots, n-1$, $y_n = \xi_n\sqrt{\lambda}$ (see (2,0)) and by (1,4) we can express this point in the local coordinate system $\bar{x}_1, \ldots, \bar{x}_n : t = \lambda\tau, \bar{x}_i = 0, i = 1, \ldots$ $\ldots, n - 1, \bar{x}_n = \xi_n \sqrt{\lambda}$. The point $[\tau, 0, \ldots, 0, \xi_n]$ and the number λ correspond to $[t, x^*]$ by (2,14) to (2,16).

In the sequel we shall use the transformation (2,0) with λ given by (2,14) to (2,16).

The solution $\bar{u}(t, \bar{x})$ can be written in the form

$$\begin{aligned}\bar{u}(t, \bar{x}) &= v(t, y) = v_\lambda(\tau, \xi) = \\ &= v_0(\tau, \xi) + \sqrt{(\lambda)}\, \Delta_0(\tau, \xi) + \lambda\, \varrho_0(\tau, \xi) + \lambda \sqrt{(\lambda)}\, R_\lambda(\tau, \xi)\,.\end{aligned}$$

The transformations $[t, \bar{x}] \to (t, y)$ and $(t, y) \to (\tau, \xi)$ are given by (1,4) or (2,0), respectively and the second derivatives can be calculated

$$(2,17) \qquad \begin{aligned}\frac{\partial^2 \bar{u}}{\partial \bar{x}_i\, \partial \bar{x}_j}(t, x^*) = &-\frac{1}{\sqrt{(\lambda)}}\left[\left(\frac{\partial v_0}{\partial \xi n}(\tau, \xi_n) + \sqrt{(\lambda)}\, \frac{\partial \Delta_n}{\partial \xi_n}(\tau, \xi) + \right.\right. \\ &\left.\left. + \lambda \frac{\partial \varrho_0}{\partial \xi_n}(\tau, \xi)\right) \frac{\partial^2 \bar{h}}{\partial \bar{x}_i\, \partial \bar{x}_j}(0) - \sqrt{(\lambda)}\, \frac{\partial^2 \varrho_0}{\partial \xi_i\, \partial \xi_j}(\tau, \xi)\right] + \\ &+ \lambda \frac{\partial R_\lambda}{\partial \xi_n}(\tau, \xi)\, \frac{\partial^2 \bar{h}}{\partial \bar{x}_i\, \partial \bar{x}_j}(0) + \sqrt{(\lambda)}\, \frac{\partial^2 R_\lambda}{\partial \xi_i\, \partial \xi_j}(\tau, \xi) \quad \text{for} \quad i \neq n \neq j\,,\end{aligned}$$

$$(2,18) \qquad \begin{aligned}\frac{\partial^2 \bar{u}}{\partial \bar{x}_i\, \partial \bar{x}_n}(t, x^*) = &\frac{1}{\sqrt{(\lambda)}}\left(\frac{\partial \Delta_i}{\partial \xi_n}(\tau, \xi) + \sqrt{(\lambda)}\, \frac{\partial^2 \varrho_0}{\partial \xi_i\, \partial \xi_n}(\tau, \xi)\right) + \\ &+ \sqrt{(\lambda)}\, \frac{\partial^2 R_\lambda}{\partial \xi_i\, \partial \xi_n}(\tau, \xi)\end{aligned}$$

for $i \neq n$,

$$(2,19) \qquad \begin{aligned}\frac{\partial^2 \bar{u}}{\partial \bar{x}_n^2}(t, x^*) = &\frac{1}{\lambda}\left[\frac{\partial^2 v_0}{\partial \xi_n^2}(\tau, \xi_n) + \sqrt{(\lambda)}\, \frac{\partial^2 \Delta_n}{\partial \xi_n^2}(\tau, \xi) + \lambda\, \frac{\partial^2 \varrho_0}{\partial \xi_n^2}(\tau, \xi)\right] + \\ &+ \sqrt{(\lambda)}\, \frac{\partial^2 R_\lambda}{\partial \xi_n^2}(\tau, \xi)\end{aligned}$$

where the point $\xi = [\xi_1, \ldots, \xi_n]$ corresponds to the point x^* by (2,0) while the point $[\tau, 0, \ldots, 0, \xi_n]$ and number λ correspond to the point $[t, x^*]$ by (2,14), (2,15) and (2,16). Now we substitute expressions (2,2), (2,3) into the right-hand sides of (2,17) to (2,19) and use the estimates (2,8), (2,12), (2,13). These estimates can be used because of the choice of λ. Thus we have

$$(2,20) \qquad \begin{aligned}&\left|\frac{\partial^2 \bar{u}}{\partial \bar{x}_i\, \partial \bar{x}_j}(t, x^*) - \frac{1}{\sqrt{(\lambda)}} \sqrt{\left(\frac{2}{\pi\tau}\right)}\, e^{-\xi_n^2/2\tau}\, \frac{\partial^2 \bar{h}}{\partial \bar{x}_i \partial \bar{x}_j}(0)\right| \leqq \\ &\leqq c_1\left(1 + \frac{\xi_n^3}{\tau^{3/2}}\right) e^{-\xi_n^2/2\tau} + \sqrt{(\lambda\tau)}\, c_1\left(1 + \frac{\xi_n^6}{\tau^3}\right) e^{-\xi_n^2/2\tau} +\end{aligned}$$

$$+ c_1\left(1 + \frac{\xi_n^7}{\tau^3\sqrt{\tau}}\right) e^{-\xi_n^2/2\tau} + \lambda K(p)\, c_1\tau^{3/2} e^{-(c_2/2)(\xi_n^2/2\tau)} +$$

$$+ \sqrt{(\lambda)}\, K(p)\, c_1\tau^{3/2} e^{-(c_2/2)(\xi_n^2/2\tau)}, \quad \text{for} \quad i \neq n \neq j,$$

$$(2,21) \qquad \left|\frac{\partial^2\bar{u}}{\partial\bar{x}_i\,\partial\bar{x}_n}(t, x^*) - \frac{1}{\sqrt{(2\pi\lambda)}}\frac{\partial\Lambda^0_{nn}}{\partial y_i}(0, 0)\left(\frac{1}{\sqrt{(\tau)}} - \frac{\xi_n^2}{\tau\sqrt{(\tau)}}\right) e^{-(\xi_n^2/2\tau)}\right| \leqq$$

$$\leqq c_1\left(1 + \frac{\xi_n^7}{\tau^3\sqrt{(\tau)}}\right) e^{-(\xi_n^2/2\tau)} + \sqrt{(\lambda)}\, K(p)\, c_1\tau^{3/2} e^{-(c_2/2)(\xi_n^2/2\tau)}, \quad \text{for} \quad i \neq n,$$

$$(2,22) \qquad \left|\frac{\partial^2\bar{u}}{\partial\bar{x}_n^2}(t, x^*) - \frac{1}{\lambda}\frac{\xi_n}{\tau}\sqrt{\frac{2}{\pi\tau}}\, e^{-\xi_n^2/2\tau} - \frac{4a_n^0(0, 0)}{\sqrt{(2\pi\tau\lambda)}}\, e^{-\xi_n^2/2\tau}\right| \leqq$$

$$\leqq \frac{c_1}{\sqrt{(\lambda\tau)}}\left(\frac{\xi_n}{\sqrt{(\tau)}} + \frac{\xi_n^4}{\tau^2}\right) e^{-\xi_n^2/2\tau} + c_1\left(1 + \frac{\xi_n^7}{\tau^3\sqrt{(\tau)}}\right) e^{-\xi_n^2/2\tau} +$$

$$+ \sqrt{(\lambda)}\, K(p)\, c_1\tau^{3/2} e^{-(c_2/2)(\xi_n^2/2\tau)}$$

where the constants c_1, c_2 are independent of both the point P and of the number p. The constant $K(p)$ appeared first in (2,12), (2,13).

The matrix $\partial^2\bar{u}/\partial\bar{x}_i\,\partial\bar{x}_j(t, x^*)$ is positive definite if and only if the determinants of $\partial^2\bar{u}/\partial\bar{x}_i\,\partial\bar{x}_j(t, x^*)$, $i, j = 1, \ldots, k$ are positive for $k = 1, \ldots, n$. Multiply the rows of these matrices by $\sqrt{(\lambda\pi\tau/2)}\, e^{\xi_n^2/2\tau}$. Denote by d_{ij} the elements of the resulting matrices and by D_k the determinants of d_{ij}, $i, j = 1, \ldots, k$. With respect to (2,20) to (2,22) we obtain

$$\left|d_{ij} - \frac{\partial^2\bar{h}}{\partial\bar{x}_i\,\partial\bar{x}_j}(0)\right| \leqq \sqrt{(\lambda)}\, c_1\sqrt{(\tau)}\left(1 + \frac{\xi_n^7}{\tau^3\sqrt{(\tau)}}\right) + \lambda\tau c_1\left(1 + \frac{\xi_n^6}{\tau^3}\right) +$$

$$+ \lambda\sqrt{(\lambda)}\, K(p)\, c_1\tau^2 e^{(1-c_2/2)(\xi_n^2/2\tau)} + \lambda K(p)\, c_1\tau^2 e^{(1-c_2/2)(\xi_n^2/2\tau)},$$

$$\text{for} \quad i, j = 1, \ldots, n-1,$$

$$\left|d_{in} - \frac{1}{2}\frac{\partial\Lambda^0_{nn}}{\partial y_i}(0, 0)\left(1 - \frac{\xi_n^2}{\tau}\right)\right| \leqq \sqrt{(\lambda)}\, c_1\sqrt{(\tau)}\left(1 + \frac{\xi_n^7}{\tau^3\sqrt{(\tau)}}\right) +$$

$$+ \lambda K(p)\, c_1\tau^2 e^{(1-c_2/2)(\xi_n^2/2\tau)}, \quad \text{for} \quad i \neq n,$$

$$\left|d_{nn} - \frac{1}{\sqrt{(\lambda)}}\frac{\xi_n}{\tau} - 2a_n^0(0, 0)\right| \leqq c_1\left(\frac{\xi_n}{\sqrt{(\tau)}} + \frac{\xi_n^4}{\tau^2}\right) + c_1\sqrt{(\lambda\tau)}\left(1 + \frac{\xi_n^7}{\tau^3\sqrt{(\tau)}}\right) +$$

$$+ \lambda K(p)\, c_1\tau^2 e^{(1-c_2/2)(\xi_n^2/2\tau)}.$$

The number c_1 can be greater than the constant c_1 in the previous formulae but it is independent again of both P and p. Denote $\hat{d}_{ij} = \partial^2\bar{h}/\partial\bar{x}_i\,\partial\bar{x}_j(0)$ for $i, j = 1, \ldots$

$\ldots, n-1$, $\hat{d}_{in} = \hat{d}_{ni} = \frac{1}{2}(\partial \Lambda^0_{nn}/\partial y_i)(0, 0)$ for $i \neq n$, $\hat{d}_{nn} = 2a^0_n(0, 0)$. Due to Hypothesis (C) all determinants $\hat{D}_k$, $k = 1, \ldots, n-1$ are positive and due to Hypothesis (D) also the determinant $\hat{D}_n$ is positive (we have to take account of that

$$(\partial \bar{\Lambda}_{nn}/\partial \bar{x}_i)(0, 0) = (\partial \Lambda^0_{nn}/\partial y_i)(0, 0) \text{ and } 2a^0_n(0, 0) = 2\bar{a}_n(0, 0) - \operatorname{tr} \Gamma)$$

where $\hat{D}_k$ is the determinant of the matrix $\hat{d}_{ij}$, $i, j = 1, \ldots, k$. Thus there exists a number $\delta_0 > 0$ so that the inequalites $|d_{ij} - \hat{d}_{ij}| < \delta_0$ $i \neq n \neq j$, $d_{nn} > \hat{d}_{nn} - \delta_0$ imply that the determinants D_k are positive and in particular that $D_{n-1} > \frac{1}{2} \min_{P \in \dot{D}} \det \{\partial^2 \bar{h}/\partial \bar{x}_i \, \partial \bar{x}_j(0)\}^{n-1}_{i,j=1}$ (the last expression will be denoted by Z). This assertion is trivial for D_k, $k = 1, \ldots, n-1$, but the determinant D_n needs special consideration. The assertion follows from the fact than D_n can be written as

$$D_n = d_{nn} D_{n-1} - \sum_{i,j=1}^{n-1} (-1)^{i+j} d_{in} d_{jn} D_{ij} \tag{2,23}$$

where D_{ij} is the subdeterminant corresponding to the element d_{ij}. Further, we choose $\delta_1 > 0$ so that $(\partial \Lambda^0_{nn}/\partial y_i)(\xi_n^2/\tau) < \delta_0$, $c_1(\xi_n/\sqrt{\tau} + \xi_n^4/\tau^2) < \delta_0/2$ for $0 \leqq \xi_n < \delta_1$, $\tau = p - \xi_n$ and a number $\alpha > 0$ so that

$$\alpha < K(p)\, c_1 \,, \quad \alpha p^2 < \frac{\delta_0}{6} \,, \quad \sqrt{\left(\frac{\alpha p c_1}{K(p)}\right)} \left(1 + \frac{\xi_n^7}{\tau^3 \sqrt{\tau}}\right) e^{-(1 - c_2/2)(\xi_n^2/4\tau)} < \frac{\delta_0}{6}$$

and

$$\left[\frac{\xi_n}{\tau} \sqrt{\left(\frac{K(p)\, c_1}{\alpha}\right)} e^{(1 - c_2/2)(\xi_n^2/4\tau)} + 2a^0_n(0, 0) - c_1 \left(\frac{\xi_n}{\sqrt{(\tau)}} + \frac{\xi_n^4}{\tau^2}\right) - \frac{\delta_0}{3}\right] Z > \tag{2,24}$$

$$> \frac{M}{4} \left(\sum_{i=1}^{n-1} \left|\frac{\partial \Lambda^0_{nn}}{\partial y_i}(0, 0)\right| \left|1 - \frac{\xi_n^2}{\tau}\right| + \frac{2\delta_0}{3}\right)^2 \quad \text{for} \quad \xi_n \geqq \delta_1 \,, \quad \tau = p - \xi_n$$

where $M = \max \{|D_{ij}| : P \in \dot{D}, \, i, j = 1, \ldots, n-1\}$.

Define

$$\lambda_p(\tau) = \frac{\alpha}{K(p)\, c_1} e^{-(1 - c_2/2)(\xi_n^2/2\tau)} \quad \text{where} \quad \xi_n = p - \tau \,. \tag{2,25}$$

With respect to the choice of numbers $\delta_0, \delta_1, \alpha$ we obtain $|d_{ij} - \hat{d}_{ij}| < (2\delta_0/3)$ for $i, j \neq n$ and $\lambda = \lambda_p(\tau)$, i.e. the determinants D_k, $k = 1, \ldots, n-1$ are positive. Let us consider the last one, D_n. Suppose $0 \leqq \xi_n \leqq \delta_1$. Since

$$|d_{in} - \hat{d}_{in}| \leqq \frac{1}{2} \left|\frac{\partial \Lambda^0_{nn}}{\partial y_i} \frac{\xi_n^2}{\tau}\right| + \frac{2\delta_0}{6}$$

we have $|d_{in} - \hat{d}_{in}| < \delta_0$.

Further, $d_{nn} \geqq \hat{d}_{nn} - \delta_0$ (the term $(1/\sqrt{\lambda})(\xi_n/\tau)$ being nonnegative) so that D_n is nonegative for ξ_n satisfying the above inequality.

Suppose $\xi_n \geqq \delta_1$, i.e. $\tau \leqq p - \delta_1$. In this case we have

$$|d_{ij} - \hat{d}_{ij}| < 2\delta_0/3\,, \quad \left|d_{in} - \hat{d}_{in} + \frac{1}{2}\frac{\partial A^0_{nn}}{\partial y_i}\frac{\xi_n^2}{\tau}\right| \leqq \frac{\delta_0}{3} \quad \text{and}$$

$$d_{nn} \geqq \hat{d}_{nn} + \frac{1}{\sqrt{(\lambda)}}\frac{\xi_n}{\tau} - c_1\left(\frac{\xi_n}{\tau} + \frac{\xi_n^4}{\tau^2}\right) - \frac{\delta_0}{3}\,.$$

Due to (2,24) the right hand side of (2,23) is positive so that the determinant D_n is positive in this case, too. We summarize these results in a lemma.

