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CHARACTERIZATIONS OF GRAPHS HAVING ORIENTATIONS SATISFYING LOCAL DEGREE RESTRICTIONS

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1. INTRODUCTION

Throughout the short history of graph theory several authors have discussed problems concerning orientations of undirected graphs. These have included, for example, enumeration of the number of orientations of given undirected graphs as dealt with in the monograph of F. HARARY and E. M. PALMER [14] (some of these results also appear in [13]). However the type of problems most frequently considered are all special instances of the following.

Problem. *Given a property* P *that oriented (antisymmetric) graphs may possess, characterize those undirected (symmetric) graphs having an orientation with property* P.

Our purpose in this article is twofold; we first wish to provide a convenient reference to and show the interdependence between the many results and open questions concerning graph orientation problems and secondly, we wish to add to this body of information through consideration of a specific orientation problem. The problem is that of characterizing those graphs having an orientation D in which the outdegree and indegree of each point p_j of D satisfy $r_j \leqq \mathrm{od}(p_j) \leqq s_j$ and $u_j \leqq \mathrm{id}(p_j) \leqq v_j$ respectively for a specified collection $\{(r_j, s_j, u_j, v_j)\}$ of 4-tuples of non-negative integers.

2. TERMINOLOGY

In this section we make precise some of the terminology we will use throughout the remainder of the paper. Any term used later without definition will have the meaning given in [11] or [12].

A *walk* of an undirected (directed) graph is a sequence of points $p_1, \ldots, p_n$ in which each $p_i p_{i+1}$, $i = 1, \ldots, n - 1$, is a line (arc) of the graph. If $p_1 = p_n$ the walk is *closed*. A *path* is a walk in which no point appears twice. A *cycle* C_n is a closed walk in which $p_1 = p_n$, $n \geqq 3$, and no other point appears twice. The *girth* of a graph is the number of points in a smallest cycle (if such exists). K_n, the *complete* graph on n points, has any two distinct points adjacent. A *semicylce* of a diagraph is a sequence of points $p_1, \ldots, p_n$, $n \geqq 3$ in which $p_1 = p_n$, no point distinct from p_1 appears twice, and either $p_i p_{i+1}$ or $p_{i+1} p_i$ is an arc of the digraph for $i = 1, \ldots, n-1$. The *converse* of an oriented graph is obtained by replacing each arc pq by the arc qp. A *strong* orientation is an orientation in which there is a path from any point p to any distinct point q. A digraph has an *n-arc strong* orientation if after the deletion of any $n-1$ arcs there is still a path from any point p to any distinct point q. We will use $\mathrm{od}(p)$ and $\mathrm{id}(p)$ to represent the outdegree and indegree respectively of a point p of a digraph. A digraph is *transitive* if it contains the arc ac whenever it contains the arcs ab and bc. At the other extreme, a digraph is a *basis* digraph if the deletion of any arc pq results in a graph having no path from point p to point q. The relation between basis digraphs and partial orders is discussed in chapters 9 and 10 of [23].

3. A BRIEF SURVEY OF ORIENTATION PROBLEMS

We believe the following list covers most publications dealing with the general problem with which we are concerned; however, we hold no hope that the list is complete and will welcome any missing items being brought to our attention.

Perhaps one of the most attractive results is that of T. GALLAI [6] who settled a conjecture of P. ERDÖS by showing that a graph G has an orientation containing no path of length k if and only if G is k-colorable.

FRANK HARARY, EDGAR PALMER and CEDRIC SMITH [15] proved that a graph has an orientation that is not self-converse if and only if G is not one of K_1, K_2, C_3, C_4 or C_5.

In an article by P. ERDÖS, L. GERENCSÉR and A. MÁTÉ [2] it is shown that for each partition of the point set of a graph G into two sets A and B there is an orientation D of G such that for any two points a in A and b in B there is a point c of A for which the distance in D from c to b is at most the distance in G between a and b.

In [3, p. 99] P. ERDÖS mentions the following open problem due to OYSTEIN ORE. Characterize those graphs having an orientation containing no cycle and further containing no semicycle which becomes a cycle if one of its arcs is reversed.

