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PRÜFER d-GROUPS

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In a previous paper [3] we studied a ring-like system called a multiring (introduced by T. NAKANO [4]) which differs from the usual concept of rings by admitting a multivalued addition. We applied ideal-theoretical methods to the theory of m-rings (multirings) and d-groups to define Prüfer d-groups and we obtained several different characterizations of a special type of Prüfer d-groups.

In this paper we extend and generalize some results of [3], especially, we show eight different conditions equivalent to the property "a d-group is a Prüfer d-group". Further, we deal with the existence of an extension of a valuation m-ring of a d-group G to a valuation m-ring of a d-group G' which is integral over G and we prove that the integral closure of a Prüfer d-group is a Prüfer d-group. Finally, we characterize archimedean simply ordered d-groups, d-groups of principal m-ideals and Bezout d-groups.

1. INTRODUCTION

Our notation will be in general that of [3]. In particular, a *d-group* is a partially ordered commutative group G with an element $0 \notin G$, which admits a multivalued addition $\oplus$ such that

(1) $a \oplus b = b \oplus a$,

(2) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$,

(3) $a \in b \oplus c$ implies $b \in a \oplus c$,

(4) $a(b \oplus c) = ab \oplus ac$,

(5) $0 \in a \oplus b$ if and only if $a = b$,

(6) $a, b \geqq c$ and $x \in a \oplus b$ imply $x \geqq c$ for any $a, b, c \in G$.

An *m-ring* is a commutative semigroup $(M, .)$ that admits a multivalued addition $\oplus$ and satisfies (1)–(5). In this paper all m-rings are required to obey the cancellation law and the existence of identity element.

Let A be an m-ring, $U(A)$ its group of units. Then all the quotients ab^{-1} with $a, b \in A$, $b \neq 0$ form a group $Q(A)$. It is easy to see that the factor group $D(A) = Q(A)/U(A)$ is partially ordered and becomes a d-group. $D(A)$ is called a *d-group relative to A.*

A subset J of an m-ring A is called an *m-ideal* of A provided that $a \oplus b \subseteq J$, $ar \in J$ for any $a, b \in J$, $r \in A$, and it is called a *prime m-ideal* provided that $ab \in J$ implies $a \in J$ or $b \in J$ for each $a, b \in A$.

An m-ring A is called *local* provided that a sum of non-units does not contain a unit, and A is called a *valuation m-ring* provided that $D(A)$ is simply ordered. The unique maximal m-ideal of A is denoted by $M(A)$.

A d-group G is called a *Prüfer d-group* provided that a quotient m-ring

$$(G_+)_P = \{gh^{-1} : g \in G_+,\ h \in G_+ - P\}$$

(where $G_+ = \{g \in G : g \geqq 1\}$) is a valuation m-ring for each prime m-ideal P of G_+.

An element p of a d-group G is called *integral* over an m-subring A of G if there exist elements $a_0, \ldots, a_n \in A$, $n \geqq 0$ such that

$$p^{n+1} \in a_n p^n \oplus \ldots \oplus a_0 .$$

An m-subring A of G is called *integrally closed* in G provided that every element of G integral over A is contained in A.

2. PRÜFER d-GROUPS

In this section we deal with an extension and generalization of [3]; Theorem 8. In particular, we show eight different characterizations of Prüfer d-groups.

First we shall prove several lemmas. In what follows, by $\mathfrak{M}(G)$ ($\mathfrak{B}(G)$) we shall denote the set of directed prime d-convex subgroups (prime m-ideals) of $G(G_+)$. For definition see [4].

Lemma 2.1. *Let G be a d-group. Then there exists a one-to-one map* ψ *of* $\mathfrak{M}(G)$ *onto* $\mathfrak{B}(G)$ *such that*

$$H_1 \subseteqq H_2 \Leftrightarrow \psi(H_1) \supseteq \psi(H_2)$$

for $H_1, H_2 \in \mathfrak{M}(G)$. *Further, if G is directed, then for any* $H \in \mathfrak{M}(G)$ *we have*

$$D((G_+)_{\psi(H)}) \cong G/H .$$

Proof. Let $P \in \mathfrak{B}(G)$. Then the quotient subgroup $\varphi(P)$ of the semigroup $G_+ - P$ is a directed subgroup of G, thus it is d-convex by [4]; Lemma 5, and $\varphi(P) \in \mathfrak{M}(G)$

by [4]; Lemma 6. On the other hand, by [3]; Lemma 4 we obtain that $\psi(H) = G_+ - (H \cap G_+)$ is a prime m-ideal of G_+ for any $H \in \mathfrak{M}(G)$. Now it is easy to see that ψ and φ are mutually inverse bijections. Suppose that G is directed. Then for $gH \in (G/H)_+$ we may find $g_1 \geqq 1$, $h \in H \cap G_+$ such that $g = g_1 h^{-1} \in (G_+)_{\psi(H)}$ and it is easy to see that this map may be extended onto a required isomorphism.