Lemma 2. *Let the conditions of Theorem* 1 *be fulfilled. Assume that p is a number* $0<p<1$. *The solution* $\bar{u}(t, \bar{x})$ *is a strictly convex function as a function of* $\bar{x}_1, \ldots, \bar{x}_n$ *at those point* $[t, \bar{x}]$ *where* $t = \lambda_p(\tau)\,\tau$, $\bar{x}_i = 0$ *for* $i = 1, \ldots, n-1$, $\bar{x}_n = (p - \tau)\,.\,(\lambda_p(\tau))^{1/2}$, *the parameter* τ *being an arbitrary number* $0 < \tau \leqq p$ *and* $\lambda_p(\tau)$ *being given by* (2,25) *with* $\xi_n = p - \tau$.

Lemma 2 implies that the function $u(t, x)$ is strictly convex (the relations between x and $\bar{x}$ are linear) on a certain set which can be described as the boundary of a subregion of Q. The precise description and basic properties of the region and of $u(t, x)$ are given in the next section.

3

Denote $N_p(P) = \{[t, y] : t = \tau\lambda,\ y_i = 0 \text{ for } i = 1, \ldots, n-1,\ y_n = (p - \tau)\sqrt{(\lambda)},\ 0 \leqq \lambda \leqq \lambda_p(\tau),\ 0 \leqq \tau \leqq p\}$. By means of the transformation (1,4) and the transformation mentioned in Hypothesis (D) or Remark 1 the points $[t, y] \in N_p(P)$ can be transformed to the original coordinate system $x_1, \ldots, x_n$. The resulting set will be denoted by $N_p^+(P)$. Of course, both the sets $N_p(P)$ and $N_p^+(P)$ represent the same set of points (epressed in two different coordinate systems). Put $M_p = \bigcup_{P \in \dot{D}} N_p^+(P)$ and $Q_p = (0, L) \times D - M_p$.

Lemma 3. *The set* M_p *is closed. The boundary of* Q_p *is smooth for sufficiently small p, i.e. the set* $Q_p \cap \{[\tau, x] : x \in D,\ 0 < \tau < p\}$ *is locally of the type* $(\bar{E})$ *(see* [4]*). We have* $Q = \bigcup_{p>0} Q_p$. *If* $u(t, x)$ *is the bounded solution of* (0,1) *fulfilling* (0,2) *and* (0,3) *then*

$$\lim_{t\to 0}\ \sup_{[t,x]\in Q_p} \partial^2 u/\partial x_i\, \partial x_j(t, x) = 0 \quad \textit{for every} \quad p > 0\,, \quad i, j = 1, \ldots, n\,.$$

The set described in Lemma 2 on which $u(t, x)$ is strictly convex is just the side-boundary of $Q_p \cap \{[\tau, x] : x \in D,\ \tau \in \langle 0, p \rangle\}$. (It can be easily shown that the points $[t, 0, \ldots, 0, y_n]$ where $t = \lambda\tau$, $y_n = \xi_n\sqrt{(\lambda)}$, $\tau + \xi_n = p$, $\lambda > \lambda_p(\tau)$ do not belong to $N_p(P)$.) At every point $P \in \dot{D}$ there exists a local coordinate system $\bar{x}_1, \ldots, \bar{x}_n$

(Hypothesis (D)). The direction of the coordinate $\bar{x}_n$ is determined uniquely with respect to the original coordinate system $x_1, \ldots, x_n$. The direction of $\bar{x}_n$ depends smoothly on P. The set $N_p^+(P)$ lies in the plane which is determined by $\bar{x}_n$ and is parallel to the t-axis. The Hypothesis (B) ensures that for sufficiently small p the sets $N_p^+(P)$ are disjoint for different P. We have always emphasized that the constants $c_1, c_2, K(p)$ do not depend on P. It can be easily proved that the set M_p is closed and that the boundary of Q_p is smooth in the sense required. (2,25) implies $\lambda_p(\tau) \leqq 1$ so that $[t, x] \in N_p^+(P)$ yields $t \leqq \lambda_p(\tau)\,\tau \leqq \tau \leqq p$ and $0 \leqq y_n \leqq (p - \tau)(\lambda_p(\tau))^{1/2} \leqq p$ where y corresponds to x by the transformation used in (D) and (1,4). Thus we have $\bigcap_{p>0} M_p \subset \langle 0, L) \times \dot{D}$ and $\bigcup_{p>0} Q_p = Q$.

Now we shall prove the other part of Lemma 3. Let $0 < p < 1$ be chosen. Let $[t^0, x^0]$ be a point from Q_p. The construction of Q_p guarantees the existence of a point P, $P \in \dot{D}$ and the corresponding coordinate system $\bar{x}_1, \ldots, \bar{x}_n$ such that the point x fulfils $\bar{x}_1^0 = \bar{x}_2^0 = \ldots = \bar{x}_{n-1}^0 = 0$ where $\bar{x}_i^0$ are the coordinates of x^0 in $\bar{x}_1, \ldots, \bar{x}_n$.

Analogously as above let x^* denote the point $[0, \ldots, 0, \bar{x}_n^0]$. Let the point $[\tau, 0, \ldots, 0, \xi_n]$ and the number λ be determined for $[t^0, x^*]$ by (2,14), (2,15) and (2,16). Assume

$$\text{(i)} \qquad \bar{x}_n^0 + t^0 \leqq p\,.$$

If the point $[t^0, x^0]$ is written in the form $t^0 = \lambda\tau$, $\bar{x}_1^0 = \bar{x}_2^0 = \ldots = \bar{x}_{n-1}^0 = 0$, $\bar{x}_n^0 = (p - \tau)\sqrt{\lambda}$ and if we take into account $[t^0, x^*] \notin N_p(P)$ we obtain $\lambda \geqq \lambda_p(\tau)$. The inequality $\bar{x}_n^0 + t^0 \leqq p$ implies $\lambda \leqq 1$.

Let us consider the inequalities (2,20) to (2,22). Replace all λ in the numerators by 1 and all λ in the denominators by $\lambda_p(\tau)$. In this way we obtain

$$\left|\frac{\partial^2 \bar{u}}{\partial \bar{x}_i\, \partial \bar{x}_j}(t^0, x^*)\right| \leqq \frac{L(p)}{\tau}\, e^{-(c_3/2)((p-\tau)^2/\tau))} \tag{3,1}$$

where $L(p)$, c_3 are constants independent of P.

By virtue of $t \geqq \lambda_p(\tau)\,\tau$ and $0 < \lambda_p(\tau) < 1$ for $\tau > 0$ we obtain $\tau \to 0$ for $t \to 0$. Inequality (3,1) then implies

$$\limsup_{t^0 \to 0} \{|\partial^2 \bar{u}/\partial \bar{x}_i\, \partial \bar{x}_j(t^0, x^*)|;\ [t^0, x^0] \in Q_p,\ t^0 + \bar{x}_n^0 \leqq p\} = 0\,.$$

(ii) If $t^0 + \bar{x}_n^0 > p$, then

$$\limsup_{t^0 \to 0} \{|\partial^2 \bar{u}/\partial \bar{x}_i\, \partial \bar{x}_j(t^0, x^*)|;\ [t^0, x^0] \in Q_p,\ t^0 + \bar{x}_n^0 > p\} = 0$$

follows from the continuity of the second derivatives and from (0,2). Lemma 3 is proved.

In the next section we shall need an auxiliary statement about convex functions.

Remark 4. *Let $f(x_1, x_2, x_3)$ be defined in a neighbourhood of the origin and let f have continuous derivatives of the third order. Let f be convex in the neighbourhood. If*

$$\partial^2 f/\partial x_1^2(0, 0, 0) = \partial^2 f/\partial x_2^2(0, 0, 0) = 0\,,$$

then

$$\partial^3 f/\partial x_1\, \partial x_2\, \partial x_3(0, 0, 0) = 0\,.$$

Proof. Certainly

$$\partial^2 f/\partial x_1^2(0, 0, x_3) = \int_0^{x_3} \partial^3 f/\partial x_1^2\, \partial x_3(0, 0, \eta)\, \mathrm{d}\eta = x_3 \int_0^1 \partial^3 f/\partial x_1^2\, \partial x_3(0, 0, x_3 t)\, \mathrm{d}t\,.$$

The Lebesgue theorem implies

$$\lim_{x_3 \to 0} \int_0^1 \partial^3 f/\partial x_1^2\, \partial x_3(0, 0, x_3 t)\, \mathrm{d}t = \partial^3 f/\partial x_1^2\, \partial x_3(0, 0, 0)\,. \quad \text{i.e.}$$

$$\partial^3 f/\partial x_1^2\, \partial x_3(0, 0, 0) = \lim_{x_3 \to 0} x_3^{-1}\, \partial^2 f/\partial x_1^2(0, 0, x_3)\,.$$

By virtue of convexity of f we easily deduce

$$\partial^3 f/\partial x_1^2\, \partial x_3(0, 0, 0) = 0 \tag{3,2}$$

and similarly

$$\partial^3 f/\partial x_2^2\, \partial x_3(0, 0, 0) = 0\,. \tag{3,3}$$

Since $(\partial^2 f/\partial x_1\, \partial x_2(0, 0, 0))^2 \leqq \partial^2 f/\partial x_1^2(0, 0, 0)\, \partial^2 f/\partial x_2^2(0, 0, 0) = 0$ we can write $\partial^2 f/\partial x_1\, \partial x_2(0, 0, x_3) = x_3 \int_0^1 \partial^3 f/\partial x_1\, \partial x_2\, \partial x_3(0, 0, x_3 t)\, \mathrm{d}t$. The matrix $A : A_{ij} = \partial^2 f/\partial x_i\, \partial x_j(0, 0, x_3)$, $i = j = 1, 2$ can be written as

$$\begin{pmatrix} x_3 \int_0^1 \partial^3 f/\partial x_1^2\, \partial x_3(0, 0, x_3 t)\, \mathrm{d}t & x_3 \int_0^1 \partial^3 f/\partial x_1\, \partial x_2\, \partial x_3(0, 0, x_3 t)\, \mathrm{d}t \\ x_3 \int_0^1 \partial^3 f/\partial x_1\, \partial x_2\, \partial x_3(0, 0, x_3 t)\, \mathrm{d}t & x_3 \int_0^1 \partial^3 f/\partial x_2^2\, \partial x_3(0, 0, x_3 t)\, \mathrm{d}t \end{pmatrix}$$

As the determinant of this matrix has to be nonegative we have

$$\left(\int_0^1 \partial^3 f/\partial x_1\, \partial x_2\, \partial x_3(0, 0, x_3 t)\, \mathrm{d}t\right)^2 \leqq \int_0^1 \partial^3 f/\partial x_1^2\, \partial x_3(0, 0, x_3 t)\, \mathrm{d}t\, \times$$

$$\times \int_0^1 \partial^3 f/\partial x_2^2\, \partial x_3(0, 0, x_3 t)\, \mathrm{d}t$$

and for $x_3 \to 0$ we obtain

$$(\partial^3 f/\partial x_1\, \partial x_2\, \partial x_3(0, 0, 0))^2 \leqq \partial^3 f/\partial x_1^2\, \partial x_3(0, 0, 0)\, \partial^3 f/\partial x_2^2\, \partial x_3(0, 0, 0)\,.$$

The statement of Remark 4 follows immediately from (3,2) and (3,3).

4

In this section we shall study the convexity of $u(t, x)$ in the interior of Q under the condition that $u(t, x)$ is convex on the side boundary of Q. For the purposes of the section we shall need the following notation.

Let H be a subregion of Q. Denote by $H(t_0)$ the intersection of H with the set $\{[t, x] : t > t_0, x \in D\}$ and by $H(t_0, t_1)$, $t_0 < t_1 \leqq L$ the intersection of H with the set $\{[t, x] : t_0 < t < t_1, x \in D\}$. The parabolic boundary of $H(t_0, t_1)$ will be denoted by $\partial H(t_0, t_1)$, i.e.

$$\partial H(t_0, t_1) = \dot{H}(t_0) - \{t_1\} \times D\,, \quad \partial H(t_0) = \dot{H}(t_0) - \{L\} \times D\,.$$

Lemma 4. *Let Hypothesis* (A), (E) *be fulfilled and let $u(t, x)$ be the bounded solution of* (0,1) *fulfilling* (0,2) *and* (0,3). *If the derivatives up to the second order are continuous in $H(t_0)$ and if $u(t, x)$ is convex on $\partial H(t_0)$, then there exists a positive number δ such that $u(t, x)$ is a convex function of x in $H(t_0, t_0 + \delta)$.*

We prove Lemma 4 by contradiction. Since the violation of convexity at one point would not imply any contradiction we add a small convex term to the right hand side of the given parabolic equation.

Denote by $u^{(1)}(t, x)$ the solution of the equation

$$\frac{\partial u}{\partial t} = \frac{1}{2} \sum_{i,j} A_{ij}(t, x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_i a_i(t, x) \frac{\partial u}{\partial x_i} + \sum_i x_i^2$$

fulfilling $u^{(1)}(t_0, x) = \sum_i x_i^2$ in the half-space $t > t_0$, $x \in R^n$. The coefficients of the equation were extended to the whole half-space at the beginning of the proof of Theorem 1. Put $u_\varepsilon(t, x) = u(t, x) + \varepsilon\, u^{(1)}(t, x)$ where $\varepsilon > 0$ and $u(t, x)$ is the bounded solution of (0,1) fulfilling (0,2) and (0,3). The function $u_\varepsilon(t, x)$ is evidently a solution of

$$\frac{\partial u}{\partial t} = \frac{1}{2} \sum_{i,j} A_{ij}(t, x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_i a_i(t, x) \frac{\partial u}{\partial x_i} + \varepsilon \sum x_i^2 \tag{4,1}$$

fulfilling the initial condition

$$u_\varepsilon(t_0, x) = u(t_0, x) + \varepsilon \sum x_i^2\,. \tag{4,2}$$

As $u^{(1)}(t_0, x)$ is strictly convex and the second partial derivatives of $u^{(1)}$ are continuous, there exists a positive number δ such that $u^{(1)}(t, x)$ is convex on $\langle t_0, t_0 + \delta\rangle \times \times \bar{D}$. We shall prove that this δ has the properties from Lemma 4.

Since $u(t, x)$ is convex on $\partial H(t_0)$ and $u^{(1)}(t, x)$ is strictly convex on $\{[t, x] : t_0 \leqq \leqq t \leqq t_0 + \delta, x \in \bar{D}\}$ the function $u_\varepsilon(t, x)$ is strictly convex on $\partial H(t_0, t_0 + \delta)$.