According to a theorem of P. W. KASTELEYN [16] a planar representation G of a planar graph may be oriented so that for every cycle C of G, the number of arcs that are oriented in the clockwise sense is of opposite parity to the number of points enclosed by C. CHARLES H. C. LITTLE [18] has extended Kasteleyn's theorem to non-planar graphs.

H. E. ROBBINS [24] has shown that a graph G has a strong orientation if and only if G is connected and bridgeless. OYSTEIN ORE [23, p. 148] asked for a characterization of those graphs having an orietation in which any two points (arcs) lie in a cycle. C. ST. J. A. NASH-WILLIAMS [21] generalized Robbins' result by showing that a graph G has an n-arc strong orientation if and only if G is $2n$-line connected. He then generalized this result to the following [22]. Every graph has an orientation D such that for each pair of distinct points p and q the minimum number of arcs in D whose deletion eliminates all paths from p to q in D is at least the greatest integer in half the minimum number of lines whose deletion from G eliminates all paths from p to q in G.

OYSTEIN ORE [23, p. 155] has asked for a characterization of graphs having an orientation which is a basis digraph. C. E. HAFF, U. S. R. MURTY, and R. C. WILTON [9] showed that any graph with girth larger than its chromatic number can be so oriented. More importantly they also gave an example of a graph with girth 4 having no such orientation. They subsequently found but did not publish a simpler example of such a graph. They further remarked that they knew of no solution for the analogous problem of characterizing graphs having an orientation corresponding to a lattice. LAURENCE R. ALVAREZ [1] has, however, characterized graphs having an orientation which is the basis of a modular lattice of finite length and then extended this characterization to cover distributive lattices of finite length.

OYSTEIN ORE [23, p. 163] has posed the further problem of determining, in the class of graphs without triangles, those graphs, with the smallest number of lines, that do not have an orientation which is the basis graph of a partial order. K. M. MOSESJAN [19] has shown this to be a Myclelski graph [20].

A. GHOUILA-HOURI [7] and later P. C. GILMORE and A. J. HOFFMAN [8] showed that a graph has a transitive orientation if and only if every closed walk $p_1, p_2, \ldots$ $\ldots, p_m = p_1$ of G in which m is odd and $p_i = p_j$ implies $p_{i+1} \neq p_{j+1}$ for $i \neq j$, contains a point p_k for which the line $p_k p_{k+2}$ is in G. T. GALLAI [5] then studied this characterization in depth. E. S. WOLK [25] has characterized graphs having a transitive orientation D satisfying the further property that whenever uw and vw are distinct arcs of D then D also contains uv or vu.

S. L. HAKIMI [10] has shown that a graph G whose points are labeled $p_1, p_2, \ldots, p_n$ and with whose points are associated non-negative integers $d_1, d_2, \ldots, d_n$ respectively has an orientation in which $\mathrm{od}(p_i) = d_i$, $i = 1, 2, \ldots, n$ if and only if for each subgraph H of G, $\sum_H d_i \geqq N(H)$ with equality holding when $H = G$. Here $\sum_H d_i$ is the sum of all those d_i for which p_i is in H and $N(H)$ is the number of lines of H. He further showed the orientation is unique if and only if G is acyclic. A. LEMPEL [17] has shown that every graph G has an orientation in which every point has outdegree at most $[\frac{1}{2}(d + 1)]$ where d is the maximum average degree of induced subgraphs of G.

A further motivation for our investigation is the following result of BOHDAN ZELINKA [26] independently discovered later by G. SIMMONS [4]. They showed that if a (connected) graph had at least as many lines as it had points then it was possible to assign each point to an incident line so that no two points were assigned to the same line. In our terminology we could then say that a graph G had an orientation in which the outdegree of each point is at least one if and only if G had at least as many lines as points. We will see that this is a special case of one of the corollaries to our main theorem.