Proposition 2.2. *Let G be a d-group and let A be an m-ideal of G_+. Then*

$$A = \bigcap\{AH : H \in \mathfrak{M}(G)\} .$$

Proof. It is clear that $A \subseteqq \bigcap\{AH : H \in \mathfrak{M}(G)\}$. We suppose that $z \in AH$ for each $H \in \mathfrak{M}(G)$. Since H is directed, for any $H \in \mathfrak{M}(G)$ there exist $a_H \in A$, $h_H \in H \cap G_+$ such that

$$z = a_H h_H^{-1} .$$

Hence by Lemma 2.1, $z \in (G_+)_{\psi(H)}$ for any $H \in \mathfrak{M}(G)$. Now we put

$$B = \{y \geqq 1 : yz \in A\} .$$

It is clear that B is an m-ideal of G_+ and $B \nsubseteq \psi(H)$ for each $H \in \mathfrak{M}(G)$. Hence B is not contained in any prime m-ideal of G_+. Thus $B = G_+$ and $z \in A$.

Let G be a d-group. A subset $F \subset G$ is called a *fractional m-ideal* provided that there exist an m-ideal A of G_+ and $g \in G$ such that $F = Ag^{-1} = \{ag^{-1} : a \in A\}$. An m-ideal A of G_+ is called *invertible* provided that there exists a fractinal m-ideal F such that $A \,.\, F = G_+$. In what follows, we shall denote by $(a_1, \ldots, a_n)_G$ an m-ideal of G_+ generated by the family $\{a_1, \ldots, a_n\} \subseteqq G_+$.

For the proof of the main theorem we need a generalization of [4]; Theorem 6. Namely, we shall not assume that all d-convex subgroups in [4]; Theorem 6 are directed.

Theorem 2.3. *Let G be a directed d-group. Then*

$$\bigcap\{H : H \in \mathfrak{M}(G)\} = \{1\} .$$

Proof. The proof of this theorem is a modification of the original one. Let $p \in \bigcap\{H : H \in \mathfrak{M}(G)\}$ and suppose that $p \neq 1$. Zorn's lemma shows the existence of a directed d-convex subgroup H of G such that H is a maximal (in the set of directed d-convex subgroups of G) in the sense that

$$H \cap [p^{-1}) = \emptyset ,$$

where $[x) = \{g \in G : g \geqq x\}$. Now, by [4]; Lemma 8 we obtain that H is prime, hence $p^{-1} \in H$, a contradiction. Thus $p = 1$.

Theorem 2.4. *Let G be a directed d-group. Then the following conditions are equivalent*:

(1) $\{G/H : H \in \mathfrak{M}(G)\}$ *is a realization of* G. (For definition see [4].)

(2) *G is a Prüfer d-group.*

(3) *G_+ is integrally closed in G and for each m-subring A such that $G_+ \subseteqq A \subset G$, there exists $\mathfrak{B} \subseteqq \mathfrak{B}(G)$ such that $A = \bigcap\{(G_+)_P : P \in \mathfrak{B}\}$.*

(4) *Each m-subring A such that $G_+ \subseteqq A \subset G$ is integrally closed in G.*

(5) *A factor d-group G/H is simply ordered for each $H \in \mathfrak{M}(G)$.*

(6) *Each finitely generated m-ideal of G_+ is invertible.*

(7) *Each m-ideal with a basis of two elements of G_+ is invertible.*

(8) *G_+ is integrally closed in G and for each $a, b \in G_+$ there exists an integer $n > 1$ such that $(a, b)_G^n = (a^n, b^n)_G$.*

(9) *G_+ is integrally closed in G and for each $a, b \in G_+$ there exists an integer $n > 1$ such that $a^{n-1}b \in (a^n, b^n)_G$.*

Proof. (1) ⇒ (2). Let $P \in \mathfrak{B}(G)$. Then by Lemma 2.1 we have $D((G_+)_P) \cong G/\psi^{-1}(P)$. Since $G/\psi^{-1}(P)$ is simply ordered, it follows that $(G_+)_P$ is a valuation m-ring. Therefore G is a Prüfer d-group.

(2) ⇒ (3). In [3]; Theorem 8 we have proved that each m-subring A such that $G_+ \subseteqq A \subset G$ is a Prüfer m-ring (i.e. $D(A)$ is a Prüfer d-group). Now we may assume that A is the integral part of the d-group $D(A)$. Hence, by Proposition 2.2, $A = \bigcap\{AH : H \in \mathfrak{M}(D(A))\}$ and from the proof of Lemma 2.1 it is easy to see that $AH = A_{\psi(H)}$, where

$$\psi : \mathfrak{M}(D(A)) \to \mathfrak{B}(D(A))$$

is the map from Lemma 2.1. Thus

$$A = \bigcap\{A_P : P \in \mathfrak{B}(D(A))\}$$

and A_P is a valuation m-ring. Since $P \cap G_+ \in \mathfrak{B}(G)$ and $(G_+)_{P \cap G_+}$ is a valuation m-ring for each $P \in \mathfrak{B}(D(A))$, it follows that there exists $P' \in \mathfrak{B}(G)$ such that $A_P = (G_+)_{P'}$. Thus $A = \bigcap(G_+)_{P'}$.