Assume that $u_\varepsilon(t, x)$ is not strictly convex in $H(t_0, t_0 + \delta)$, i.e. there exists a point $[t^0, x^0] \in H(t_0, t_0 + \delta)$ $(t_0 < t^0 < t_0 + \delta)$ such that $u_\varepsilon(t, x)$ is strictly convex in $H(t_0, t^0)$ but not at the point $[t^0, x^0]$. Denote by $U_\varepsilon(t, x)$ the matrix of the type $n \times n$

whose elements are $\partial^2 u_\varepsilon/\partial x_i\,\partial x_j(t, x)$. Denote by $D_\varepsilon(t, x)$ the determinant of this matrix. The properties specified above yield that $D_\varepsilon(t, x) > 0$ in $H(t_0, t^0)$, $D_\varepsilon(t^0, x) \geqq \geqq 0$ for $[t^0, x] \in H$ and by Lemma 1, $D_\varepsilon(t^0, x^0) = 0$. Denote by k the rank of the matrix $U_\varepsilon(t^0, x^0)$. Evidently $0 \leqq k < n$. Certainly there exists a unitary matrix T such that $T^{\mathrm{T}} U_\varepsilon(t^0, x^0)\, T$ is a diagonal matrix and the transformation $x = Tz$, $u_\varepsilon(t, x) = \tilde{u}(t, z)$ maps (4,1) into

$$\frac{\partial \tilde{u}}{\partial t} = \frac{1}{2} \sum_{i,j} \tilde{A}_{ij}(t, z) \frac{\partial^2 \tilde{u}}{\partial z_i\,\partial z_j} + \sum_i \tilde{a}_i(t, z) \frac{\partial \tilde{u}}{\partial z_i} + \varepsilon \sum_i z_i^2 .$$

Since the number ε is fixed it will be omitted in the sequel. The regions D, Q_p, $H(t_0, t)$ are mapped onto $\tilde{D}$, $\tilde{Q}_p$, $\tilde{H}(t_0, t)$, respectively and the point $[t^0, x^0]$ to $[t^0, z^0]$. We have

$$\text{(4,3)} \qquad \begin{aligned} &\partial^2 \tilde{u}/\partial z_i^2(t^0, z^0) > 0 \quad \text{for} \quad i = 1, \ldots, k \quad \text{and} \\ &\partial^2 \tilde{u}/\partial z_i^2(t^0, z^0) = 0 \quad \text{for} \quad i = k + 1, \ldots, n . \end{aligned}$$

Denote by $\tilde{U}_{k+1}(t, z)$ the matrix of the type $(k + 1) \times (k + 1)$ (where k is the rank of U_ε) with the elements $\partial^2 \tilde{u}/\partial z_i\,\partial z_j(t, z)$, $i, j = 1, \ldots, k + 1$. Let $\tilde{D}_{k+1}(t, z)$ denote the determinant of this matrix. The function $\tilde{D}_{k+1}(t, z)$ fulfils the parabolic equation

$$\begin{aligned} \frac{\partial D}{\partial t} = {} & \frac{1}{2} \sum_{i,j=1}^{n} \tilde{A}_{ij}(t, z) \frac{\partial^2 D}{\partial z_i\,\partial z_j} + \sum_{i=1}^{n} \tilde{a}_i(t, z) \frac{\partial D}{\partial z_i} + 2 \sum_{s=1}^{k+1} \frac{\partial \tilde{a}_s}{\partial z_s}(t, z)\, D + \\ & + \sum_{i,j=1}^{n} \sum_{l,s=1}^{k+1} \frac{\partial \tilde{A}_{ij}}{\partial z_l} \frac{\partial^3 \tilde{u}}{\partial z_i\,\partial z_j\,\partial z_s} A_{sl} + 2 \sum_{i=k+2}^{n} \sum_{l=1}^{k+1} \sum_{s=1}^{k+1} \frac{\partial \tilde{a}_i}{\partial z_l} \frac{\partial^2 \tilde{u}}{\partial z_i\,\partial z_s} A_{sl} + \\ & + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 \tilde{u}}{\partial z_i\,\partial z_j} \zeta(\tilde{A}_{ij}) + \sum_{i=1}^{n} \frac{\partial \tilde{u}}{\partial z_i} \zeta(\tilde{a}_i) - \\ & - \frac{1}{2} \sum_{i,j=1}^{n} \sum_{\substack{r,l=1 \\ r \neq l}}^{k+1} \sum_{\substack{p,q=1 \\ p \neq q}}^{k+1} \tilde{A}_{ij} \frac{\partial^3 \tilde{u}}{\partial z_i\,\partial z_r\,\partial z_p} \frac{\partial^3 \tilde{u}}{\partial z_j\,\partial z_l\,\partial z_q} A_{pr,ql} + 2\varepsilon \sum_{l=1}^{k+1} A_{ll} \end{aligned}$$

where $\zeta(f) = \sum_{l,s=1}^{k+1} A_{ls}\, \partial^2 f/\partial z_l\,\partial z_s$. A_{ls} being the algebraic complement of $\tilde{U}_{k+1}(t, z)$ with respect to the element $\partial^2 \tilde{u}/\partial z_l\,\partial z_s$,

$$A_{pr,ql} = (-1)^{p+q+r+l} B_{pr,ql} \quad \text{for} \quad p < q\,, \quad r < l \quad \text{or} \quad p > q\,, \quad r > l\,,$$

$$A_{pr,ql} = (-1)^{p+q+r+l+1} B_{pr,ql} \quad \text{for} \quad p < q\,, \quad r > l \quad \text{or} \quad p > q\,, \quad r < l$$

and $B_{pr,ql}$ is the subdeterminant of $\tilde{U}_{k+1}$ resulting from it by omitting the rows p, q and the columns r, l. Obviously at the point $[t^0, z^0]$ we have $\tilde{D}_{k+1}(t^0, z^0) = 0$, $A_{k+1,k+1}(t^0, z^0) = \prod_{s=1}^{k} \partial^2 \tilde{u}/\partial z_s^2(t^0, z^0)$ while all other $A_{ls}(t^0, z^0)$ equal zero.

$$A_{pp;k+1,k+1} = A_{k+1,k+1;pp} = \left(\frac{\partial^2 \tilde{u}}{\partial z_p^2}\right)^{-1} \prod_{s=1}^{k} \frac{\partial^2 \tilde{u}}{\partial z_s^2},$$

$$A_{p,k+1;k+1,p} = A_{k+1,p;p,k+1} = -\left(\frac{\partial^2 \tilde{u}}{\partial z_p^2}\right)^{-1} \prod_{s=1}^{k} \frac{\partial^2 \tilde{u}}{\partial z_s^2}$$

for $p < k + 1$ and all other terms of this form equal zero. As was stated above, $\tilde{D}_{k+1}(t^0, z^0) = 0$, $\tilde{D}_{k+1}(t, z) > 0$ for $t < t^0$ so that $\tilde{D}_{k+1}(t^0, z) \geqq 0$. These relations imply on the one hand $\partial \tilde{D}/\partial z_i(t^0, z^0) = 0$ and on the other that the matrix $\partial^2 \tilde{D}/\partial z_i \, \partial z_j(t^0, z^0)$ is positive semi-definite. Since $\partial^2 \tilde{u}/\partial z_{k+1}^2(t^0, z) \geqq 0$ and from (4,3) $\partial^2 \tilde{u}/\partial z_{k+1}^2(t^0, z^0) = 0$ we obtain $\partial^3 \tilde{u}/\partial z_i \, \partial z_{k+1}^2(t^0, z^0) = 0$. Since $T^{\mathrm{T}} U_\varepsilon(t^0, x^0) \, T$ is a diagonal matrix, (4,3) yields $\partial^2 \tilde{u}/\partial z_s \, \partial z_{k+1}(t^0, z^0) = 0$ for $s \geqq k + 2$. The linearity of a_i implies $\zeta(\tilde{a}_i) = 0$. Moreover, by virtue of Remark 4 the equation for $\tilde{D}_{k+1}$ can be reduced at $[t^0, z^0]$ to

$$\frac{\partial D}{\partial t}(t^0, z^0) = \frac{1}{2} \sum_{i,j=1}^{n} \tilde{A}_{ij} \frac{\partial^2 D}{\partial z_i \, \partial z_j} + \frac{1}{2} \sum_{i=1}^{k} \frac{\partial^2 \tilde{A}_{ii}}{\partial z_{k+1}^2} \frac{\partial^2 \tilde{u}}{\partial z_i^2} \prod_{s=1}^{k} \frac{\partial^2 \tilde{u}}{\partial z_s^2} +$$

$$+ \sum_{i,j=1}^{k} \frac{\partial \tilde{A}_{ij}}{\partial z_{k+1}} \frac{\partial^3 \tilde{u}}{\partial z_i \, \partial z_j \, \partial z_{k+1}} \prod_{s=1}^{k} \frac{\partial^2 \tilde{u}}{\partial z_s^2} +$$

$$+ \sum_{i,j=1}^{k} \tilde{A}_{ij} \sum_{p=1}^{k} \frac{\partial^3 \tilde{u}}{\partial z_i \, \partial z_p \, \partial z_{k+1}} \frac{\partial^3 \tilde{u}}{\partial z_j \, \partial z_p \, \partial z_{k+1}} \left(\frac{\partial^2 \tilde{u}}{\partial z_p^2}\right)^{-1} \prod_{s=1}^{k} \frac{\partial^2 \tilde{u}}{\partial z_s^2} + 2\varepsilon \prod_{s=1}^{k} \frac{\partial^2 \tilde{u}}{\partial z_s^2}.$$

Due to (4,3) the numbers $\partial^2 \tilde{u}/\partial z_i^2(t^0, z^0)$ are positive for $i = 1, \ldots, k$. Denote $\alpha_i = \sqrt{(\partial^2 \tilde{u}/\partial z_i^2)}$ and $\beta_{ij} = (1/\alpha_i \alpha_j)(\partial^3 \tilde{u}/\partial z_i \, \partial z_j \, \partial z_{k+1})$. If (E) or (0,5) is fulfilled then (see Remark 2)

$$(4,4) \qquad \frac{1}{2} \sum_{i=1}^{k} \frac{\partial^2 \tilde{A}_{ii}}{\partial z_{k+1}^2} \alpha_i^2 + \sum_{i,j=1}^{k} \frac{\partial \tilde{A}_{ij}}{\partial z_{k+1}} \alpha_i \alpha_j \beta_{ij} + \sum_{i,j=1}^{k} \tilde{A}_{ij} \alpha_i \alpha_j \sum_{p=1}^{k} \beta_{ip} \beta_{jp} \geqq 0$$

so that $\partial \tilde{D}_{k+1}/\partial t(t^0, z^0) > 0$ which contradicts the assumption that $D_{k+1}(t, z) > 0$ in $H(t_0, t)$ and $D_{k+1}(t^0, z^0) = 0$. We have proved the convexity of $u_\varepsilon(t, x)$ in $H(t_0, t_0 + \delta)$. Since the number ε was an arbitrary positive number and the choice of δ was independent of ε, Lemma 4 is proved.

The proof of Theorem 1 is now easy. We choose a number $p > 0$ by Lemma 3. We shall apply Lemma 4 with $H = Q_p$. The solution $u(t, x)$ is certainly convex on $\partial H(0)$ by Lemma 2 and the statement after Lemma 3. Lemma 3 asserts that the second derivatives of $u(t, x)$ are continuous on $H(0)$. This means that we can apply Lemma 4 and we know that $u(t, x)$ is convex in some $H(0, \delta)$. First we shall prove that $u(t, x)$ is convex in the whole $H(0)$. If this statement were not true then there would exist a number $t_0 \geqq \delta$ such that $u(t, x)$ is strictly convex in $H(0, t_0)$ but not in any $H(0, t_0 + \eta)$, $\eta > 0$. Since $u(t_0, x)$ is convex in x and $u(t, x)$ is a solution of $(0,1)$ fulfilling a consistent boundary problem the second derivatives of u are continuous and hence Lemma 4 can be applied to $H(t_0)$. Using this lemma we obtain that $u(t, x)$ is convex in some $H(t_0, t_0 + \delta)$, $\delta > 0$ and this is a contradiction with the definition of t_0. We have proved that $u(t, x)$ is convex in Q_p. Due to Lemma 3 we have $Q = \bigcup_p Q_p$ so that $u(t, x)$ is convex in the whole Q. Theorem 1 is proved.

5

Theorem 1 is a modification of Theorem 2 [5] to a multidimensional case. In [5] a method was suggested which enables applications of Theorem 2 [5] to the case of nonlinear drift coefficients. We can proceed similarly in the case of multidimensional problem. We outline briefly the main points of the method. We choose arbitrarily a point $[\bar{t}, \bar{x}]$ from the region Q. We transform equation $(0,1)$ onto $(5,4)$ by a transformation $x = \psi(\xi)$, $u(t, x) = v(t, \xi)$ so that the corresponding drift coefficients are linear in ξ. Theorem 2 provides the corresponding relations for the components of $\psi(\xi)$. Obviously, we need the transformation ψ to be one-to-one and to have a nonzero Jacobian. Let $\varphi(x)$ be the inverse transformation. According to Theorem 2 the components of $\varphi(x)$ fulfil equation $(5,5)$ which is simpler than $(5,3)$. If the transformed equation $(5,4)$ fulfils the conditions of Theorem 1 then solution $v(t, \xi)$ is convex as a function of ξ. The relation between the second derivatives of $u(t, x)$ and $v(t, \xi)$ is given by $(5,6)$ in Lemma 5. Since $v(t, \xi)$ is convex it is sufficient to guarantee that the last term in $(5,6)$ is nonnegative. We need this condition to be fulfilled only at the point $[\bar{t}, \bar{x}]$. It means that the transformation $\varphi(x)$ may depend on the sign of $\partial u/\partial x_k(\bar{t}, \bar{x})$, i.e., on the point $[\bar{t}, \bar{x}]$, and on the direction l.

The method can be described shortly: the functions $\varphi_i(x)$ are solutions of $(5,5)$ so that the transformation $\xi = \varphi(x)$ is one-to-one with nonzero Jacobian on the whole D and so that the expression

$$\sum_k \frac{\partial u}{\partial x_k} \sum_i \frac{\partial \psi_k}{\partial \xi_i} \frac{\mathrm{d}^2 \varphi_i}{\mathrm{d} l^2}$$

at the point $[\bar{t}, \bar{x}]$ is nonnegative.

In article [5] this method was used to derive Theorems 3, 4 [5]. The method is however much more complicated in a multidimensional case. Nevertheless, it can be used under some circumstances. We shall use this approach in the case when the drift coefficients differ little from linear functions, i.e. when the drift coefficients are $-\lambda_i x_i + \varepsilon\, a_i(x)$ where ε is a small parameter. We shall preserve the assumptions of Theorem 5 [1] which guarantee the convexity of $u(t, x)$ on the boundary set $(0, t_0) \times \dot{D}$ for sufficiently small t_0. We shall be able to formulate conditions (Theorem 3) under which the solution $u(t, x)$ is convex in the whole $(0, t_0) \times D$ for small ε and t_0. It means that Theorem 3 gives conditions under which the matrix function

$A(t, x)$ is strongly maximal with respect to $-\lambda_i x_i + \varepsilon\, a_i(x)$ and with respect to $(0, t_0) \times D$ for small ε and t_0.

First a theorem is formulated where the relations for the transformation $\varphi(x)$ are given guaranteeing that (0,1) is transformed onto an equation with linear drift coefficients.

Theorem 2. *Let a parabolic differential equation*

$$\frac{\partial u}{\partial t} = \frac{1}{2} \sum_{i,j} A_{ij}(t, x) \frac{\partial^2 u}{\partial x_i\, \partial x_j} + \sum_i a_i(t, x) \frac{\partial u}{\partial x_i} \tag{5,1}$$

be given in $Q = (0, L) \times D$. A transformation $x = \psi(\xi)$, $u(t, x) = v(t, \xi)$ transforms (5,1) onto

$$\frac{\partial v}{\partial t} = \frac{1}{2} \sum_{i,j} \tilde{A}_{ij}(t, \xi) \frac{\partial^2 v}{\partial \xi_i\, \partial \xi_j} + \sum_s \left[\tilde{a}_s(t, \xi) - \right.$$
$$\left. - \frac{1}{2} \sum_{p,r,k} \tilde{A}_{pr}(t, \xi) \frac{\partial^2 \psi_k}{\partial \xi_p\, \partial \xi_r} \left(\frac{\mathrm{d}\psi}{\mathrm{d}\xi} \right)^{-1}_{sk} \right] \frac{\partial v}{\partial \xi_s} \tag{5,2}$$

where $\mathrm{d}\psi/\mathrm{d}\xi$ is the matrix $(\mathrm{d}\psi/\mathrm{d}\xi)_{i,j} = \partial \psi_i / \partial \xi_j$, $i, j = 1, \ldots, n$; if $a(t, x)$, $\tilde{a}(t, \xi)$ are considered as column vectors then $\tilde{a}(t, \xi) = (\mathrm{d}\psi/\mathrm{d}\xi)^{-1}\, a(t, \psi(\xi))$ and $\tilde{A}(t, \xi) = (\mathrm{d}\psi/\mathrm{d}\xi)^{-1} A(t, \psi(\xi))\, ((\mathrm{d}\psi/\mathrm{d}\xi)^{-1})^{\mathrm{T}}$. The matrix $\tilde{A}(t, \xi)$ is positive definite if the transformation ψ has a nonzero Jacobian. Conditions

$$\tilde{a}_s(t, \xi) = \frac{1}{2} \sum_{p,r,k} \tilde{A}_{pr}(t, \xi) \frac{\partial^2 \psi_k}{\partial \xi_p\, \partial \xi_r} \left(\frac{\mathrm{d}\psi}{\mathrm{d}\xi} \right)^{-1}_{sk} + \alpha_s(t) + \sum_l \beta_{ls}(t)\, \xi_l \tag{5,3}$$

ensure that equation (5,2) is of the type

$$\frac{\partial v}{\partial t} = \frac{1}{2} \sum_{i,j} \tilde{A}_{ij}(t, \xi) \frac{\partial^2 v}{\partial \xi_i\, \partial \xi_j} + \sum_s \left(\alpha_s(t) + \sum_l \beta_{ls}(t)\, \xi_l \right) \frac{\partial v}{\partial \xi_s}\,. \tag{5,4}$$

Assume that the inverse transformation $\xi = \varphi(x)$ to $x = \psi(\xi)$ exists. Then the condition (5,3) can be rewritten as

$$\frac{1}{2} \sum_{i,j} A_{ij}(t, x) \frac{\partial^2 \varphi_l}{\partial x_i\, \partial x_j} + \sum_i a_i(t, x) \frac{\partial \varphi_l}{\partial x_i} = \alpha_l(t) + \sum_k \beta_{kl}(t)\, \varphi_k(t) \tag{5,5}$$

and in this case

$$\tilde{A}(t, \xi) = \frac{\mathrm{d}\varphi}{\mathrm{d}x} (\varphi^{-1}(\xi))\, A(t, \varphi^{-1}(\xi)) \left(\frac{\mathrm{d}\varphi}{\mathrm{d}x} (\varphi^{-1}(\xi)) \right)^{\mathrm{T}},$$

$$\tilde{a}_k(t, \xi) = \sum_i a_i(t, \varphi^{-1}(\xi)) \frac{\partial \varphi_k}{\partial x_i} (\varphi^{-1}(\xi)) + \frac{1}{2} \sum_{i,j} A_{ij}(t, \varphi^{-1}(\xi)) \frac{\partial^2 \varphi_k}{\partial x_i\, \partial x_j} (\varphi^{-1}(\xi))\,.$$

To prove Theorem 2 we substitute the transformation into (5,1). Since the proof consists in some tedious calculations it is omitted. We shall omit the proof of both the following lemmas for the same reason. The next lemma deals with the relation between the second derivatives of u and v.