4. ORIENTATIONS WITH LOCAL DEGREE BOUNDS

The results discussed in the last paragraphs suggested consideration of the following problem: "Given the collection of 4-tuples of non-negative integers $\{(r_j, s_j, u_j, v_j)\}$, $(r_j \leqq s_j$ and $u_j \leqq v_j)$, characterize those graphs having an orientation in which $r_j \leqq \mathrm{od}(p_j) \leqq s_j$ and $u_j \leqq \mathrm{id}(p_j) \leqq v_j$ for each point p_j in G." We obtain this characterization in the Main Theorem.

We will have a need for more terminology than that introduced in section 2, it and new notation will be introduced when needed.

Given a graph G the set of all nonempty subsets of the point set of G will be denoted by $\mathscr{G}$ and G_i will designate an arbitrary member of $\mathscr{G}$. As usual $|S|$ will denote the cardinality of the set S. The number of lines in the subgraph $\langle G_i \rangle$ induced on G_i by G is denoted by $K(G_i)$ while $L(G_i)$ is the total number of lines of G incident to at least one point of G_i. The collection of lines of G incident to exactly one point of G_i is called the *boundary* of G_i and is denoted by $B(G_i)$. Obviously we have the following.

Property i.

$$L(G_i) = K(G_i) + |B(G_i)|$$

and

$$\sum_{p_e \in G_i} \deg(p_e) = 2\,K(G_i) + |B(G_i)| = K(G_i) + L(G_i)\,.$$

This and further properties will be used in the sequel without specific reference. Also all graphs will be assumed to be connected.

Suppose now that G has been oriented. The sum of $\mathrm{od}(p)$, $(\mathrm{id}(p))$, over all points p in G_i will be designated by $\mathrm{od}(G_i)$, $(\mathrm{id}(G_i))$ respectively. On the other hand $\mathrm{od}(\langle G_i \rangle)$ and $\mathrm{id}(\langle G_i \rangle)$ will denote the sum of the outdegrees and indegrees in $\langle G_i \rangle$ of all points of G_i. The number of lines in the boundary of G_i that are lines from some point of G_i will be denoted by $\mathrm{od}(B(G_i))$. The notation $\mathrm{id}(B(G_i))$ is defined analogously in terms of lines to some point of G_i. The following equalities are obvious consequences of these definitions.

Property ii.

$$\mathrm{od}(G_i) = \mathrm{od}(\langle G_i \rangle) + \mathrm{od}(B(G_i))$$

and

$$\mathrm{id}(G_i) = \mathrm{id}(\langle G_i \rangle) + \mathrm{id}(B(G_i))\,.$$

Property iii.

$$|B(G_i)| = \mathrm{od}(B(G_i)) + \mathrm{id}(B(G_i))\,.$$

Property iv.

$$\mathrm{od}(\langle G_i \rangle) = \mathrm{id}(\langle G_i \rangle) = K(G_i)\,.$$

Property v.

$$L(G_i) \geqq \mathrm{od}(G_i) \geqq K(G_i)$$

and

$$L(G_i) \geqq \mathrm{id}(G_i) \geqq K(G_i)\,.$$

For purposes which become clear in some of the proofs it is convenient to introduce further notation.

$$G_0(G_i) = \{p \text{ in } G\text{: there is a path from a point in } G_i \text{ to } p\}$$

$$G_I(G_i) = \{p \text{ in } G\text{: there is a path from } p \text{ to a point in } G_i\}\,.$$

Since a point p is a path we trivially have the following.

Property vi.

$$G_i \subseteqq G_0(G_i) \quad \text{and} \quad G_i \subseteqq G_I(G_i)\,.$$

If G_i is the singleton $\{p\}$ we may use the notation $G_0(p)$ and $G_I(p)$.

We observe that $G_0(G_i)$ and $G_I(G_i)$ depend not only on G_i but also on the particular orientation of G. Since there will be no equivocation concerning the orientation in question when the sets are used, we do not encumber the notation with an orientation index.

We conclude our list of properties with the following, none of which is difficult to prove.