(3) ⇒ (4). Let A be an m-ring such that $G_+ \subseteqq A \subset G$. Hence there exists $\mathfrak{B} \subseteqq \mathfrak{B}(G)$ such that

$$A = \bigcap\{(G_+)_P : P \in \mathfrak{B}\} .$$

Since $D((G_+)_P) \cong G/\psi^{-1}(P)$ (Lemma 2.1) and G_+ is integrally closed in G, it follows ([3]; Proposition 10) that $(G/\psi^{-1}(P))_+$ is integrally closed in $G/\psi^{-1}(P)$. Hence $(G_+)_P$ is integrally closed in G by [3]; Lemma 6. Therefore A is integrally closed in G.

(4) $\Rightarrow$ (2). The proof of this implication is quite the same as the proof of the implication (3) $\Rightarrow$ (2) of [3]; Theorem 8.

(2) $\Rightarrow$ (5). Let $H \in \mathfrak{M}(G)$. Since $(G_+)_{\psi(H)}$ is a valuation m-ring and $G/H \cong \cong D((G_+)_{\psi(H)})$ (Lemma 2.1), it follows that G/H is simply ordered for each $H \in \mathfrak{M}(G)$.

(5) $\Rightarrow$ (6). We show first that X/H is an m-ideal of $(G/H)_+$ for each m-ideal X of G_+ and for each $H \in \mathfrak{M}(G)$. In fact, let xH, $yH \in X/H$, $zH \in xH \oplus yH$. Hence there exist h_1, $h_2 \in H$ such that

$$z \in xh_1 \oplus yh_2 \, .$$

Since G/H is simply ordered, we may assume that $xH \geqq yH$. Thus $x = yhg$ for some $h \in H$, $g \geqq 1$. Hence

$$z \in y(hh_1g \oplus h_2) \subseteqq y(G_+H) \subseteqq XH \, .$$

Thus $z = ah'$ for some $a \in X$, $h' \in H$ and

$$zH = aH \in X/H \, .$$

Now let $gH \geqq H$, $xH \in X/H$. Then we have $gh^{-1} \geqq 1$ for some $h \in H$ and $gxH = = xgh^{-1}H \in X/H$. Thus X/H is an m-ideal.

Further, assume that $A = (a_1, \ldots, a_n)_G$ is an m-ideal of G_+. We set

$$B = \{g \geqq 1 : ga_k \geqq a_1 \text{ for } k = 1, \ldots, n\} \, .$$

It is easy to see that B is an m-ideal of G_+. We shall prove that

$$A \, . \, B = [a_1) = \{g \geqq 1 : g \geqq a_1\} \, .$$

In fact, by Proposition 2.2 it suffices to prove that

$$A \, . \, B/H = [a_1)/H$$

for each $H \in \mathfrak{M}(G)$.

First we shall show that

$$B/H = \{bH \geqq H : ba_kH \geqq a_1H \text{ for } k = 1, \ldots, n\} \, .$$

In fact, suppose that $bH \in (G/H)_+$ such that $ba_kH \geqq a_1H$ for $k = 1, \ldots, n$. Then there exist $h_k \in H$, $k = 1, \ldots, n$, $h_0 \in H$ such that

$$ba_kh_k \geqq a_1 \, , \quad b \geqq h_0 \, ; \quad k = 1, \ldots, n \, .$$

Since H is directed, there exists $h \in H$ such that

$$h \geqq h_k, \; h_0^{-1} \, ; \quad k = 1, \ldots, n \, .$$

Thus

$$(bh)\, a_k \geqq bh_ka_k \geqq a_1 \, , \quad bh \geqq 1 \, ; \quad k = 1, \ldots, n \, .$$

Therefore $bh \in B$ and $bH = (bh)\, H \in B/H$. The converse inclusion is trivial.

Now, since G/H is simply ordered, for each $H \in \mathfrak{M}(G)$ there exists $a_H \in \{a_1, \ldots, a_n\}$ such that

$$A/H = [a_H H) .$$

Hence

$$A . B/H = \{zgH : za_H H \geqq a_1 H, \; gH \geqq a_H H\} .$$

Since $a_1 H \geqq a_H H$, it follows that there exists $zH \geqq H$ such that

$$a_1 H = a_H zH \geqq a_H H$$

and we obtain

$$[a_1)/H \subseteqq A . B/H .$$

The converse inclusion is trivial. Therefore $[a_1) = A . B$ and we obtain

$$(B . [a_1^{-1})) . A = G_+ .$$

Thus A is an invertible m-ideal of G_+.