Lemma 5. *Let $u(t, x)$, $v(t, \xi)$ fulfil the assumption of Theorem 2. If l is an arbitrary nonzero vector then*

$$(5,6)\qquad \frac{\mathrm{d}^2 u}{\mathrm{d}l^2}(t, x) = \frac{\mathrm{d}^2 v}{\mathrm{d}k^2}(t, \xi) + \sum_i \frac{\partial v}{\partial \xi_i}(t, \xi)\frac{\mathrm{d}^2 \varphi_i}{\mathrm{d}l^2} = \frac{\mathrm{d}^2 v}{\mathrm{d}k^2} + \\ + \sum_s \frac{\partial u}{\partial x_s}\sum_i \frac{\partial \psi_s}{\partial \xi_i}\frac{\mathrm{d}^2 \varphi_i}{\mathrm{d}l^2}$$

where the vector k is $(\mathrm{d}\varphi/\mathrm{d}x)\, l$ and the points x, ξ are related by $\xi = \varphi(x)$.

6

We shall need still some information about the change of the boundary of the region D and, in particular, about the convexity of the region transformed.

Lemma 6. *Let the region D fulfil Hypothesis (B) and let a transformation $\xi = \varphi(x)$ on $\bar{D}$ be given so that $\varphi(x)$ has continuous second derivatives up to the boundary, φ is one-to-one and its Jacobian is nonzero. Suppose that the boundary $\dot{D}$ can be described by $[x_1, x_2, \ldots, x_{n-1}, h(x_1, \ldots, x_{n-1})]$ in a neighbourhood of an arbitrary point P on the boundary, the function h being continuous and with continuous second derivatives. Then $\bar{P} = \varphi(P) \in \dot{\varphi}(D)$, the region $\varphi(D)$ fulfils Hypothesis (B) and if the matrix A defined below is regular then there exists a neighbourhood of $\bar{P}$ so that the boundary of $\varphi(D)$ can be expressed by $\xi_n = \xi_n(\xi_1, \ldots, \xi_{n-1})$ and the first and second derivatives of ξ_n are given by*

$$(6,1)\qquad \frac{\mathrm{d}\xi_n}{\mathrm{d}\xi} = (A^{\mathsf{T}})^{-1}\, a\,,$$

$$(6,2)\qquad \frac{\mathrm{d}^2\xi_n}{\mathrm{d}\xi^2} = (A^{\mathsf{T}})^{-1}\Bigg[\Phi^{(n)} - \sum_{s=1}^{n-1}\Phi^{(s)}((A^{\mathsf{T}})^{-1}\, a)_s + \hat{\Phi}^{(n)} - \\ - \sum_{s=1}^{n-1}\hat{\Phi}^{(s)}((A^{\mathsf{T}})^{-1}\, a)_s + hh^{\mathsf{T}}\left(\frac{\partial^2\varphi_n}{\partial x_n^2} - \sum_{s=1}^{n-1}\frac{\partial^2\varphi_s}{\partial x_n^2}((A^{\mathsf{T}})^{-1}\, a)_s\right) + \\ + H\left(\frac{\partial\varphi_n}{\partial x_n} - \sum_{s=1}^{n-1}\frac{\partial\varphi_s}{\partial x_n}((A^{\mathsf{T}})^{-1}\, a)_s\right)\Bigg]A^{-1}$$

where A, $\Phi^{(s)}$, $\hat{\Phi}^{(s)}$, H are matrices of the type $(n-1) \times (n-1)$ with the elements

$$A_{ij} = \frac{\partial \varphi_i}{\partial x_j}(P) + \frac{\partial \varphi_i}{\partial x_n}(P) \frac{\partial h}{\partial x_j}(P), \quad \Phi^{(s)}_{i,j} = \frac{\partial^2 \varphi_s}{\partial x_i \, \partial x_j}(P),$$

$$\hat{\Phi}^{(s)}_{i,j} = \frac{\partial^2 \varphi_s}{\partial x_i \, \partial x_n}(P) \frac{\partial h}{\partial x_j}(P) + \frac{\partial^2 \varphi_s}{\partial x_j \, \partial x_n}(P) \frac{\partial h}{\partial x_i}(P),$$

$$H_{ij} = \frac{\partial^2 h}{\partial x_i \, \partial x_j}(P), \quad i, j = 1, \ldots, n-1$$

while h, a are $(n-1)$-dimensional column vectors with the elements

$$h_i = \frac{\partial h}{\partial x_i}(P), \quad a_i = \frac{\partial \varphi_n}{\partial x_i}(P) + \frac{\partial \varphi_n}{\partial x_n}(P) \frac{\partial h}{\partial x_i}(P), \quad i = 1, \ldots, n-1.$$

Remark 5. Lemma 6 implies that the region $\varphi(D)$ is convex at the point $\bar{P} \in \dot{\varphi}(D)$ provided

$$\Phi^{(n)} - \sum_{s=1}^{n-1} \Phi^{(s)}((A^{\mathrm{T}})^{-1} a)_s + \hat{\Phi}^{(n)} - \sum_{s=1}^{n-1} \hat{\Phi}^{(s)}((A^{\mathrm{T}})^{-1} a)_s +$$
$$+ hh^{\mathrm{T}} \left(\frac{\partial^2 \varphi_n}{\partial x_n^2} - \sum_{s=1}^{n-1} \frac{\partial^2 \varphi_s}{\partial x_n^2} ((A^{\mathrm{T}})^{-1} a)_s \right) + H \left(\frac{\partial \varphi_n}{\partial x_n} - \sum_{s=1}^{n-1} \frac{\partial \varphi_s}{\partial x_n} ((A^{\mathrm{T}})^{-1} a)_s \right)$$

is positive definite. We do not assume that $x_1, x_2, \ldots, x_n$ is a local coordinate system in Lemma 6 so that the numbers $h_i = \partial h / \partial x_i(P)$ may be nonzero.

The application of the above described method requires some more definitions and notation.

Definition. Let D be a given region. Denote by D_δ the *δ-neighbourhood of D*, i.e. $D_\delta = \{x : \|x - y\| < \delta, \ y \in D\}$. Let a real function $f(x)$ be defined on $\bar{D}$. The function $f(x)$ is called *real analytic* in $\bar{D}$ if it can be extended to a D_δ so that the extension can be developed into a power series in a neighbourhood of every point of $\bar{D}$. Define, as usual, the Banach space $C_{2+\alpha}(D)$ as the class of all functions defined on D with Hölder continuous second derivatives,

$$\|f\|_0^D = \sup_D |f(x)|, \quad H_\alpha(D) = \sup_D \frac{|f(x) - f(y)|}{\|x - y\|^\alpha}, \quad \|f\|_\alpha^D = \|f\|_0^D + H_\alpha(D),$$

$$\|f\|_{2+\alpha}^D = \|f\|_0^D + \sum_i \left\| \frac{\partial f}{\partial x_i} \right\|_0^D + \sum_{i,j} \left\| \frac{\partial^2 f}{\partial x_i \, \partial x_j} \right\|_\alpha^D.$$

The upper index D will be omitted if there is no danger of ambiguity. The space $C^0_{2+\alpha}(D)$ is the set of all functions $f(x) \in C_{2+\alpha}(D)$ satisfying $f(x) = 0$ for $x \in \dot{D}$. The norm in the space $C^0_{2+\alpha}(D)$ is the same as in the space $C_{2+\alpha}(D)$.

Let an elliptic operator L be given in D. We say that zero is not an eigenvalue of L if $Lf = 0$, $f \in C^0_{2+\alpha}(D)$ implies $f(x) \equiv 0$.

7

In the present section we have to subject the coefficients A_{ij}, a_i to much stronger assumptions than Hypothesis (A).

Hypothesis (F). *Assume that the coefficients $A_{ij}(x)$, $a_i(x)$ are real functions independent of t which are real analytic in $\bar{D}$. The matrix function $A(x)$ is uniformly positive definite in D.*

Similarly, Hypothesis (E) will be substituted by Hypothesis (G).

Hypothesis (G). *The matrix functions*

$$M^{\mathrm{T}}(x)\, N_l(x) + N_l^{\mathrm{T}}(x)\, M(x) + \sqrt{(2)}\, P_l(x)\,,$$

$$M^{\mathrm{T}}(x)\, N_l(x) + N_l^{\mathrm{T}}(x)\, M(x) - \sqrt{(2)}\, P_l(x) \quad (\textit{see } (0,4))$$

are uniform positive definite matrices with respect to $x \in D$ and to every unit vector l.

In virtue of Remark 2 these conditions are stronger than the condition on $A(x)$ formulated in Hypothesis (E).

We shall consider a parabolic differential equation where the drift coefficients slightly differ from linear functions:

$$\frac{\partial u}{\partial t} = \frac{1}{2} \sum_{i,j} A_{ij}(x) \frac{\partial^2 u}{\partial x_i\, \partial x_j} + \sum_i (-\lambda_i x_i + \varepsilon\, a_i(x))\, \frac{\partial u}{\partial x_i} \tag{7,1}$$

with the initial condition

$$u(0, x) = 0 \quad \text{for} \quad x \in D \tag{7,2}$$

and with the boundary condition

$$u(t, x) = 1 \quad \text{for} \quad t > 0\,, \quad x \in \dot{D}\,. \tag{7,3}$$

If $\varepsilon = 0$, equation (7,1) assumes the form

$$\frac{\partial u}{\partial t} = \frac{1}{2} \sum_{i,j} A_{ij}(x) \frac{\partial^2 u}{\partial x_i\, \partial x_j} - \sum_i \lambda_i x_i \frac{\partial u}{\partial x_i}\,. \tag{7,4}$$

Theorem 3. *Let Hypotheses* (B), (C), (D) *for* (7,4) *and* (F), (G) *be fulfilled. If* 0 *is not an eigenvalue of the elliptic operators*

$$(7,5) \qquad L_k = \frac{1}{2}\sum_{i,j} A_{ij}(x)\frac{\partial^2}{\partial x_i\,\partial x_j} - \sum_i \lambda_i x_i \frac{\partial}{\partial x_i} + \lambda_k\,, \quad k = 1, \dots, n$$

then there exist $t_0 > 0$ and $\varepsilon_0 > 0$ such that the bounded solution of (7,1) fulfilling (7,2), (7,3) is convex as a function of x for $0 < t < t_0$, $x \in D$, $|\varepsilon| < \varepsilon_0$.

First of all, we shall say something about the location of points $[\lambda_1, \dots, \lambda_n]$ at which the operators L_k have not zero eigenvalue. Namely, we shall show that these points are not spread too densely. Obviously, given numbers $\lambda_1, \lambda_2, \dots, \lambda_{s-1}, \lambda_{s+1}, \dots \dots, \lambda_n$ then there exist only countably many values λ_s without an accumulation point such that L_k have zero eigenvalue. Even in the case $\lambda = \lambda_1 = \lambda_2 = \dots = \lambda_n$ we can prove

Lemma 7. *Let the functions $A_{ij}(x)$, $(\partial a_i/\partial x_k)(x)$, $(\partial c/\partial x_k)(x)$ be Hölder continuous in D. If the matrix function $A(x)$ is uniformly positive definite in D, then there exists at most countably many values λ without an accumulation point such that the operator*

$$(7,6) \qquad L_\lambda = \frac{1}{2}\sum_{i,j} A_{ij}(x)\frac{\partial^2}{\partial x_i\,\partial x_j} - \lambda\sum_i a_i(x)\frac{\partial}{\partial x_i} + \lambda\, c(x)$$

has zero eigenvalue.

Proof of Lemma 7. Denote $L_0 f = \frac{1}{2}\sum_{i,j} A_{ij}(x)\,\partial^2 f/\partial x_i\,\partial x_j$ and $L_1^+ f = \sum_i a_i(x)\,.\,\partial f/\partial x_i - c(x) f$. If zero is an eigenvalue of L_λ, for a certain λ then there exists a nontrivial $f \in C^0_{2+\alpha}(D)$ such that $L_0 f = \lambda L_1^+ f$. Since the operator L_0 considered on $C^0_{2+\alpha}(D)$ has an inverse operator L_0^{-1} we can rewrite the last equation as $\lambda T f = f$ where $T = L_0^{-1}L_1^+$, $T: C_{1+\alpha}(D) \to C^0_{2+\alpha}(D)$. This means that the number λ is an eigenvalue of T. Due to (5,39) [6] we have $\|f\|_{2+\alpha} \leqq C(\|L_0 f\|_\alpha + \|f\|_0)$ and with respect to (2,4) [6], $\|f\|_{2+\alpha} \leqq C_1\|L_0 f\|_\alpha$ for $f \in C^0_{2+\alpha}(D)$. These inequalities imply $\|Tf\|_{2+\alpha} \leqq \leqq C_2\|f\|_{1+\alpha}$ for $f \in C_{1+\alpha}(D)$. We proved that T is a continuous linear operator from $C_{1+\alpha}(D)$ into $C^0_{2+\alpha}(D)$. If T is considered as a transformation $C_{1+\alpha}(D) \to C_{1+\alpha}(D)$, then it is a compact operator and it can have mostly countably many eigenvalues without an accumulation point. The lemma is proved.

Remark 6. *Assume that 0 is not an eigenvalue of the operators L_k defined in (7,5). Then there exists a number $\delta_0 > 0$ such that 0 is not an eigenvalue of L_k in D_δ for $0 \leqq \delta < \delta_0$.*

Proof. Suppose that there exist nontrivial $f_n \in C^0_{2+\alpha}(D_{1/n})$ fulfilling $L_k f_n = 0$ (k fixed). These functions can be normalized by $\sup\{|f_n| : x \in D_{1/n}\} = 1$. The proof of Theorem 5 [6] suggests that C is independent of n. We have

$$(7,7) \qquad \|f_n\|^{D_{1/n}}_{2+\alpha} \leqq C\,.$$

This inequality yields that there exists a converging subsequence $f_n \to f_0$. The function f_0 is a solution of $L_k f_0 = 0$ and due to (7,7) $f_0 \in C^0_{2+\alpha}(D)$ and $\sup\{|f_0| : x \in D\} = 1$. This contradicts the assumption that 0 is not an eigenvalue of L_k in D.

The proof of Theorem 3 will be splitted into several lemmas. Hypothesis (F) is essentially used only in the next lemma.