Property vii.

$$\mathrm{od}(G_0(G_i)) = K(G_0(G_i))\,. \quad \mathrm{id}(G_0(G_i)) = L(G_0(G_i))\,,$$

$$\mathrm{od}(G_I(G_i)) = L(G_I(G_i))\,. \quad \mathrm{id}(G_I(G_i)) = K(G_I(G_i))\,.$$

The following additional notation is needed for the statements and proofs of the ensuing results.

$$G_i^+ = G_i^+\{(x_j, y_j)\} = \{p_j \text{ in } G_i : \deg(p_j) \geqq x_j + y_j\}\,,$$

$$G_i^- = G_i^-\{(x_j, y_j)\} = \{p_j \text{ in } G_i : \deg(p_j) \leqq x_j + y_j\}$$

where $G_i^+ \cap G_i^- = \emptyset$. Note that those points where equality is satisfied must be chosen to fall in just one of the two sets, just which set will depend on what is needed in a particular instance. The *seam* on G_i with respect to $\{(x_j, y_j)\}$, denoted by $S_i = S_i\{(x_j, y_j)\}$ is the collection of all lines have one endpoint in $G_i^+\{(x_j, y_j)\}$ and the other in $G_i^-\{(x_j, y_j)\}$.

Main Theorem. *Let $\{(r_j, s_j, u_j, v_j)\}$ be a given collection of 4-tuples of non-negative integers ($r_j \leqq s_j$ and $u_j \leqq v_j$). There exists an orientation of a given graph G such that $r_j \leqq \mathrm{od}(p_j) \leqq s_j$ and $u_j \leqq \mathrm{id}(p_j) \leqq v_j$ for each point p_j in G if and only if*

(i) $$r_j + u_j \leqq \deg(p_j) \leqq s_j + v_j \quad \textit{for each } p_j \textit{ in } G\,,$$

(ii) $$\sum_{p_e \in G_i^+\{(r_j,v_j)\}} v_e - K(G_i^+\{(r_j, v_j)\}) + L(G_i^-\{(r_j, v_j)\}) - \sum_{p_e \in G_i^-\{(r_j,v_j)\}} r_e \geqq |S_i\{(r_j, v_j)\}|$$

for each G_i in $\mathscr{G}$, and

(iii) $$\sum_{p_e \in G_i^+\{(s_j,u_j)\}} s_e - K(G_i^+\{(s_j, u_j)\}) + L(G_i^-\{(s_j, u_j)\}) - \sum_{p_e \in G_j^-\{(s_j,u_j)\}} u_e \geqq$$
$$\geqq |S_i\{(s_j, u_j)\}|$$

for each G_i in $\mathscr{G}$.

Proof. The degree inequality is easily seen to be necessary since $\deg(p_j) = \mathrm{od}(p_j) + \mathrm{id}(p_j)$. Also, by the directional duality principle [11], showing either of the remaining inequalities suffices. Suppose G has such an orientation; let us show that for any G_i in $\mathscr{G}$ the last inequality is satisfied. Now

$$K(G_i^+\{(s_j, u_j)\}) \leqq \mathrm{od}(G_i^+\{(s_j, u_j)\}) - \alpha \leqq \sum_{p_e \in G_j} s_e - \alpha$$

and

$$L(G_i^-\{(s_j, u_j)\}) \geqq \mathrm{id}(G_i^-\{(s_j, u_j)\}) + \beta \geqq \sum_{p_e \in G_i^+} u_e + \beta$$

where α is the number of lines of $S_i\{(s_j, u_j)\}$ oriented outward from points in G_i^+ and β is the same for G_i^-. Consequently we find that

$$\sum_{p_e \in G_i^+} s_e - K(G_i^+\{(s_j, u_j)\}) + L(G_i^-\{(s_j, u_j)\}) - \sum_{p_e \in G_i^-} u_e \geqq \alpha + \beta = |S_i\{(s_j, u_j)\}|\,.$$

Thus the given conditions are necessary.

To show sufficiency we choose an orientation h so that

$$F(h) = \sum_{p_e \in G} \max\{0, r_e - \mathrm{od}(p_e), \mathrm{od}(p_e) - s_e, u_e - \mathrm{id}(p_e), \mathrm{id}(p_e) - v_e\}$$

is minimal. We show that $F(h) = 0$, that is, the graph admits the desired orientation. If $F(h) > 0$ then for some point p_j one of the entries in $F(h)$ will be positive.