(6) $\Rightarrow$ (7). Trivial.

(7) $\Rightarrow$ (8). It is clear that $(a, b)_G^3 = (a^3, a^2b, ab^2, b^3)_G = (a, b)_G . (a^2, b^2)_G$. Since $(a, b)_G$ is invertible, it follows that $(a, b)_G^2 = (a^2, b^2)_G$.

(8) $\Rightarrow$ (9). Trivial.

(9) $\Rightarrow$ (1). Let $H \in \mathfrak{M}(G)$ and suppose that $gH \in G/H$. Since G is directed, there exists $a \geqq 1$ such that $ag \geqq 1$. Hence there exists an integer $n > 0$ such that

$$a^n g \in (a^n, (ag)^n)_G .$$

Thus we have

$$a^n g \in u_1 a^n \oplus u_2 a^n g^n$$

for some $u_1 \geqq 1$, $u_2 \geqq 1$ and using (3) from the definition of a d-group we obtain $u_1 = gu_1'$ for some

$$u_1' \in 1 \oplus u_2 g^{n-1} .$$

Since G/H is local and

$$H \in u_1' H \oplus u_2 g^{n-1} H ,$$

it follows that $H = u_1' H$ or $H = u_2 g^{n-1} H$. In the first case we have $H \leqq u_1 H = gu_1' H = gH$; in the second case we have $(g^{-1})^{n-1} H = u_2 H \geqq H$. Suppose that $(g^{-1})^{n-1} H > H$. Since G/H is local, we have

$$(g^{-1})^{n-1} H \oplus H = \{H\} .$$

Thus $(g^{-1}) H$ is integral over $(G/H)_+$. Since G_+ is integrally closed, it follows by [3]; Proposition 10 that $(g^{-1}) H \geqq H$.

Suppose that $(g^{-1})^{n-1} H = H$. Again $(g^{-1}) H$ is integral over $(G/H)_+$ and we obtain $gH \leqq H$. Therefore G/H is simply ordered for each $H \in \mathfrak{M}(G)$. Now Theorem 2.3 implies that $\{G/H : H \in \mathfrak{M}(G)\}$ is a realization of G.

From the above theorem we obtain a characterization of Prüfer integral domains.

Recall that for an integral domain A the family

$$\bar{A} = \{\bar{x} = \{x, -x\} : x \in A\}$$

is an m-ring with respect to the addition

$$\bar{x} \oplus \bar{y} = \{\overline{x + y}, \overline{x - y}\}$$

and multiplication

$$\bar{x} \,.\, \bar{y} = \overline{xy}\,.$$

Proposition 2.5. *Let A be an integral domain. Then A is a Prüfer domain if and only if $\{D(\bar{A})/H : H \in \mathfrak{M}(D(\bar{A}))\}$ is a realization of the d-group $D(\bar{A})$.*

Proof. Let A be a Prüfer domain. Since $\bar{A}_P = \overline{A_P}$ for each prime ideal P of A, we obtain that $\bar{A}$ is a Prüfer m-ring (i.e. $D(\bar{A})$ is a Prüfer d-group) and by Theorem 2.4 the set $\{D(\bar{A})/H : H \in \mathfrak{M}(D(\bar{A}))\}$ is a realization of $D(\bar{A})$.

Conversely, let $\{D(\bar{A})/H : H \in \mathfrak{M}(D(\bar{A}))\}$ be a realization of $D(\bar{A})$. We may assume that $\bar{A} = D(\bar{A})_+$. Then by Lemma 2.1, $D(\bar{A})/H \cong D(\bar{A}_{\psi(H)}) = D(\overline{A_P})$ for $\bar{P} = \psi(H)$. Thus $\overline{A_P}$ is a valuation m-ring. Now it is easy to see that A_P is a valuation ring and applying the bijection from Lemma 2.1 we obtain that A is a Prüfer domain.

3. INTEGRAL EXTENSIONS OF d-GROUPS

Let G be a d-group, $\mathcal{G}$ a d-group integral over G. We shall consider in this section the existence of extensions of valuation m-rings of G to valuation m-rings of $\mathcal{G}$, the rank of this extension and an extension of a Prüfer d-group.

Proposition 3.1. *Let G be a d-group, $\mathcal{G}$ a d-group integral over G such that $\mathcal{G}_+$ is integral over G_+ and let R be a valuation m-ring of G containing G_+. Then there exists a valuation m-ring $\mathcal{R}$ of $\mathcal{G}$ such that*

$$\mathcal{R} \cap G = R\,.$$

Proof. We show first that the proposition holds if G is a simply ordered d-group and $R = G_+$. In fact, set

$$M = \{g \in G : g > 1\}\,,$$

$\mathcal{J} = \{\alpha \in \mathcal{G}_+ :$ there exists $m \in M$ such that $\alpha \geqq m\}$. It is easy to see that $\mathcal{J}$ is an m-ideal of $\mathcal{G}_+$ and $M \subseteqq \mathcal{J}$. Suppose that $\mathcal{J} = \mathcal{G}_+$. Then there exists $m \in M$ such that $m^{-1} \geqq 1$. Since $\mathcal{G}_+$ is integral over G_+ and m is a non-unit of G_+, we obtain a contradiction with [5]; Lemma 1.