8

Lemma 8. *Let $A_{ij}(x)$ $a_i(x)$, $c(x)$, $f(x)$ be real analytic in a region D_δ and let $A(x)$ be uniformly positive definite. If there exists at least one solution of*

$$(8,1) \qquad \frac{1}{2}\sum_{i,j} A_{ij}(x)\frac{\partial^2\varphi}{\partial x_i\,\partial x_j} + \sum_i a_i(x)\frac{\partial\varphi}{\partial x_i} + c(x)\,\varphi = f(x)$$

belonging to $C_{2+\alpha}(D_\delta)$ then there exists a positive number M such that to every point $x_0 \in D$ and to every n-dimensional unit vector l, solutions $\varphi(x)$, $\chi(x)$ of (8,1) can be found fulfilling $\mathrm{d}^2\varphi/\mathrm{d}l^2(x_0) = 1$, $\|\varphi\|^{D_{\delta/2}}_{2+\alpha} \leqq M$ and $\mathrm{d}^2\chi/\mathrm{d}l^2(x_0) = -1$, $\|\chi\|^{D_{\delta/2}}_{2+\alpha} \leqq M$, respectively.

Proof. Put $l = [1, 0, \ldots, 0]$. First we shall prove the following statement: *Given a point $x_0 \in D_\delta$, there exists a solution $\varphi(x)$ of*

$$(8,2) \qquad \frac{1}{2}\sum_{i,j} A_{ij}(x)\frac{\partial^2\varphi}{\partial x_i\,\partial x_j} + \sum_i a_i(x)\frac{\partial\varphi}{\partial x_i} + c(x)\,\varphi = 0$$

defined on the whole D_δ and fulfilling $\partial^2\varphi/\partial x_1^2(x_0) \neq 0$.

Let $x_0 \in D_\delta$ be given. Due to Kovalevska's theorem there exists a solution $\varphi_0(x_1, \ldots, x_n)$ of (8,2) defined in a small ball $\|x - x_0\| < r$ and fulfilling $\varphi_0(x_1, \ldots, x_{n-1}, (x_0)_n) = x_1^2/2$, $(\partial\varphi_0/\partial x_n)(x_1, \ldots, x_{n-1}, (x_0)_n) = 0$. The adjoint equation to (8,2) has also real analytic coefficients. According to the analyticity theorem [6] every solution of the adjoint equation is real analytic. It means that the adjoint equation has the property of weak extensibility. Owing to theorems of Malgrange and Laxe [6] we know that (8,2) has Runge's property, i.e. to every $\xi > 0$ there exists a solution $\varphi_\xi(x)$ defined on D_δ fulfilling $|\varphi_0(x) - \varphi_\xi(x)| < \xi$ for $\|x - x_0\| < r/2$. By virtue of (5,31) [6] we obtain $\lim_{\xi\to 0} \partial^2\varphi_\xi/\partial x_1^2(x_0) = \partial^2\varphi_0/\partial x_1^2(x_0) = 1$. The statement is proved.

Denote $M(x_0, l_0) = \inf\{\|\varphi\|^{D_{\delta/2}}_{2+\alpha}\}$ where the greatest lower bound is taken over all solutions of (8,2) which are defined on D_δ and fulfil $\mathrm{d}^2\varphi/\mathrm{d}l_0^2(x_0) = 1$. Due to the above statement this set is nonvoid. The function $M(x, l)$ is finite for every $x \in D_{\delta/2}$, l, $\|l\| = 1$. We shall prove that $M(x, l)$ is upper-semicontinuous. Let x_0, l_0 be given. Given a positive number ξ we can find a solution φ_ξ (8,2) fulfilling $\mathrm{d}^2\varphi_\xi/\mathrm{d}l_0^2(x_0) = 1$ and $\|\varphi_\xi\|^{D_{\delta/2}}_{2+\alpha} \leqq M(x_0, l_0) + \xi$. Since the second derivative is continuous, there exists a number $\delta > 0$ such that $|\mathrm{d}^2\varphi_\xi/\mathrm{d}l_0^2(x_0) - \mathrm{d}^2\varphi_\xi/\mathrm{d}l_1^2(x_1)| < \xi$ for $\|x_1 - x_0\| < \delta$,

$\|l_0 - l_1\| < \delta$. The function $w_\xi(x) = \varphi_\xi(x)/[\mathrm{d}^2\varphi_\xi/\mathrm{d}l_1^2(x_1)]$ is a solution of (8,2) fulfilling $\mathrm{d}^2 w_\xi/\mathrm{d}l_1^2(x_1) = 1$ and $\|w_\xi(x)\|_{2+\alpha}^{D_{\delta/2}} \leqq (M(x_0, l_0) + \xi)/(1 - \xi)$. Hence $M(x_1, l_1) \leqq (M(x_0, l_0) + \xi)/(1 - \xi)$. The upper semi-continuity implies boundedness so that $M(x, l) \leqq M_0$ for some M_0, $x \in \bar{D}$, $\|l\| = 1$.

Denote by $\bar{\varphi}(x)$ a solution of (8,1) belonging to $C_{2+\alpha}(D_\delta)$. Let η be a positive number and $x_0 \in D$. Denote by $\varphi_\eta(x)$ a solution of (8,2) fulfilling $\mathrm{d}^2\varphi_\eta/\mathrm{d}l^2(x_0) = 1$, $\|\varphi_\eta\|_{2+\alpha}^{D_{\delta/2}} \leqq M_0 + \eta$ and $w_\eta(x) = \bar{\varphi}(x) + (1 - \mathrm{d}^2\bar{\varphi}/\mathrm{d}l^2(x_0))\,\varphi_\eta(x)$.

The function $w_\eta(x)$ is certainly a solution of (8,1) fulfilling

$$\mathrm{d}^2 w_\eta/\mathrm{d}l^2(x_0) = 1 \quad \text{and} \quad \|w_\eta\|_{2+\alpha}^{D_{\delta/2}} \leqq \|\bar{\varphi}\|_{2+\alpha}^{D_{\delta/2}} + \Big(1 + \max_{D_{\delta/2}, \|l\| = 1} |\mathrm{d}^2\bar{\varphi}/\mathrm{d}l^2(x)|\Big)\,.$$

$$.\,(M_0 + \eta)\,.$$

The function χ can be expressed in the form $\chi(x) = \bar{\varphi}(x) - (1 + \mathrm{d}^2\bar{\varphi}/\mathrm{d}l^2(x_0))\,\varphi_\eta(x)$. As $\mathrm{d}^2\bar{\varphi}/\mathrm{d}l^2(x)$ is bounded on the compact set $\bar{D}_{\delta/2}$ the lemma is proved with the constant M:

$$M > \|\bar{\varphi}\|_{2+\alpha}^{D_{\delta/2}} + \left(1 + \max_{x,l}\left|\frac{\mathrm{d}^2\bar{\varphi}}{\mathrm{d}l^2}(x)\right|\right) M_0\,.$$

9

The outlined method suggests that we need to transform equation (7,1) onto a parabolic equation with linear drift coefficients. The existence of such transformation is ensured by

Lemma 9. *Let all the assumptions of Theorem* 3 *be fulfilled. There exist positive numbers* ε_0, δ_0 *such that for every* $0 < \varepsilon \leqq \varepsilon_0$, $0 < \delta \leqq \delta_0$ *the elliptic equations*

$$(9,1) \qquad \frac{1}{2}\sum_{i,j} A_{ij}(x)\frac{\partial^2\varphi}{\partial x_i\,\partial x_j} + \sum_i(-\lambda_i x_i + \varepsilon\, a_i(x))\frac{\partial\varphi}{\partial x_i} + \lambda_k\varphi = 0\,,$$

$$k = 1, \ldots, n$$

have solutions $\varphi_\varepsilon^{(k)}(x)$ *belonging to* $C_{2+\alpha}(D_\delta)$ *and fulfilling* $(\partial/\partial\varepsilon)\,(\mathrm{d}^2\varphi_\varepsilon^{(k)}/\mathrm{d}l^2)\,(x_0)/_{\varepsilon=0} = 1$ *for every* $x_0 \in D$, $\|l\| = 1$. *The functions* $\varphi_\varepsilon^{(k)}(x)$ *can be written in the form*

$$(9,2) \qquad \varphi_\varepsilon^{(k)}(x) = x_k + \sum_{s=1}^{\infty}\varepsilon^s\,\varphi^{(k,s)}(x)$$

where $\varphi^{(k,s)}(x)$ *are solutions of*

$$(9,3) \qquad \frac{1}{2}\sum_{i,j} A_{ij}(x)\frac{\partial^2\varphi}{\partial x_i\,\partial x_j} - \sum_i \lambda_i x_i\frac{\partial\varphi}{\partial x_i} + \lambda_k\varphi = -\sum_j a_j(x)\frac{\partial\varphi^{(k,s-1)}}{\partial x_j}(x)\,,$$

$$\text{for} \quad s \geqq 1$$

fulfilling $\varphi^{(k,s)}(x) = 0$ *for* $x \in \dot{D}_\delta$, $s > 1$, $\varphi^{(k,0)}(x) = x_k$. *Finally, there exist constants* C, M *independent of* x_0, l *so that*

$$\|\varphi^{(k,s)}\|_{2+\alpha}^{D_\delta} \leqq CM^s .$$

Remark 7. It can be proved in the same way that there also exist solutions $\varphi_\varepsilon^{(-k)}(x)$ fulfilling

$$\frac{\partial}{\partial \varepsilon} \frac{\mathrm{d}^2 \varphi_\varepsilon^{(-k)}}{\mathrm{d}l^2} (x_0)/_{\varepsilon=0} = -1 .$$

Also for $-\varepsilon_0 \leqq \varepsilon < 0$ there exist solutions fulfilling $(\partial/\partial\varepsilon)(\mathrm{d}^2\varphi_\varepsilon^{(\pm k)}/\mathrm{d}l^2)(x_0)/_{\varepsilon=0} = \pm 1$ and the other conditions of Lemma 9.

Proof of Lemma 9. With respect to (9,3) the function $\varphi^{(k,1)}(x)$ fulfils

$$(9,4) \qquad \frac{1}{2} \sum_{i,j} A_{ij}(x) \frac{\partial^2 \varphi}{\partial x_i \partial x_j} - \sum_i \lambda_i x_i \frac{\partial \varphi}{\partial x_i} + \lambda_k \varphi = -a_k(x) .$$

We choose a number δ_0 as in Remark 6. Due to this Remark zero is not an eigenvalue of the operator L_k defined in (7,5) in D_δ. Theorem 1 Chap. 4 [6] yields that zero is not an eigenvalue of L_k in our sense if and only if it is not an eigenvalue in the sense of [6]: there does not exist a nontrivial solution of the generalized Dirichlet problem $L_k\varphi = 0$ in D_δ (see § 4.6 [6]). This fact enables us to use Theorems 4 and 6 Chap. 4 [6]. Hence there exists a generalized solution of (9,4) in D_δ. Due to Theorem 1 [6] the solution has continuous second derivatives and due to the analyticity Theorem it is real analytic. Thus we can apply Lemma 8 which implies that there exists a solution $\varphi^{(k,1)}(x)$ fulfilling $(\mathrm{d}^2\varphi^{(k,1)}/\mathrm{d}l^2)(x_0) = 1$ and $\|\varphi^{(k,1)}\|_{2+\alpha}^{D_\delta} \leqq M$. Applying Theorems 4, 6, Theorem 1 and the analyticity theorem from [6] once more we arrive at the conclusion that equations (9,3) have real analytic solutions for every $s > 1$.

To prove Lemma 9 it is sufficient to derive some estimates for $\|\varphi^{(k,s)}\|_{2+\alpha}^{D_\delta}$. Consider an equation

$$(9,5) \qquad L_k\varphi = f \quad \text{where} \quad \varphi = 0 \quad \text{on} \quad \dot{D}_\delta ; \quad f \in C_\alpha(D_\delta)$$

and L_k is defined in (7,5). Put $T\varphi = -\lambda_k(L_k - \lambda_k)^{-1} \varphi$ (k is fixed). The operator T maps $C_\alpha(D_\delta)$ into $C_{2+\alpha}^0(D_\delta)$ and

$$(9,6) \qquad \|T\varphi\|_{2+\alpha}^{D_\delta} \leqq C\|\varphi\|_\alpha^{D_\delta}$$

Equation (9,5) can be rewritten as

$$(9,7) \qquad \varphi = T\varphi - \frac{1}{\lambda_k} Tf , \quad \varphi \in C_{2+\alpha}^0(D_\delta) , \quad f \in C_\alpha(D_\delta) .$$

As $\varphi = T\varphi$ is equivalent to $L_k\varphi = 0$ and zero is not an eigenvalue of L_k, we obtain that zero is not an eigenvalue of the operator $I - T$ where I is the identical operator and $I - T$ is considered an operator from $C^0_{2+\alpha}(D_\delta)$ into $C^0_{2+\alpha}(D_\delta)$. Remark 8, which will be formulated and proved later, implies that $I - T$ has a continuous inverse operator $(I - T)^{-1}$ (as a transformation from $C^0_{2+\alpha}(D_\delta)$ to $C^0_{2+\alpha}(D_\delta)$. Thus using (9,7) we conclude

$$\|\varphi\|^{D_\delta}_{2+\alpha} = \frac{1}{|\lambda_k|} \|(I - T)^{-1} Tf\|^{D_\delta}_{2+\alpha} \leqq C_1 \|Tf\|^{D_\delta}_{2+\alpha}$$

and with respect to (9,6),

$$(9,8) \qquad \|\varphi\|^{D_\delta}_{2+\alpha} \leqq C_2 \|f\|^{D_\delta}_{\alpha} .$$

In our case $f = -\sum_i a_i(x)\,(\partial\varphi^{(k,s-1)}/\partial x_i)\,(x)$ and since $\|a_i\|_\alpha \leqq K, \|\varphi^{(k,1)}\|^{D_\delta}_{2+\alpha} \leqq M$ (see Lemma 8) we obtain

$$(9,9) \qquad \|\varphi^{(k,s)}\|^{D_\delta}_{2+\alpha} \leqq (C_2 K)^{s-1}\, M .$$

The last inequality implies that for $0 < \varepsilon_0 < (C_2 K)^{-1}$ the series is convergent. Since $(\mathrm{d}^2\varphi^{(k,1)}/\mathrm{d}l^2)\,(x_0) = 1$, inequality (9,8) also implies $(\partial/\partial\varepsilon)\,(\mathrm{d}^2\varphi^{(k)}_\varepsilon/\mathrm{d}l^2)\,(x_0)/_{\varepsilon=0} = 1$. Lemma 9 is proved.

The next remark will be formulated in a more general wording than it is needed for the proof of Lemma 9. Nevertheless, we shall need this Remark when formulating Theorem 3 for parabolic equations which slightly differ from those fulfilling the assumptions of Theorem 3.

Remark 8. *Let*

$$L^{(p)} = \frac{1}{2} \sum_{i,j} A^{(p)}_{ij}(x) \frac{\partial^2}{\partial x_i\, \partial x_j} - \sum_i \lambda_i x_i \frac{\partial}{\partial x_i} + \lambda$$

be a sequence of elliptic operators where $A^{(p)}_{ij} \in C_\alpha(D)$. Assume the operators $L^{(p)}$ to be uniformly bounded: $\|A^{(p)}_{ij}\|_\alpha \leqq K$ and uniformly positive definite:

$$\sum_{i,j} A^{(p)}_{ij}(x)\, \alpha_i \alpha_j \geqq m \sum_i \alpha_i^2, \quad m > 0 .$$

Further, let the coefficients converge uniformly in D: $A^{(p)}_{ij}(x) \to A^{(0)}_{ij}(x)$, $p \to \infty$. If zero is not an eigenvalue of $L^{(0)}$, then for sufficiently great p zero is not an eigenvalue of $L^{(p)}$, the operators $I - T^{(p)}$ are continuous transformations from $C^0_{2+\alpha}(D)$ into $C^0_{2+\alpha}(D)$, there exist inverse transformations $(I - T^{(p)})^{-1}$ from $C^0_{2+\alpha}(D)$ into $C^0_{2+\alpha}(D)$ and the norms $(I - T^{(p)})^{-1}$ are uniformly bounded. (The operators $T^{(p)}$ are defined by $T^{(p)}f = -\lambda(L^{(p)} - \lambda)^{-1} f$, $T^{(p)}f \in C^0_{2+\alpha}(D)$, for $f \in C_\alpha(D)$).