Suppose $r_j > \mathrm{od}(p_j)$ for some point p_j in G and consider $G_I = G_I(p_j)$. If p_e is in $G_I^+ = G_I^+(p_j)\{(r_j, v_j)\}$ then $\deg(p_e) \geqq r_e + v_e$, so if $\mathrm{id}(p_e) < v_e$ then $\mathrm{od}(p_e) > r_e$ and replacing a $p_e - p_j$ path with its converse reduces $F(h)$, a contradiction. Thus $\mathrm{id}(p_e) \geqq v_e$ and we find that

$$K(G_I^+) = \mathrm{id}(G_I^+) - \alpha \geqq \sum_{p_e \in G_i^+} v_e - \alpha\,,$$

where α is the number of seamlines oriented as inlines to points in G_i^+. If the point p_j is in G_i^+ then $\mathrm{id}(p_j) > v_j$ and the above inequality is strict.

Similarly, if p_e is in $G_I^-(p_j)\{(r_j, v_j)\}$ then $\deg(p_e) \leqq r_e + v_e$, so that $\mathrm{od}(p_e) > r_e$ implies $\mathrm{id}(p_e) < v_e$ and replacing a $p_e - p_j$ path with its converse reduces $F(h)$, again a contradiction. Thus $\mathrm{od}(p_e) \leqq r_e$ and we find that $L(G_I^-) = \mathrm{od}\,(G_I^-) + \beta \leqq \leqq \sum_{p_e \in G_I^-} r_e + \beta$ where β is the number of seamlines oriented as inlines to points in G_I^-. If p_j is in G_I^- then, since $\mathrm{od}(p_j) < r_j$, the above inequality is strict. Now p_j is in either G_I^+ or G_I^-, thus at least one of the inequalities is strict so we find

$$|S_I(p_j)| = \alpha + \beta > \sum_{p_e \in G_I^+} v_e - K(G_I^+) + L(G_k^-) - \sum_{p_e \in G_I^-} r_e$$

contrary to the hypothesis. Thus $\mathrm{od}(p_j) \geqq r_j$ for each point p_j in G. By the dual argument we also find that $\mathrm{id}(p_j) \geqq u_j$ for each p_j in G.

Thus if $F(h) > 0$, for some point p_j in G either $\mathrm{od}(p_j) > s_j$ or $\mathrm{id}(p_j) > v_j$. Suppose $\mathrm{od}(p_j) > s_j$ and consider $G_0 = G_0(p_j)$.

If p_e is in $G_0^- = G_0^-(p_j)\{(s_j, u_j)\}$ then $\deg(p_e) \leqq s_e + u_e$ and $\mathrm{id}(p_e) \geqq u_e$. If $\mathrm{id}(p_e) > u_e$ then $\mathrm{od}(p_e) < s_e$ and replacing a $p_j - p_e$ path by its converse reduces $F(h)$, a contradiction. So $\mathrm{id}(p_e) = u_e$ and $L(G_0^-) = \mathrm{id}(G_0^-) + \beta = \sum_{p_e \in G_0^-} u_e + \beta$ where β is the number of seamlines oriented as outlines from points in G_0^-.

Now p_j is in G_0^+ for if it were in G_0^- this would require that $\mathrm{id}(p_j) < u_j$ contrary to the dual of the prior case. If p_e is in $G_0^+ = G_0^+(p_j)\{(s_j, u_j)\}$, then $\deg(p_e) \geqq \geqq s_e + u_e$ and $\mathrm{od}(p_e) \geqq s_e$. Since $\mathrm{od}(p_e) < s_e$ implies $\mathrm{id}(p_e) > u_e$ and replacing a $p_j - p_e$ path with its converse reduces $F(h)$, again a contradiction. We now have

$$K(G_0^+) = \mathrm{od}(G_0^+) - \alpha > \sum_{p_e \in G_0^+} s_e - \alpha$$

where α is the number of seamlines oriented as outlines to points in G_0^+. Note that the inequality is strict because p_j is in G_0^+. Consequently we find that

$$|S_0(p_j)| = \alpha + \beta > \sum_{p_e \in G_0^+} s_e - K(G_0^+) + L(G_0^-) - \sum_{p_e \in G_0^-} u_e$$

contrary to hypothesis. Thus for each p_j in G, $\mathrm{od}(p_j) \leqq s_j$ and by the dual argument $\mathrm{id}(p_j) \leqq v_j$. That is, $F(h) = 0$ and the proof of the theorem is complete.