Hence there exists a maximal m-ideal $\mathcal{M}$ of $\mathcal{G}_+$ such that

$$\mathcal{J} \subseteqq \mathcal{M},$$

and we have

$$M \subseteqq \mathcal{J} \cap G_+ \subseteqq \mathcal{M} \cap G_+ \subseteqq M.$$

Therefore $M = \mathcal{M} \cap G_+$.

Now by [3]; Proposition 3 there exists a valuation m-ring $\mathcal{R}$ of $\mathcal{G}$ such that

$$M(\mathcal{R}) \cap \mathcal{G}_+ = \mathcal{M}.$$

Let $x \in \mathcal{R} \cap G$ and suppose that $x < 1$. Then $x^{-1} \in M = M(\mathcal{R}) \cap G_+$, thus $x = (x^{-1})^{-1} \notin \mathcal{R}$, a contradiction. Thus $\mathcal{R} \cap G \subseteqq G_+$ and since the converse inclusion is trivial, the proposition holds in this case.

Now, to prove the proposition in a general case, we put

$$G' = D(R), \quad \mathcal{G}' = D(R'),$$

where R' is the integral closure of R in $\mathcal{G}$. First we show that the canonical homomorphism

$$G/U(R) \to \mathcal{G}/U(R')$$

is injective. Indeed, suppose that $g \in U(R') \cap G$ and $g \notin U(R)$. If $g \in R$, we have $g^{-1} \notin R$, $g^{-1} \in U(R') \subseteqq R'$, a contradiction. If $g \notin R$, we have $g^{-1} \in R$, g integral over R and by [5]; Lemma 1 we obtain a contradiction. Thus $g \in U(R)$ and we may regard $D(R)$ as a d-subgroup of $D(R')$. It is clear that $D(R')$ is integral over $D(R)$. Now, according to the first part of this proof, there exists a valuation m-ring $\mathcal{R}'$ of $\mathcal{G}'$ such that

$$\mathcal{R}' \cap G' = G'_+.$$

Put

$$\mathcal{R} = \{\alpha \in \mathcal{G} : \alpha\, U(R') \in \mathcal{R}'\}.$$

Then $\mathcal{R}$ is a valuation m-ring and

$$\mathcal{R} \cap G = R.$$

Using Proposition 3.1 we obtain the "lying-over theorem" for prime m-ideals. (See [1].)

Proposition 3.2. *Let G be a d-group, $\mathcal{G}$ a d-group integral over G and such that $\mathcal{G}_+$ is integral over G_+ and let P be a prime m-ideal of G_+. Then there exists a prime m-ideal $\mathcal{P}$ of $\mathcal{G}_+$ such that*

Proof. By [3]; Proposition 3 there exists a valuation m-ring R of G such that $M(R) \cap G_+ = P$. By Proposition 3.1 there exists a valuation m-ring $\mathcal{R}$ of $\mathcal{G}$ such that

$$\mathcal{R} \cap G = R, \quad M(\mathcal{R}) \cap G = M(R).$$

Put $\mathscr{P} = M(\mathscr{R}) \cap \mathscr{G}_+$. Then

$$\mathscr{P} \cap G_+ = M(\mathscr{R}) \cap \mathscr{G}_+ \cap G_+ = M(\mathscr{R}) \cap G_+ = P .$$

If R is a valuation m-ring, the ordinal type of the set of proper $(\neq R)$ prime m-ideals of R (ordered under $\supseteqq$) is called the rank of R and is denoted by $r(R)$. By Lemma 2.1 $r(R)$ equals the ordinal type of the set of directed prime d-convex subgroups of $D(R)$ ordered under $\subseteqq$.

We shall use the following notation: We set

$$[G' : G] \leqq n$$

for d-groups G', G if G is a d-subgroup of G' and for any $g'_1, \ldots, g'_{n+1} \in G'$ there exist $a_1, \ldots, a_{n+1} \in G$ such that

$$0 \in g'_1 a_1 \oplus \ldots \oplus g'_{n+1} a_{n+1} .$$

Proposition 3.3. *Let $[G' : G] \leqq n$. Then G' is integral over G.*

Proof. Trivial.