Proof. The first step is to prove that zero is not an eigenvalue of $L^{(p)}$ for sufficiently great p. This proof is very similar to that of Remark 6. Suppose there exists a subsequence of $L^{(p)}$ so that 0 is an eigenvalue of $L^{(p)}$. We denote the subsequence by $L^{(p)}$ again. It means that there exists a sequence $f^{(p)}(x)$ of solutions of

$L^{(p)}f^{(p)} = 0$, $f^{(p)} \in C^0_{2+\alpha}(D)$, $\max_x |f^{(p)}(x)| = 1$. By Theorem 5 Chap. 5 [6] we obtain $\|f^{(p)}\|^D_{2+\alpha} \leqq C$. As in the proof of Remark 6 we can choose a subsequence of $f^{(p)}$ converging to a certain $f^{(0)}(x)$. Certainly $L^{(0)}f^{(0)} = 0$, $f^{(0)} \in C^0_{2+\alpha}(D)$ and $\max_x |f^{(0)}(x)| = 1$. This contradicts to the assumption that zero is not an eigenvalue of $L^{(0)}$.

Now, as the second step we shall prove that $(I - T^{(p)})$ maps $C^0_{2+\alpha}(D)$ onto $C^0_{2+\alpha}(D)$ for sufficiently great p. Choose a number p so that 0 is not an eigenvalue of $L^{(p)}$. We shall assume for the moment that the coefficients $A^{(p)}_{ij}$ are real analytic. Due to (9,6) $I - T^{(p)}$ is a continuous operator. Denote by B the image of $C^0_{2+\alpha}(D)$ given by $I - T^{(p)}$. Certainly B is a Banach subspace of $C^0_{2+\alpha}(D)$. The equation $f - T^{(p)}f = g$ is equivalent to $L^{(p)}f = (L^{(p)} - \lambda)\, g$ for $f, g \in C^0_{2+\alpha}(D)$. If $g \in C_\infty(\bar{D})$ then $(L^{(p)} - \lambda)\, g \in C_\infty(D)$ and due to the assumption about the eigenvalues and due to Theorems 4,6 Chap. 4 [6] the equation $L^{(p)}f = (L^{(p)} - \lambda)\, g$ has a weak solution f, $f \in H^1_0$ (the well-known class H^1_0 is defined for example in [6]). Using Theorem 1 [6] we conclude $f \in C_\infty(D)$ and by Theorem 5 [6], $f \in C^0_{2+\alpha}(D)$. This implies $(I - T^{(p)})f = g$ so that $B = C^0_{2+\alpha}(D)$. If the coefficients $A^{(p)}_{ij}$ are not real analytic in D, then they can be approximated by real analytic coefficients $A^{(p,s)}_{i,j}$ which converge uniformly to $A^{(p)}_{ij}$. We proved that to every s and $g \in C^0_{2+\alpha}(D)$ there exists a solution of $L^{(p,s)}f^{(p,s)} = (L^{(p,s)} - \lambda)\, g, f^{(p,s)} \in C^0_{2+\alpha}(D)$. Since $\|f^{(p,s)}\|^D_{2+\alpha} \leqq C\|g\|^D_{2+\alpha}$ we can choose a subsequence of $f^{(p,s)}$ converging to a $f^{(p)}$ in $C^0_{2+\alpha}(D)$ which is a solution of $L^{(p)}f^{(p)} = (L^{(p)} - \lambda)\, g$.

Since 0 is not an eigenvalue of $L^{(p)}$ and therefore it is not an eigenvalue of $I - T^{(p)}$ and since $I - T^{(p)}$ maps $C^0_{2+\alpha}(D)$ onto itself we obtain by the closed graph theorem that the inverse operators $(I - T^{(p)})^{-1}$ are continuous. Nonetheless, we need to prove the uniform boundedness of the inverse operators. To this purpose we shall formulate an auxiliary statement.

Statement. $T^{(p)}$ *are „uniformly compact" as transformations* $C_\alpha(D) \to C_{2+\alpha}(D)$, *i.e., given a sequence of* $f^{(p)} \in C_\alpha(D)$ *so that* $\|f^{(p)}\|_\alpha \leqq 1$, *there exists a subsequence* $f^{(i(p))}$ *so that* $T^{(i(p))}f^{(i(p))}$ *converges in the norm of* $C_{2+\alpha}(D)$ *to a* $g^{(0)} \in C_{2+\alpha}(D)$. *If moreover* $f^{(p)} \to f^{(0)}$, *then* $T^{(p)}f^{(p)} \to T^{(0)}f^{(0)}$ *in the norm of* $C_{2+\alpha}(D)$.

Proof of the statement. Consider $g^{(p)} = T^{(p)}f^{(p)}$. This equation can be rewritten as

$$(L^{(p)} - \lambda)\, g^{(p)} = -\lambda f^{(p)}\,. \tag{9,10}$$

Since $L^{(p)} - \lambda$ are uniformly elliptic operators we have $\|g^{(p)}\|_{2+\alpha} \leqq C\|f^{(p)}\|_\alpha \leqq C$ *where the constant C is independent of p. Hence we can choose a subsequence* $g^{(i(p))}$ *converging to a* $g^{(0)} \in C_{2+\alpha}(D)$. *Consequently, there exists a subsequence* $f^{(i(p))}$ *so that* $T^{(i(p))}f^{(i(p))} \to g^{(0)}$ in the norm of $C_{2+\alpha}(D)$. If moreover $f^{(p)} \to f^{(0)}$, then using *(9,10)* we have $(L^{(0)} - \lambda)\, g^{(0)} = -\lambda f^{(0)}$, i.e. $T^{(0)}f^{(0)} = g^{(0)}$. The statement is proved.

If the operators $(I - T^{(p)})^{-1}$ are not uniformly bounded, then there exists a sequence $f^{(p)} \in C^0_{2+\alpha}(D)$, $f^{(p)} \to 0$ such that $g^{(p)} = (I - T^{(p)})^{-1} f^{(p)}$ fulfil $\|g^{(p)}\|^D_{2+\alpha} = 1$. The last relation can be modified to $g^{(p)} = T^{(p)}g^{(p)} + f^{(p)}$. Using the statement and the fact that $f^{(p)} \to 0$ in $C_{2+\alpha}(D)$ we obtain that a subsequence of $g^{(p)}$ converges to a certain $g^{(0)} \in C_{2+\alpha}(D)$. Using the last assertion of the statement we have $g^{(0)} = T^{(0)}g^{(0)}$. Since 0 is not an eigenvalue of $T^{(0)}$ we obtain $g^{(0)} = 0$, which contradicts the assumption $\|g^{(p)}\|_{2+\alpha} = 1$. Remark 8 is proved. The proof of Remark 8 completes the proof of Lemma 9.

10

We shall need still the convergence of the series $\sum \varepsilon^s \|\varphi^{(k,s)}\|_{3+\alpha}$.

Lemma 10. *Let L be an elliptic operator*

$$Lf = \frac{1}{2}\sum_{i,j} A_{ij}(x)\frac{\partial^2 f}{\partial x_i\,\partial x_j} + \sum a_i(x)\frac{\partial f}{\partial x_i} + c(x)f.$$

Assume that the coefficients of L belong to $C_{l+\alpha}(D_\delta)$ where l is nonnegative integer, $D^{(1)}$, $D^{(2)}$ are regions fulfilling $D^{(1)} \subset \overline{D^{(1)}} \subset D^{(2)} \subset \overline{D^{(2)}} \subset D_\delta$. We have

$$\|f\|^{D^{(1)}}_{l+2+\alpha} \leqq C(\|Lf\|^{D^{(2)}}_{l+\alpha} + \|f\|^{D^{(2)}}_0) \tag{10,1}$$

for $f \in C_{l+2+\alpha}(D_\delta)$ where the constant C depends only on l, on the regions $D^{(1)}$, $D^{(2)}$, on the norms of the coefficients of L in $C_{l+\alpha}$ and on the coefficient of ellipticity of L.

Remark 9. Lemmas 9 and 10 imply that to every nonnegative integer l there exists a positive number ε_0 so that

$$\sum_n \varepsilon^n \|\varphi^{(k,n)}\|^D_{l+\alpha} \tag{10,2}$$

is convergent for $|\varepsilon| \leqq \varepsilon_0$.

Lemma 10 is a generalization of Theorem 3 Chap. 5 [6] since the proof can be without change applied to the case of elliptic operators of order m.

Proof of Lemma 10. If $l = 0$, then the lemma is a consequence of Theorem 3 [6]. Suppose that the lemma is true for l. Choose a region $D^{(3)}$ so that $\overline{D^{(1)}} \subset D^{(3)} \subset \overline{D^{(3)}} \subset D^{(2)}$. Applying (10,1) to $D^{(1)}$, $D^{(3)}$ instead to $D^{(1)}$, $D^{(2)}$ we obtain

$$\left\|\frac{\partial f}{\partial x_i}\right\|^{D^{(1)}}_{l+2+\alpha} \leqq C_1\left(\left\|L\frac{\partial f}{\partial x_i}\right\|^{D^{(3)}}_{l+\alpha} + \left\|\frac{\partial f}{\partial x_i}\right\|^{D^{(3)}}_0\right) \quad i = 1, \dots, n$$

for $f \in C_{l+3+\alpha}(D_\delta)$. Since

$$L\frac{\partial f}{\partial x_i} = \frac{\partial}{\partial x_i}Lf - \frac{1}{2}\sum_{k,p}\frac{\partial A_{kp}}{\partial x_i}\frac{\partial^2 f}{\partial x_k\,\partial x_p} - \sum_k \frac{\partial a_k}{\partial x_i}\frac{\partial f}{\partial x_k} - \frac{\partial c}{\partial x_i}f$$

we obtain

$$\left\|\frac{\partial f}{\partial x_i}\right\|_{l+2+\alpha}^{D^{(1)}} \leqq C_1\left(\|Lf\|_{l+1+\alpha}^{D^{(3)}} + C_2\|f\|_{l+2+\alpha}^{D^{(3)}} + \left\|\frac{\partial f}{\partial x_i}\right\|_0^{D^{(3)}}\right)$$

where C_2 depends on the norms of the coefficients in $C_{l+1+\alpha}(D_\delta)$. Applying (10,1) to the regions $D^{(3)}$, $D^{(2)}$ we conclude

$$\left\|\frac{\partial f}{\partial x_i}\right\|_{l+2+\alpha}^{D^{(1)}} \leqq C_1(\|Lf\|_{l+1+\alpha}^{D^{(3)}} + C_2C_3(\|Lf\|_{l+\alpha}^{D^{(2)}} + \|f\|_0^{D^{(2)}}) +$$
$$+ C_4\|f\|_0^{D^{(2)}}) \leqq C_5(\|f\|_{l+1+\alpha}^{D^{(2)}} + \|f\|_0^{D^{(2)}}).$$

With respect to the definition of $C_{l+3+\alpha}$ the last inequality implies that (10,1) is true also for $l + 1$. Lemma 10 is proved.

11

Now, all is prepared for the proof of Theorem 3. Assume 0 is not an eigenvalue of the operators L_k given by (7,5). According to Remark 6 we choose a number $\delta_0 > 0$ so that 0 is not an eigenvalue of L_k in any region D_δ for $0 \leqq \delta \leqq \delta_0$. A positive number ε_0 is chosen such that Lemma 9 and Remark 9 are valid. Let a point $[\bar{t}, x_0] \in \in Q$ and an n-dimensional unit vector l be given. Denote

(11,1) $\quad \hat{\varphi}_\varepsilon^{(i)}(x) = \varphi_\varepsilon^{(\pm i)}(x)$ *where the sign* $+$ *is taken if* $\partial u/\partial x_i(\bar{t}, x_0) \geqq 0$,

the sign $-$ *is taken if* $\partial u/\partial x_i(\bar{t}, x_0) < 0$.

The functions $\varphi_\varepsilon^{(i)}(x)$ are given by Lemma 9 and by Remark 7 for $\varepsilon < 0$ or $-i$. Actually, the functions $\hat{\varphi}_\varepsilon^{(i)}(x)$ depend on $\bar{t}$ but since this dependence affects neither the radii of convergence nor the norm of $\hat{\varphi}_\varepsilon^{(i)}$ in $C_{2+\alpha}(D)$, the parameter $\bar{t}$ will not be explicitly marked in $\hat{\varphi}_\varepsilon^{(i)}(x)$. According to Theorem 2 the transformation $\xi = \hat{\varphi}_\varepsilon(x)$, $u(t, x) = v(t, \xi)$ maps equation (7,1) into

$$\frac{\partial v}{\partial t} = \frac{1}{2}\sum_{i,j}\tilde{A}_{ij}^{(\varepsilon)}(\xi)\frac{\partial^2 v}{\partial\xi_i\,\partial\xi_j} - \sum_i \lambda_i\xi_i\frac{\partial v}{\partial\xi_i} \tag{11,2}$$

and the region D onto $\hat{\varphi}_\varepsilon(D)$ so that the function $v(t, \xi)$ is the bounded solution of (11,2) fulfilling $v(0, \xi) = 0$ for $\xi \in \hat{\varphi}_\varepsilon(D)$ and $v(t, \xi) = 1$ for $t > 0$, $\xi \in \tilde{\varphi}_\varepsilon(D)$. The transformation $\xi = \hat{\varphi}_\varepsilon(x)$ converges to the identical transformation $\xi = \hat{\varphi}_0(x) = +x$ for $\varepsilon \to 0$. The Jacobian of $\hat{\varphi}_\varepsilon(x)$ converges to the Jacobian of $\hat{\varphi}_0(x)$ which equals one. Obviously there exists an inverse transformation $x = \psi_\varepsilon(\xi)$ for sufficiently small ε.

We shall derive formula (11,3) for $\psi_\varepsilon^{(i)}(\xi)$.

Let z be an n-dimensional vector with components $+1$ or -1. Denote $\varphi_\varepsilon^{(i)}(z, x) = = \varphi_\varepsilon^{(\pm i)}(x)$ where the sign $+$ is taken if $z_i = 1$ and the sign $-$ is taken if $z_i = -1$.

The Jacobians of transformations $\xi = \varphi_\varepsilon(z, x)$ converge to one for $\varepsilon \to 0$. Certainly there exist inverse transformations $x = \psi_\varepsilon(z, \xi)$ for sufficiently small ε. Using (10,2) it can be written

$$\psi_\varepsilon^{(i)}(z, \xi) = \xi_i + \sum_{p=1}^{l-3} \varepsilon^p \hat{\psi}^{(i,p)}(z, \xi) + \varepsilon^{l-3} \hat{\gamma}^{(i)}(z, \xi, \varepsilon)$$

where $\hat{\psi}^{(i,p)}(z, \xi) \in C_{3+\alpha}(D_{\delta_0})$, $\|\hat{\gamma}\|_{3+\alpha}^{D_{\delta_0}} \to 0$ for $\varepsilon \to 0$ uniformly with respect to z.

Let $z(\varepsilon)$ be defined by: $z_i(\varepsilon) = +1$ if $\partial u / \partial x_i(\bar{t}, x_0) \geqq 0$ and $z_i(\varepsilon) = -1$ if $\partial u / \partial x_i(\bar{t}, x_0) < 0$. We have $\hat{\varphi}_\varepsilon(x) = \varphi_\varepsilon(z(\varepsilon), x)$ and $\psi_\varepsilon(\xi) = \psi_\varepsilon(z(\varepsilon), \xi)$ such that

$$\psi_\varepsilon^{(i)}(\xi) = \xi_i + \sum_{p=1}^{l-3} \varepsilon^p \psi^{(i,p)}(\xi) + \varepsilon^{l-3} \gamma^{(i)}(\xi, \varepsilon) \tag{11,3}$$

where $\psi^{(i,p)}(\xi)$ can depend on ε nevertheless $\|\psi^{(i,p)}\|_{3+\alpha}^{D_{\delta_0}}$ are uniformly bounded and $\|\gamma^{(i)}\|_{3+\alpha}^{D_{\delta_0}} \to 0$ for $\varepsilon \to 0$.

Using the property of $\psi_\varepsilon(\xi)$ function $\tilde{A}^{(\varepsilon)}(\xi)$ can be written as

$$\tilde{A}^{(\varepsilon)}(\xi) = \left(\frac{d\psi_\varepsilon}{d\xi}\right)^{-1} A(\psi_\varepsilon(\xi)) \left(\left(\frac{d\psi_\varepsilon}{d\xi}\right)^{-1}\right)^{\mathrm{T}}.$$

Now we shall apply Theorem 1 to equation (11,2) which is considered in the region $\hat{\varphi}_\varepsilon(D)$. We shall successively verify the assumptions of Theorem 1. Choosing for example $l = 4$ in (11,3) we see that Hypothesis (F) together with (11,3) ensure Hypothesis (A) of Theorem 1. Hypotheses (B) and (C) (for small ε) directly follow from Hypotheses (B), (C) of Theorem 3 due to Lemma 6.