Since conditions (ii) and (iii) of the Main Theorem simplify considerably when one or more of the four indegree-outdegree conditions is eliminated we list some of these cases to facilitate use of the theorem. It should be noted that the directional duals of these results are, of course, also valid. For the sake of brevity we list only the essential changes in the hypothesis and (ii) and (iii) of the conclusion.

Corollary 1.

i) $\mathrm{od}(p_j) \geqq r_j$ *iff* $L(G_i) \geqq \sum_{p_e \in G_i} r_e$.

ii) $\mathrm{od}(p_j) \leqq s_j$ *iff* $K(G_i) \leqq \sum_{p_e \in G_i} s_e$.

iii) $r_j \leqq \mathrm{od}(p_j) \leqq s_j$ *iff* $L(G_i) \geqq \sum_{p_e \in G_i} r_e$ *and* $K(G_i) \leqq \sum_{p_e \in G_i} s_e$.

iv) $\mathrm{od}(p_j) \geqq r_j$ *and* $\mathrm{id}(p_j) \geqq u_j$ *iff* $L(G_i) \geqq \max\{\sum_{p_e \in G_i} r_e, \sum_{p_e \in G_i} u_e\}$.

v) $\mathrm{od}(p_j) \leqq s_j$ *and* $\mathrm{id}(p_j) \leqq v_j$ *iff* $K(G_i) \leqq \min\{\sum_{p_e \in G_i} s_e, \sum_{p_e \in G_i} v_e\}$.

vi) $\mathrm{od}(p_j) \leqq s_j$ *and* $\mathrm{id}(p_j) \geqq u_j$ *iff* $\sum_{p_e \in G_i^+} s_e - K(G_i^+) + L(G_i^-) - \sum_{p_e \in G_i^-} u_e \geqq |S_i|$.

vii) $r_j \leqq \mathrm{od}(p_j) \leqq s_j$ *and* $\mathrm{id}(p_j) \geqq u_j$ *iff* $\sum_{p_e \in G_i^+} s_e - K(G_i^+) + L(G_i^-) -$
$- \sum_{p_e \in G_i^-} u_e \geqq |S_i|$, $L(G_i) \geqq \sum_{p_e \in G_i} r_e$ *and* $\deg(p_j) \geqq r_j + u_j$.

viii) $r_j \leqq \mathrm{od}(p_j) \leqq s_j$ *and* $\mathrm{id}(p_j) \leqq v_j$ *iff* $\sum_{p_e \in G_i^+} v_e - K(G_i^+) + L(G_i^-) -$
$- \sum_{p_e \in G_i^-} r_e \geqq |S_i|$, $K(G_i) \geqq \sum_{p_e \in G_i} s_e$ *and* $\deg(p_j) \leqq s_j + v_j$.

Proof. Rather than give a detailed proof of each part we simply point out that they may be obtained from the Main Theorem by assigning r_j and u_j the value 0 when appropriate and assigning s_j and v_j sufficiently large values when appropriate so that, after propert choice of the G_i^+, and G_i^-, the appropriate G_i^+ and G_i^- are empty.

Another special case of interest is that in which $r_j = s_j$ for each j; here we have the following result.