Proposition 3.4. *For simply ordered d-groups G, G' such that G' is integral over G, the factor group G'/G is a torsion group.*

Proof. We show first that the proposition holds for

$$[G' : G] \leqq n .$$

In fact, let $a \in G'$ and suppose that $a^i \notin G$ for $i = 1, \ldots, n + 1$. Then there exist $g_0, \ldots, g_n \in G$ such that

$$0 \in g_n a^n \oplus \ldots \oplus g_0 .$$

Since G is simply ordered, there exists an index i, $0 \leqq i \leqq n$ such that

$$g_i \leqq g_k \quad \text{for} \quad k = 0, \ldots, n .$$

Then we have

$$0 \in g'_n a^n \oplus \ldots \oplus a^i \oplus \ldots \oplus g'_0$$

for some $g'_k \in G_+$, $k = 0, \ldots, n$. Since $g'_k a^k \neq g'_j a^j$ for $k \neq j$, by [3]; Lemma 1 we obtain that

$$a^i \geqq \min \{g'_k a^k : k \neq i\} = g'_i a^j$$

for some j, $0 \leqq j \leqq n$. Thus $a^{i-j} \in G$, a contradiction. Now let $\{G'_i\}_{i \in I}$ be the set of simply ordered d-subgroups of G' such that for any $i \in I$ there exists an integer n_i with

$$[G'_i : G] \leqq n_i .$$

Since G' is integral over G, we have

$$G' = \bigcup\{G'_i : i \in I\} .$$

Therefore G'/G is a torsion group.

Lemma 3.5. *Let G be a simply ordered d-group and let H be a d-convex subgroup of G such that G/H is a torsion group. Then $r(G_+) = r(H_+)$.*

Proof. For $H' \in \mathfrak{M}(H)$ we set

$$f(H') = \{g \in G: \text{ there exists an integer } n \geqq 1 \text{ such that } g^n \in H'\}$$

and for $K \in \mathfrak{M}(G)$ we set

$$g(K) = K \cap H .$$

It is easy to see that $f(H') \in \mathfrak{M}(G)$, $g(K) \in \mathfrak{M}(H)$ and f, g are mutually inverse. The rest follows by Lemma 2.1.

Proposition 3.6. *Let G be a d-group, $\mathscr{G}$ a d-group integral over G and let R be a valuation m-ring of G. Then $r(R) = r(\mathscr{R})$ for any valuation m-ring $\mathscr{R}$ of $\mathscr{G}$ such that*

$$\mathscr{R} \cap G = R .$$

Proof. We may regard the d-group $D(R)$ as a d-convex subgroup of $D(\mathscr{R})$. Now it is easy to see that $D(\mathscr{R})$ is integral over $D(R)$. Hence by Proposition 3.4, $D(\mathscr{R})/D(R)$ is a torsion group and by Lemma 3.5, $r(D(\mathscr{R}))_+) = r(D(R)_+)$. Thus $r(\mathscr{R}) = r(R)$.

Theorem 3.7. *Let G be a Prüfer d-group, $\mathscr{G}$ a d-group integral over G and let $\mathscr{G}_+$ be the integral closure of G_+ in $\mathscr{G}$. Then $\mathscr{G}$ is a Prüfer d-group.*

Proof. Let $\mathscr{H} \in \mathfrak{M}(\mathscr{G})$ and set

$$H = \{ab^{-1} : a,\ b \in \mathscr{H} \cap G_+\} .$$

It is clear that $H \in \mathfrak{M}(G)$. (See [4]; Lemmas 5,6.) Let $\boldsymbol{a} \in \mathscr{G}$ and suppose that $\boldsymbol{a}\mathscr{H} \ngeqq \ngeqq \mathscr{H}$. Since $\boldsymbol{a}$ is integral over G, there exist $g_1, \ldots, g_n \in G$ such that

$$\boldsymbol{a}^n \in g_1\boldsymbol{a}^{n-1} \oplus \ldots \oplus g_n .$$

By Theorem 2.4, G/H is simply ordered. If we suppose that $g_iH \geqq H$ for each i, $i = 1, \ldots, n$, we obtain that

$$g_i\mathscr{H} \geqq \mathscr{H} \quad \text{for} \quad i = 1, \ldots, n .$$

Then by [3]; Proposition 10 it is $\boldsymbol{a}\mathscr{H} \geqq \mathscr{H}$, a contradiction. Thus there exist $b_0, \ldots, b_n \in G$ such that

(1) $\qquad b_j\mathscr{H} \geqq \mathscr{H} \quad \text{for} \quad j = 0, \ldots, n\,; \qquad b_i\mathscr{H} = \mathscr{H} \quad \text{for some } i\,, \quad 0 \leqq i \leqq n$

and

$$a^n b_0 \mathscr{H} \in a^{n-1} b_1 \mathscr{H} \oplus \ldots \oplus b_n \mathscr{H}\,.$$

Assume that the above equation is of the lowest possible degree. Since $ab_0\mathscr{H}$ is integral over $(\mathscr{G}/\mathscr{H})_+$, it follows that $ab_0\mathscr{H} \geqq \mathscr{H}$.