Now we shall deal with Hypothesis (D). Let a point $P \in \dot{D}$ be chosen arbitrarily. The transformation $\xi = \hat{\varphi}_\varepsilon(x)$ maps P to P'_ε and the region D onto $\hat{\varphi}_\varepsilon(D)$. Due to Lemma 6, $P'_\varepsilon \in \dot{\hat{\varphi}}_\varepsilon(D)$. Let $x'_1, \ldots, x'_n$ be a local coordinate system fulfilling the contions which are given before Hypothesis (D) where (0,1) is replaced by (7,1) .Let

$$\frac{\partial u'}{\partial t} = \frac{1}{2} \sum_{i,j} A'_{i,j}(x') \frac{\partial^2 u'}{\partial x'_i \partial x'_j} + \sum_i a'_i(x') \frac{\partial u'}{\partial x'_i} \tag{11,4}$$

be the form of (7,1) in the $x'_1, \ldots, x'_n$ coordinate system. Certainly we have $A'(0) = I$. The relation between the coordinate systems is $x' = T(x - P)$ where T is a regular matrix of the type $n \times n$. We introduce a new coordinate system $\xi'_1, \ldots, \xi'_n$ by $\xi' = N(\xi - \hat{\varphi}_\varepsilon(P))$ where $N = T((d\hat{\varphi}_\varepsilon/dx)(P))^{-1}$. Since the transformations $x' \to x$, $x \to \xi$, $\xi \to \xi'$ are all one-to-one, there exists a one-to-one transformation $\xi' = \varphi'(x')$, i.e. $\varphi'_\varepsilon(x') = N[\hat{\varphi}_\varepsilon(T^{-1}x' + P) - \hat{\varphi}_\varepsilon(P)]$ and

$$(d\varphi'_\varepsilon/dx')(x') = N(d\hat{\varphi}_\varepsilon/dx)(T^{-1}x' + P)\, T^{-1}.$$

We shall need some properties of the transformation $\xi' = \varphi'_\varepsilon(x')$ which follow immediately from the definition:

$$\frac{d\varphi'_\varepsilon}{dx'}(0) = I, \tag{11,5}$$

$$(11{,}6)\qquad \frac{\mathrm{d}^2\varphi_\varepsilon'^{(i)}}{\mathrm{d}x'^2}(0) = \sum_k N_{ik}(T^{-1})^{\mathrm{T}} \frac{\mathrm{d}^2\hat{\varphi}_\varepsilon^{(k)}}{\mathrm{d}x^2}(P)\, T^{-1}\,.$$

If the region D can be described in a neighbourhood of P by an inequality $x_n' > h(x_1', \dots, x_{n-1}')$, then due to Lemma 6 the region $\hat{\varphi}_\varepsilon(D)$ can be described by $\xi_n' > h'(\xi_1', \dots, \xi_{n-1}')$ in a neighbourhood of $\hat{\varphi}_\varepsilon(P)$. Certainly $h'(0) = 0$, (11,5) together with (6,1) imply $(\partial h'/\partial \xi_1')(0) = 0$ and by virtue of (6,2), also

$$(11{,}7)\qquad \frac{\partial^2 h'}{\partial \xi_i'\, \partial \xi_j'}(0) = \frac{\partial^2 h}{\partial x_i'\, \partial x_j'}(0) + \frac{\partial^2 \varphi_\varepsilon'^{(n)}}{\partial x_i'\, \partial x_j'}(0)\,.$$

By $\xi' = \varphi_\varepsilon'(x')$, $u'(t, x') = v'(t, \xi')$ equation (11,4) is transformed into

$$(11{,}8)\qquad \frac{\partial v'}{\partial t} = \frac{1}{2}\sum_{i,j} \tilde{A}_{ij}'(\varepsilon, \xi')\frac{\partial^2 v'}{\partial \xi_i'\, \partial \xi_j'} + \sum_i \tilde{a}_i'(\varepsilon, \xi')\frac{\partial v'}{\partial \xi_i'}$$

where

$$(11{,}9)\qquad \tilde{A}'(\varepsilon, \xi') = \frac{\mathrm{d}\varphi_\varepsilon'}{\mathrm{d}x'}(\psi_\varepsilon'(\xi'))\, A'(\psi_\varepsilon'(\xi'))\left(\frac{\mathrm{d}\varphi_\varepsilon'}{\mathrm{d}x'}(\psi_\varepsilon'(\xi'))\right)^{\mathrm{T}},$$

$$(11{,}10)\qquad \tilde{a}_i'(\varepsilon, \xi') = \sum_k a_k'(\psi'(\xi'))\frac{\partial \varphi_\varepsilon'^{(i)}}{\partial x_k'}(\psi_\varepsilon'(\xi') + $$

$$+ \frac{1}{2}\sum_{k,l} A_{k,l}'(\psi_\varepsilon'(\xi'))\frac{\partial^2 \varphi_\varepsilon'^{(i)}}{\partial x_k'\, \partial x_l'}(\psi_\varepsilon'(\xi'))$$

and ψ_ε' is the inverse transformation to $\xi' = \varphi_\varepsilon'(x')$. Relation (11,9) yields

$$(11{,}11)\qquad \frac{\partial \tilde{A}_{nn}'}{\partial \xi_i'}(\varepsilon, 0) = \frac{\partial A_{nn}'}{\partial x_i'}(0) + 2\frac{\partial^2 \varphi_\varepsilon'^{n}}{\partial x_i'\, \partial x_n'}(0)$$

and also $\tilde{A}'(\varepsilon, 0) = A'(0) = I$.

If ε converges to zero, then due to (11,1), (9,2) it is $(\mathrm{d}\hat{\varphi}_\varepsilon/\mathrm{d}x)(P) \to I$ and $(\mathrm{d}^2\hat{\varphi}_\varepsilon^{(k)}/\mathrm{d}x^2)(P) \to 0$. This implies that the matrix N (depending on ε) converges to T for $\varepsilon \to 0$ so that by (11,6), $(\mathrm{d}^2\varphi_\varepsilon'^{(k)}/\mathrm{d}x'^2)(0)$ converges to the zero matrix.

Further (11,7) and (11,11) imply $(\partial^2 h'/\partial \xi_i'\, \partial \xi_j')(0) \to (\partial^2 h/\partial x_i'\, \partial x_j')(0)$ and $(\partial \tilde{A}_{nn}'/\partial \xi_i')(\varepsilon, 0) \to (\partial A_{nn}'/\partial x_i')(0)$, respectively, while (11,10), (11,5) ensure $\tilde{a}_n'(\varepsilon, 0) \to a_n'(0)$. We can conclude that the matrix Γ' constructed for (11,8) at P_ε' converges to the matrix Γ constructed for (11,4) at P_ε' or, which is the same by Remark 1, to the matrix constructed for (7,1) at P. Since it is assumed that $\det \Gamma > 0$ for $\varepsilon = 0$ we conclude that Hypothesis (D) is fulfilled for sufficiently small ε.

Now we shall prove that equation (11,2) fulfils Hypothesis (E). The drift coefficients are linear. In virtue of Remark 2 it is sufficient to prove that conditions (0,4) are fulfilled for sufficiently small ε.

As was stated above the matrix of diffusion coefficients of (11,2) is

$$\tilde{A}^{(\varepsilon)}(\xi) = \left(\frac{d\psi_\varepsilon}{d\xi}\right)^{-1} A(\psi_\varepsilon) \left(\left(\frac{d\psi_\varepsilon}{d\xi}\right)^{-1}\right)^{\mathsf{T}}.$$

Putting $\varepsilon = 0$ we obtain $\tilde{A}^{(0)}(\xi) = A(\xi)$. Since

$$\left.\frac{d\psi_\varepsilon}{d\xi}\right|_{\varepsilon=0} = \left.\frac{d\hat{\varphi}_\varepsilon}{dx}\right|_{\varepsilon=0} = I \quad \text{and} \quad \left.\frac{\partial^2 \hat{\varphi}_\varepsilon^{(k)}}{\partial x_p\, \partial x_q}\right|_{\varepsilon=0} = \left.\frac{\partial^3 \hat{\varphi}_\varepsilon^{(k)}}{\partial x_p\, \partial x_q\, \partial x_r}\right|_{\varepsilon=0} = 0$$

we can construct matrix functions $M^{(\varepsilon)}(\xi)$, $P_l^{(\varepsilon)}(\xi)$ and $N_l^{(\varepsilon)}(\xi)$ continuous in ξ such that $(M^{(\varepsilon)}(\xi))^{\mathsf{T}}\, M^{(\varepsilon)}(\xi) = \tilde{A}^{(\varepsilon)}(\xi)$, $P_l^{(\varepsilon)}(\xi) = d\tilde{A}^{(\varepsilon)}/dl(\xi)$ and $(N_l^{(\varepsilon)}(\xi))^{\mathsf{T}}\, N_l^{(\varepsilon)}(\xi)\, n^2 = d^2\tilde{A}^{(\varepsilon)}/dl^2(\xi)$ where $k = (d\psi_\varepsilon/d\xi)\, l$ and M, P_k, N_k are the matrices corresponding to A by Remark 2. We can find these matrix functions in the form

$$M^{(\varepsilon)}(\xi) = M(\psi_\varepsilon)\left(\left(\frac{d\psi_\varepsilon}{d\xi}\right)^{-1}\right)^{\mathsf{T}}, \quad P_l^{(\varepsilon)}(\xi) = \left(\frac{d\psi_\varepsilon}{d\xi}\right)^{-1} P_k(\psi_\varepsilon)\left(\left(\frac{d\psi_\varepsilon}{d\xi}\right)^{-1}\right)^{\mathsf{T}} + O(\varepsilon),$$

$$N_l^{(\varepsilon)}(\xi) = N_k(\psi_\varepsilon)\left(\left(\frac{d\psi_\varepsilon}{d\xi}\right)^{-1}\right)^{\mathsf{T}} + O(\varepsilon).$$

As $M^{(\varepsilon)} \to M$, $P_l^{(\varepsilon)} \to P$ and $N_l^{(\varepsilon)} \to N$ uniformly, condition (0,4) is valid for $\tilde{A}^{(\varepsilon)}(\xi)$ if ε is sufficiently small.

Since we proved that equation (11,2) fulfils Hypothesis (D) we obtain by applying Theorem 5 [1] that there exists a positive number t_0 such that $v(t, \xi)$ is strictly convex on $(0, t_0) \times \mathring{\varphi}_\varepsilon(D)$. In this way we have verified the last assumption of Theorem 1. Due to this theorem we know that $v(t, \xi)$ is convex as a function of ξ in $(0, t_0) \times \hat{\varphi}_\varepsilon(D)$.

Formula (5,6) can be modified to

$$\frac{d^2u}{dl^2}(t, x) = \frac{d^2v}{dk^2}(t, \xi) + \varepsilon \sum_k \frac{\partial u}{\partial x_k}(t, x)\left[\frac{d^2\hat{\varphi}^{(k,1)}}{dl^2}(x) + \varepsilon\left(\frac{d^2\hat{\varphi}^{(k,2)}}{dl^2} + \right.\right.$$
$$\left.\left. + \sum_i \frac{\partial \psi^{(k,1)}}{\partial x_i} \frac{d^2\hat{\varphi}^{(i,1)}}{dl^2}\right) + \varepsilon\gamma(\xi, \varepsilon)\right]$$

where $k = (d\hat{\varphi}_\varepsilon/dx)\, l$. Since $v(t, \xi)$ is convex it is $(d^2v/dk^2)(t, \xi) \geqq 0$. Formula (11,1) yields $(\partial u/\partial x_k)(\bar{t}, x_0)\,(d^2\hat{\varphi}^{(k,1)}/dl^2)(x_0) \geqq 0$ and since by Lemma 9 $(d^2\hat{\varphi}^{(k,1)}/dl^2)(x_0)$ equals one or minus one the expression

$$\frac{\partial u}{\partial x_k}(\bar{t}, x_0)\left[\frac{d^2\hat{\varphi}^{(k,1)}}{dl^2}(x_0) + \varepsilon\left(\frac{d^2\hat{\varphi}^{(k,2)}}{dl^2}(x_0) - \right.\right.$$
$$\left.\left. + \sum_i \frac{\partial \psi^{(k,1)}}{\partial x_i} \frac{d^2\hat{\varphi}^{(i,1)}}{dl^2}(x_0)\right) + \varepsilon\gamma(\hat{\varphi}_\varepsilon(x_0), \varepsilon)\right]$$

is nonnegative for small ε. The modified formula implies that $\mathrm{d}^2u/\mathrm{d}l^2(\bar{t}, x_0) \geqq 0$. Since $[\bar{t}, x_0]$ was an arbitrary point from $(0, t_0) \times D$ and l an arbitrary n-dimensional vector we have proved that $u(t, x)$ is convex as a function of x for $t \in (0, t_0)$ and $x \in D$. Theorem 3 is proved.

12

As was mentioned above the method used in the proof of Theorem 3 can be applied to slightly disturbed parabolic equations.

Remark 10. *Let $A_{ij}(x)$, $a_i(x)$, λ_i and a region D fulfil conditions of Theorem 3. Given a positive number C then there exist numbers $\nu_0 > 0$, $\varepsilon_0 > 0$, $t_0 > 0$ such that the bounded solution of*

$$(12,1) \qquad \frac{\partial u}{\partial t} = \frac{1}{2} \sum_{i,j} A^+_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_i (-\lambda_i x_i + \varepsilon a^+_i(x)) \frac{\partial u}{\partial x_i}$$

fulfilling $u(0, x) = 0$ for $x \in D$, $u(t, x) = 1$ for $t > 0$, $x \in \dot{D}$ is convex as a function of x in $(0, t_0) \times D$ and for $|\varepsilon| \leqq \varepsilon_0$ provided the coefficients $A^+_{ij}(x)$, $a^+_i(x)$ fulfil

$$|A^+_{ij}(x) - A_{ij}(x)| < \nu_0 \,, \quad \left|\frac{\partial A^+_{ij}}{\partial x_k}(x) - \frac{\partial A_{ij}}{\partial x_k}(x)\right| < \nu_0 \,,$$

$$\left|\frac{\partial^2 A^+_{ij}}{\partial x_k \partial x_l}(x) - \frac{\partial^2 A_{ij}}{\partial x_k \partial x_l}(x)\right| < \nu_0 \,, \quad \|A^+_{ij}\|_{2+\alpha} \leqq C$$

and the third derivatives $(\partial^3 A^+_{ij}/\partial x_k \partial x_l \partial x_p)(x)$ are continuous, $|a^+_i(x) - a_i(x)| < \nu_0$, $\|a^+_i\|_{2+\alpha} \leqq C$.

The proof of Remark 10 will be sketched only. Nevertheless, we need a modification of Lemma 8.