Now there are three cases to be considered.

Case 1. $b_0\mathscr{H} = \mathscr{H}$. Then $a\mathscr{H} = ab_0\mathscr{H} \geqq \mathscr{H}$, a contradiction.

Case 2. $b_0\mathscr{H} > \mathscr{H}$ and $ab_0\mathscr{H} = \mathscr{H}$. Then we have $a\mathscr{H} < ab_0\mathscr{H} = \mathscr{H}$, thus $a^{-1}\mathscr{H} \geqq \mathscr{H}$.

Case 3. $b_0\mathscr{H} > \mathscr{H}$ and $ab_0\mathscr{H} > \mathscr{H}$. Then there exists

$$b_1'\mathscr{H} \in ab_0\mathscr{H} \oplus b_1\mathscr{H}$$

such that

$$(2) \qquad b_1' a^{n-1}\mathscr{H} \in b_2 a^{n-2}\mathscr{H} \oplus \ldots \oplus b_n\mathscr{H}\,.$$

Since $\mathscr{G}/\mathscr{H}$ is a local d-group, we obtain $b_1'\mathscr{H} > \mathscr{H}$ if and only if $b_1\mathscr{H} > \mathscr{H}$. Since the equation (2) is of the degree $n - 1$ and satisfies the condition (1), we obtain a contradiction. Thus $n = 1$ and we have

$$ab_0\mathscr{H} = b_1\mathscr{H} = \mathscr{H}\,, \quad a^{-1}\mathscr{H} = b_0\mathscr{H} > \mathscr{H}\,.$$

Therefore $\mathscr{G}/\mathscr{H}$ is a simply ordered and by Theorem 2.4, $\mathscr{G}$ is a Prüfer d-group.

4. SOME PROPERTIES OF AN ORDER RELATION IN A d-GROUP

A d-group G is called a *Bezout d-group* provided that every finitely generated m-ideal of G_+ is principal, and it is called a *d-group of principal m-ideals* provided that each m-ideal of G_+ is principal.

Proposition 4.1. *Let G be a directed d-group. Then G is a Bezout d-group if and only if G is a lattice ordered group and every finitely generated m-ideal of G_+ is a filter.*

Proof. Suppose that G is a Bezout d-group. Let $a, b \in G$. Since G is directed, there exist $c, a_1, b_1 \geqq 1$ such that $a = a_1 c^{-1}$, $b = b_1 c^{-1}$. Thus there exists $d \geqq 1$ such that $(a_1, b_1)_G = [d)$. Since $d \in a_1 g \oplus b_1 q$ for some $g \geqq 1$, $q \geqq 1$, we obtain $d = a_1 \wedge b_1 = \inf\{a_1, b_1\}$. Hence $dc^{-1} = a \wedge b$ and G is an l-group. For $A = (a_1, \ldots, a_n)_G$ we have $A = [a_1 \wedge \ldots \wedge a_n)$ and A is a filter. The rest is trivial.

Proposition 4.2. *Let G be a directed d-group. Then G is a d-group of principal m-ideals if and only if G is a complete lattice ordered group satisfying the descending chain condition and every m-ideal of G_+ is a filter.*

Proof. Suppose that G is a d-group of principal m-ideals. By Proposition 4.1, G is an l-group and every finitely generated (and so every) m-ideal is a filter. Now let $\{a_i\}_{i \in I} \subseteqq G$ be such that there exists $a \in G$ such that $a \leqq a_i$ for each $i \in I$. Then $a_i a^{-1} = d_i$ $(i \in I)$ for some $d_i \geqq 1$. Let A be the m-ideal of G_+ generated by the family $\{d_i\}_{i \in I}$. Then there exists $b \geqq 1$ such that $A = [b)$ and since $b \in d_{i_1} g_1 \oplus \ldots \oplus d_{i_n} g_n$ for some $i_1, \ldots, i_n \in I$, $g_1, \ldots, g_n \in G_+$, we obtain $b = \inf \{d_i : i \in I\}$. Now $d_{i_k} g_k \geqq$ $\geqq d_{i_1} \wedge \ldots \wedge d_{i_n}$, hence $b = \inf \{d_i : i \in I\} \geqq d_{i_1} \wedge \ldots \wedge d_{i_n} \geqq b$ and we obtain $ba^{-1} = \inf \{a_i : i \in I\} = a_{i_1} \wedge \ldots \wedge a_{i_n}$. Therefore G is a complete l-group with the d.c.c. The converse is trivial.

A d-group G is called *archimedean* provided that the ordered group $G - \{0\}$ is archimedean, i.e. if $a^n < b$ for every integer n, then $a = 1$ $(a, b \in G)$. An m-subring A of a d-group G is called *completely integrally closed* provided that for any $g \in G$ such that there exists $a \in G$ with the property $ag^n \in A$ for each integer $n > 0$ it follows that $g \in A$.

We shall deal with the following properties of a d-group G:

(1) G is an archimedean d-group,

(2) there is no proper prime m-ideal of G_+,

(3) there is no proper prime d-convex subgroup of G,

(4) there is no proper d-convex subgroup of G,

(5) G_+ is completely integrally closed in G,

(6) if $g \in G$, $g \neq 1$, then $\bigcap_{n \in Z} (g^n \oplus g^n) = \{0\}$.

Proposition 4.3. *Let G be a directed d-group. Then* $(2) \Leftrightarrow (3) \Leftrightarrow (4)$, $(5) \Rightarrow (1)$. *Further, if G is a local d-group then* $(1) \Leftrightarrow (6)$ *and finally, if G is a simply ordered d-group, all the propositions are equivalent.*

Proof. $(2) \Rightarrow (3)$. This follows by Lemma 2.1.

$(3) \Rightarrow (4)$. Suppose that there is a d-convex subgroup H of G such that $H \neq \{1\}$, $H \neq G$. Then there exists an element $p > 1$ such that

$$H \cap [p) = \emptyset .$$

The Zorn's lemma shows the existence of a d-convex subgroup H' of G maximal in the sense that $H' \cap [p) = \emptyset$. By [4]; Lemma 8 we obtain that H' is a prime d-convex subgroup of G, a contradiction.

$(4) \Rightarrow (2)$. Again this follows by Lemma 2.1.

$(5) \Rightarrow (1)$. Suppose that $a^n < b$, $n \in Z$ for some $a, b \in G$. Then for each $n \in Z_+$ we have $b(a^{-1})^n > 1$ and similarly, for each $n \in Z_-$ we have $ba^{-n} > 1$. Since G_+ is completely integrally closed, we obtain $a \geqq 1$, $a^{-1} \geqq 1$. Thus $a = 1$.

Now we suppose that G is local.

(6) $\Rightarrow$ (1). Suppose that there exist $a, b \in G$, $a \neq 1$ such that $a^n < b$ for each $n \in Z$. Since G is local, we obtain $a^n \oplus b = \{a^n\}$ for $b \in Z$, hence $b \in \bigcap_{n \in Z} (a^n \oplus a^n)$, a contradiction.

(1) $\Rightarrow$ (6). Let $g \in G$, $g \neq 1$, and suppose that there exists $a \in G - \{0\}$ such that $a \in \bigcap_{n \in Z} (g^n \oplus g^n)$. Then $a \geqq g^n$ for each $n \in Z$. If we suppose that $a = g^n$ for some $n \in Z$, we have $g^n \in g^{n+1} \oplus g^{n+1}$, hence $1 \in g \oplus g$ and since G is local, we obtain $g = 1$, a contradiction. Thus $a > g^n$ for each $n \in Z$. Since G is archimedean, we have $g = 1$, a contradiction. Thus $\bigcap_{n \in Z} (g^n \oplus g^n) = \{0\}$.

Finally, we suppose that G is a simply ordered d-group and we shall prove (4) $\Rightarrow$ (5). In fact, let $g, a \in G$ be such that $ag^n \geqq 1$ for each $n \in Z_+$ and suppose that $g < 1$. Then $a > 1$. Let H be the d-convex subgroup of G generated by $g < 1$. Now, since $a^2 > 1$ and $a^2 \in H$, there exists an integer m such that

$$1 < a^2 \leqq g^m .$$

Since $g^m > 1$, it follows that $m < 0$. Further, $a \geqq g^n$ for any integer $n < 0$ and we obtain $a \geqq g^m \geqq a^2$, a contradiction. Thus $g \geqq 1$ and G_+ is completely integrally closed.

From the above proposition we obtain the following well-known corollary.

Corollary. *A non-trivial valuation ring R is completely integrally closed if and only if it is one-dimensional.*

Proof. Let G be a value group of R. Then G is a simply ordered d-group with respect to the addition

$$f \oplus g = \{h \in G : f \wedge g = f \wedge h = g \wedge h\} .$$

Suppose that R is completely integrally closed, then G_+ is completely integrally closed in G and by Proposition 4.3, G is an archimedean group. Thus $\dim R = 1$. The converse may be proved in a similar way.

References

[1] *R. Gilmer:* Multiplicative Ideal Theory, "Queens' Papers on Pure and Applied Mathematics", Kingston 1968.

[2] *M. Griffin:* Prüfer rings with zero divisors, J. reine angew. Math. *239/240* (1970), 55—67.

[3] *J. Močkoř:* A realization of d-groups, Czech. Math. J. *27 (102)* (1977), 296—312.

[4] *T. Nakano:* Rings and partly ordered systems, Math. Z. *99* (1967), 355—376.

[5] *T. Nakano:* Integrally closed integral domain, Com. Math. Univ. St. Pauli (t. *XVIII*), Tokyo 1970.

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