Lemma 11. *Let the assumptions of Remark 10 be fulfilled. There exists a number $M > 0$ such that to every point $x_0 \in D$ and to every n-dimensional unit vector l there exist solutions $\varphi^+(x)$, $\chi^+(x)$, of*

$$(12,2) \qquad L^+\varphi = \frac{1}{2} \sum_{i,j} A^+_{ij}(x) \frac{\partial^2 \varphi}{\partial x_i \partial x_j} - \sum_i \lambda_i x_i \frac{\partial \varphi}{\partial x_i} + \lambda_k \varphi = f^+(x)$$

fulfilling

$$\varphi^+ \in C_{2+\alpha}(D_{\delta/2}) \,, \quad |\mathrm{d}^2\varphi^+/\mathrm{d}l^2(x_0) - 1| < \tfrac{1}{2} \,, \quad \|\varphi^+\|^D_{2+\alpha} \leqq M$$

$$\chi^+ \in C_{2+\alpha}(D_{\delta/2}) \,, \quad |\mathrm{d}^2\chi^+/\mathrm{d}l^2(x_0) + 1| < \tfrac{1}{2} \,, \quad \|\chi^+\|^D_{2+\alpha} \leqq M \,.$$

The first part of the proof of Remark 8 proves that 0 is not an eigenvalue of (12,2) for sufficiently small ν_0. As at the beginning of the proof of Lemma 9 we can conclude that

$$(12,3) \qquad L\varphi = \frac{1}{2}\sum_{i,j} A_{ij}(x)\frac{\partial^2\varphi}{\partial x_i\,\partial x_j} - \sum_i \lambda_i x_i \frac{\partial\varphi}{\partial x_i} + \lambda_k\varphi = f(x) \quad (k \text{ fixed})$$

has a solution $\bar{\varphi}$ fulfilling $\|\bar{\varphi}\|_{2+\alpha}^{D_\delta} < \infty$. (the coefficients $A_{ij}(x)$ are real analytic in x) under the condition that f is real analytic in x. Considering (9,8) we see that the assumption concerning the analyticity of f can be substituted by the assumption $f \in C_\alpha(D_\delta)$. Similarly we can assume only that the coefficients $A_{ij}(x)$ belong to $C_\alpha(D_\delta)$. This follows again from (9,8) since the coefficient C_2 depends only on the given bounds for coefficients and on the coefficient of ellipticity of $A(x)$ (see Remark 8). Denote by $\varphi(x)$ the solution of (12,3) given by Lemma 8, fulfilling $\mathrm{d}^2\varphi/\mathrm{d}l^2(x_0) = 1$, $\|\varphi\|_{2+\alpha}^{D_{\delta/2}} \leqq M_0$ Put

$$g(x) = \frac{1}{2}\sum_{i,j}(A_{ij} - A_{ij}^{+})\frac{\partial^2\varphi}{\partial x_i\,\partial x_j} - f(x) + f^{+}(x)\,.$$

From the above considerations we know that there exists a solution $\Delta(x)$, $\Delta \in C_{2+\alpha}^0(D_\delta)$ of $L^+\Delta = g$. The function $\varphi^+ = \varphi + \Delta$ is certainly a solution of (12,2). Since the equation for Δ can be rewritten as $\Delta = -(1/\lambda_k)(I - T^+)^{-1}\, T^+ g$, we obtain $|\mathrm{d}^2\varphi^+/\mathrm{d}l^2(x_0) - 1| \leqq C_2 M_0 \nu_0$ using (9,8). The proof for χ^+ is quite analogous.

Now the solutions $\varphi_\varepsilon^+(x)$ of

$$\frac{1}{2}\sum_{i,j} A_{ij}^{+}(x)\frac{\partial^2\varphi}{\partial x_i\,\partial x_j} + \sum_i(-\lambda_i x_i + \varepsilon\, a_i^{+}(x))\frac{\partial u}{\partial x_i} + \lambda_k\varphi = 0$$

which are of the form (9,2) and which transform (12,1) onto a parbolic equation with linear drift coefficients exist due to Remark 8. The convergence of (10,2) can be ensured by Lemma 10. The rest of the proof of Remark 10 follows the same lines as that of Theorem 3.

13

We shall discuss now Hypothesis (E) in more detail in the two-dimensional case ($n = 2$). Since the matrix A of the type 2×2 has to be singular we can assume

$$A = \begin{pmatrix} a\cos\varphi & 0 \\ a\sin\varphi & 0 \end{pmatrix} \quad a > 0\,, \quad \varphi \text{ real numbers}$$

and the matrix B in the form

$$B = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}, \quad b \text{ a real number}\,.$$

The condition on A can be written as

$$\frac{a^2}{2}\left[\cos^2\varphi\,\frac{\mathrm{d}^2 A_{11}}{\mathrm{d}l^2} + 2\cos\varphi\sin\varphi\,\frac{\mathrm{d}^2 A_{12}}{\mathrm{d}l^2} + \sin^2\varphi\,\frac{\mathrm{d}^2 A_{22}}{\mathrm{d}l^2}\right] +$$

$$+ a^2 b\left[\cos^2\varphi\,\frac{\mathrm{d}A_{11}}{\mathrm{d}l} + 2\cos\varphi\sin\varphi\,\frac{\mathrm{d}A_{12}}{\mathrm{d}l} + \sin^2\varphi\,\frac{\mathrm{d}A_{22}}{\mathrm{d}l}\right] +$$

$$+ a^2 b^2\left[\cos^2\varphi\,A_{11} + 2\cos\varphi\sin\varphi\,A_{12} + \sin^2\varphi\,A_{22}\right] \geqq 0$$

where the vector l is given by $l^{\mathrm{T}} = (-\sin\varphi, \cos\varphi)$ or $l^{\mathrm{T}} = (\sin\varphi, -\cos\varphi)$. Taking into account $a \neq 0$ and the definition of l we obtain

$$(13,1)\qquad \frac{1}{2}\left[\frac{\partial^2 A_{11}}{\partial y^2}\cos^4\varphi + \frac{\partial^2 A_{22}}{\partial x^2}\sin^4\varphi + \left(\frac{\partial^2 A_{11}}{\partial x^2} + \frac{\partial^2 A_{22}}{\partial y^2} - \right.\right.$$

$$\left. - 4\frac{\partial^2 A_{12}}{\partial x\,\partial y}\right)\cos^2\varphi\sin^2\varphi + 2\left(\frac{\partial^2 A_{12}}{\partial y^2} - \frac{\partial^2 A_{11}}{\partial x\,\partial y}\right)\cos^3\varphi\sin\varphi +$$

$$\left. + 2\left(\frac{\partial^2 A_{12}}{\partial x^2} - \frac{\partial^2 A_{22}}{\partial x\,\partial y}\right)\cos\varphi\sin^3\varphi\right] + b\left[\frac{\partial A_{11}}{\partial y}\cos^3\varphi + \right.$$

$$+ \left(-\frac{\partial A_{11}}{\partial x} + 2\frac{\partial A_{12}}{\partial y}\right)\cos^2\varphi\sin\varphi + \left(-2\frac{\partial A_{12}}{\partial x} + \right.$$

$$\left.\left. + \frac{\partial A_{22}}{\partial y}\right)\cos\varphi\sin^2\varphi - \frac{\partial A_{22}}{\partial x}\sin^3\varphi\right] + b^2\left[A_{11}\cos^2\varphi + \right.$$

$$\left. + 2A_{12}\cos\varphi\sin\varphi + A_{22}\sin^2\varphi\right] \geqq 0$$

for all real b and φ. The linearity of a together with condition (13,1) can substitute Hypothesis (E) since another choice of A, B:

$$A = \begin{pmatrix} 0 & a\cos\varphi \\ 0 & a\sin\varphi \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$$

leads to the same condition (13,1).

If $b = 0$ then (13,1) yields

$$c_1\cos^4\varphi + c_2\sin^4\varphi + c_3\cos^2\varphi\sin^2\varphi + c_4\cos^3\varphi\sin\varphi + c_5\cos\varphi\sin^3\varphi \geqq 0$$

for all real φ which is equivalent to $P(\lambda) \geqq 0$ for all real λ where $P(\lambda) = c_2\lambda^4 + c_5\lambda^3 + c_3\lambda^2 + c_4\lambda + c_1$.

Statement. *Let D_4 be the discriminant of $P(\lambda)$ and D'_3 the discriminant of the derivative $P'(\lambda)$. If $D_4 > 0$, $D'_3 < 0$ then $P(\lambda) > 0$ for every real λ.*

Since the term $A_{11}\cos^2\varphi + 2A_{12}\cos\varphi\sin\varphi + A_{22}\sin^2\varphi$ is positive due to the positive definiteness of A, the term on the left-hand side of (13,1) can be treated as a quadratic polynomial.

Example 1. Suppose $A_{12}(t, x, y) = 0$, let A_{11} depend only on t and x, A_{22} only on t, y. Condition (13,1) is reduced to

$$(13,2)\qquad \frac{1}{2}\left(\frac{\partial^2 A_{11}}{\partial x^2} + \frac{\partial^2 A_{22}}{\partial y^2}\right)\cos^2\varphi\sin^2\varphi + b\left(-\frac{\partial A_{11}}{\partial x}\cos\varphi + \right.$$

$$\left. + \frac{\partial A_{22}}{\partial y}\sin\varphi\right)\cos\varphi\sin\varphi + b^2(A_{11}\cos^2\varphi + A_{22}\sin^2\varphi) \geqq 0$$

for all real φ, b. Inequality (13,2) is valid if and only if

$$\frac{\partial^2 A_{11}}{\partial x^2} + \frac{\partial^2 A_{22}}{\partial y^2} \geqq 0\,,$$

$$\left(\frac{\partial A_{11}}{\partial x}\right)^2 \leqq 2A_{11}\left(\frac{\partial^2 A_{11}}{\partial x^2} + \frac{\partial^2 A_{22}}{\partial y^2}\right), \quad \left(\frac{\partial A_{22}}{\partial y}\right)^2 \leqq 2A_{22}\left(\frac{\partial^2 A_{11}}{\partial x^2} + \frac{\partial^2 A_{22}}{\partial y^2}\right),$$

$$A_{11}\left(\frac{\partial A_{22}}{\partial y}\right)^2 + A_{22}\left(\frac{\partial A_{11}}{\partial x}\right)^2 \leqq 2A_{11}A_{22}\left(\frac{\partial^2 A_{11}}{\partial x^2} + \frac{\partial^2 A_{22}}{\partial y^2}\right).$$

Example 2. Suppose $A_{12}(t, x, y) = 0$ and let A_{11} depend only on t and y, A_{22} only on t and x. In this case condition (13,1) is reduced to

$$(13,3)\qquad \frac{1}{2}\frac{\partial^2 A_{11}}{\partial y^2}\cos^4\varphi + \frac{1}{2}\frac{\partial^2 A_{22}}{\partial x^2}\sin^4\varphi + b\left(\frac{\partial A_{11}}{\partial y}\cos^3\varphi - \frac{\partial A_{22}}{\partial x}\sin^3\varphi\right) +$$

$$+ b^2(A_{11}\cos^2\varphi + A_{22}\sin^2\varphi) \geqq 0\,.$$

It is easy to obtain sufficient conditions for (13,3), namely

$$\frac{\partial^2 A_{11}}{\partial y^2} \geqq 0\,, \quad \frac{\partial^2 A_{22}}{\partial x^2} \geqq 0\,,$$

$$\left(\frac{\partial A_{11}}{\partial y}\right)^2 \leqq 2A_{11}\frac{\partial^2 A_{11}}{\partial y^2}\,, \quad \left(\frac{\partial A_{22}}{\partial x}\right)^2 \leqq 2A_{22}\frac{\partial^2 A_{22}}{\partial x^2}\,,$$

$$A_{11}\frac{\partial^2 A_{11}}{\partial y^2}\left(\frac{\partial A_{22}}{\partial x}\right)^2 + A_{22}\frac{\partial^2 A_{22}}{\partial x^2}\left(\frac{\partial A_{11}}{\partial y}\right)^2 \leqq 2A_{11}A_{22}\frac{\partial^2 A_{11}}{\partial y^2}\frac{\partial^2 A_{22}}{\partial x^2}\,.$$

The last section is devoted to the case when the problem (0,1) to (0,3) is spherically symmetric.

Example 3. Let the region D be an n-dimensional ball $D = \{x \in R^n : |x| < R\}$ and let $A(x) = f(|x|)\, I$, $a(x) = -\alpha x$ where I is the unit matrix, $f(r)$ is a positive function with a continuous third derivative defined for $r \in \langle 0, R \rangle$ and α is a positive number.

In this case the bounded solution $u(t, x)$ of (0,1) to (0,3) depends only on t and $|x|$. Define a function $v(t, r) = u(t, x)$ where $r = \sqrt{(\sum_i x_i^2)}$. The function $v(t, r)$ fulfils the equation

$$\frac{\partial v}{\partial t} = \frac{1}{2} f(r) \frac{\partial^2 v}{\partial r^2} + \left(-\alpha r + \frac{n-1}{2r} f(r) \right) \frac{\partial v}{\partial r}. \tag{14,1}$$

With respect to (0,2), (0,3) and [4] we have $\partial u / \partial \nu\,(t, x) > 0$ for $t > 0$, $|x| = R$ where ν is the outer normal at $[t, x]$. This implies

$$\frac{\partial v}{\partial r}(t, R) > 0 \quad t > 0. \tag{14,2}$$

If we assume

$$-\alpha R + \frac{n-1}{2R} f(R) < 0 \tag{14,3}$$

then (14,1) and (14,2) yield

$$\frac{\partial^2 v}{\partial r^2}(t, R) > 0 \quad \text{for} \quad t > 0. \tag{14,4}$$

Let λ_i, $i = 1, \ldots, n$ be real number. We have obviously

$$\sum \frac{\partial^2 u}{\partial x_i\, \partial x_j}(t, x)\, \lambda_i \lambda_j = \frac{1}{r^2} \frac{\partial^2 v}{\partial r^2}(t, r)\, (\sum \lambda_i x_i)^2 +$$

$$+ \frac{1}{r^3} \frac{\partial v}{\partial r}(t, r)\, [\sum_i x_i^2 \sum_j \lambda_j^2 - (\sum \lambda_i x_i)^2].$$

The last relation proves that $u(t, x)$ is convex as a function of x at the points $[t, x]$: $t > 0$, $|x| = R$ due to (14,2) and (14,4). We have just proved that the assumption of Theorem 1 about convexity of $u(t, x)$ on the side-boundary of Q is fulfilled. We shall verify the other assumptions of Theorem 1. Hypotheses (A) to (C) are obviously fulfilled.

Let $\bar{x}_1, \ldots, \bar{x}_n$ be a local coordinate system fulfilling Hypothesis (D). Denote by $\bar{u}(t, \bar{x})$ the solution $u(t, x)$ expressed in the $(\bar{x}_1, \ldots, \bar{x}_n)$-coordinate system. We have

$$(14,5) \qquad \frac{\partial \bar{u}}{\partial t} = \frac{1}{2f(R)} f\left(\sqrt{[f(R)\sum_{i=1}^{n-1} \bar{x}_i^2 + (\bar{x}_n\sqrt{f(R)} - R)^2]}\right)\sum_{i=1}^{n} \frac{\partial^2 \bar{u}}{\partial \bar{x}_i^2} - \\ - \alpha \sum_{i=1}^{n} \bar{x}_i \frac{\partial \bar{u}}{\partial \bar{x}_i} + \frac{\alpha R}{\sqrt{(f(R))}} \frac{\partial \bar{u}}{\partial \bar{x}_n}.$$

The function $\bar{h}(\bar{x}_1, \ldots, \bar{x}_{n-1})$ is given by

$$\bar{h}(\bar{x}_1, \ldots, \bar{x}_{n-1}) = \frac{R}{\sqrt{(f(R))}} - \sqrt{\left(\frac{R^2}{f(R)} - \sum_{i=1}^{n-1} \bar{x}_i^2\right)}.$$

In this case

$$(14,6) \qquad \Gamma_{ii} = \sqrt{(f(R))}/R, \quad i = 1, \ldots, n-1,$$
$$\Gamma_{ij} = 0 \text{ for } i \neq j, \; i \neq n, \; j \neq n,$$

Equation (14,5) implies $\Gamma_{in} = \Gamma_{ni} = 0$ for $i < n$ and $\Gamma_{nn} = 2\alpha R/\sqrt{(f(R))} - (n-1)\sqrt{(f(R))}/R$. With respect to (14,3) we have $\Gamma_{nn} > 0$ and this together with (14,6) ensures that Hypothesis (D) is valid.

Remark 2 shows that we can assume (0,4) instead of Hypothesis (E). We see easily that if

$$(14,7) \qquad \frac{d^2 f}{dl^2}(|x|) \geqq 0 \quad \text{for all} \quad x \in D \quad \text{and for all unit vectors } l$$

$$(14,8) \qquad \sqrt{\left(2f(|x|)\frac{d^2 f}{dl^2}(|x|)\right)} \pm n\frac{df}{dl}(|x|) \geqq 0 \quad \text{for all} \quad x \in D$$

and for all unit vectors l, then (0,4) is fulfilled. Applying Theorem 1 we obtain the following.

Statement. *Let $D = \{x : |x| < R\}$, $A(x) = f(|x|)\,I$, $a(x) = -\alpha x$ where $f(r)$ is a positive function with a continuous third derivative defined on $\langle 0, R\rangle$. If (14,3), (14,7) and (14,8) are fulfilled then the bounded solution of (0,1) fulfilling (0,2) and (0,3) is a convex function of x in $Q = (0, L) \times D$, i.e. the matrix function $A(x)$ is strongly maximal with respect to $a(x) = -\alpha x$ and Q.*

Lemma 1 [1] implies that condition (14,3) in the statement is not only sufficient but also necessary.

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