Corollary 2. *Let G be a given graph and* $\{(r_j, u_j, v_j)\}$ *be a given collection of triples of non-negative integers. The following are equivalent:*

(i) *There is an orientation of G where* $\mathrm{od}(p_j) = r_j$ *and* $u_j \leqq \mathrm{id}(p_j) \leqq v_j$ *for each point* p_j *in G.*

(ii) *G has* $\sum_{p_e \in G} r_e$ *lines,* $L(G_i) \geqq \sum_{p_e \in G_i} r_e$ *for all* G_i *in* $\mathscr{G}$ *and* $r_j + u_j \leqq \deg(p_j) \leqq r_j + v_j$ *for each* p_j *in G.*

(iii) *G has* $\sum_{p_e \in G} r_e$ *lines,* $K(G_i) \leqq \sum_{p_e \in G_i} r_e$ *for all* G_i *in* $\mathscr{G}$ *and* $r_j + u_j \leqq \deg(p_j) \leqq$ $\leqq r_j + v_j$ *for each* p_j *in G.*

(iv) $K(G_i) \leqq \sum_{p_e \in G_i} r_e \leqq L(G_i)$ *for all* G_i *in* $\mathscr{G}$ *and* $r_j + u_j \leqq \deg(p_j) \leqq r_j + v_j$ *for each* p_j *in G.*

Proof. Set $G_i = G$ in (iv). Then since $K(G) = L(G)$ and $K(G)$ is the number of lines in G, we see that (iv) implies both (ii) and (iii).

We next show that (i) implies (iv). We first note that the restrictions in (iv) on the number of lines of G and the degrees of the points of G follows immediately from (i). Now, using parts i) and ii) of Corollary 1 we obtain the inequalities $K(G_i) \leqq \sum_{p_e \in G_i} r_e \leqq$ $\leqq L(G_i)$.

Finally we show that (ii) (and therefore (iii) by a similar argument), implies (i). Now, (ii) and part i) of Corollary 1 imply there is an orientation of G such that $\mathrm{od}(p_j) \geqq r_j$ for each p_j in G. If, in such an orientation, $\mathrm{od}(p_j) > r_j$ for some point p_j, then G has more than $\sum_{p_e \in G} r_e$ lines, contrary to (ii). Thus, $\mathrm{od}(p_j) = r_j$ for each point p_j. Also, from $r_j + u_j \leqq \deg(p_j) = \mathrm{od}(p_j) + \mathrm{id}(p_j) = r_j + \mathrm{id}(p_j) \leqq r_j + v_j$ it follows that $u_j \leqq \mathrm{id}(p_j) \leqq v_j$ and the proof of the Corollary is complete.

Corollary 2 can be used, for example, to show that any regular graph of even degree has a regular orientation.

Corollary 3. *There is an orientation of a graph G such that* $\mathrm{id}(p_j) = \mathrm{od}(p_j) = r$ *for each point p_j in G if and only if* $\deg(p_j) = 2r$ *for each p_j in G.*

Proof. The condition is clearly necessary. The identities

$$\tfrac{1}{2}[K(G_i) + L(G_i)] = \tfrac{1}{2} \sum_{p_e \in G_i} \deg(p_e) = \sum_{p_e \in G_i} r$$

yield the result

$$K(G_i) \leqq \sum_{p_e \in G_i} r \leqq L(G_i)$$

for all G_i in $\mathscr{G}$ and so, by Corollary 2, there is an orientation of G such that $\mathrm{od}(p_i) = r$ (and consequently $\mathrm{id}(p_j) = r$) for each p_j in G.

The assertion "there is an orientation on G such that $\mathrm{od}(p_j) = r_j$ and $\mathrm{id}(p_j) = u_j$ if and only if $\deg(p_j) = r_j + u_j$" suggested by Corollary 3 is not in general true. For example let $G = K_4$, $r_1 = r_2 = u_3 = u_4 = 3$ and $r_3 = r_4 = u_1 = u_2 = 0$. The degree conditions are met but K_4 has no such orientation because the "G_i conditions" of Corollary 3 are not satisfied.

Our discussion up to now has dealt only with the situation in which the indegrees and outdegrees of an orientation are restricted to lie in certain intervals. We now discuss the types of theorems that arise when each point may satisfy any one condition from a given collection of given conditions. The foregoing analysis provides all the required tools to express and prove necessary and sufficient conditions for such theorems. We illustrate with the following two examples.

Theorem. *Suppose $\{(r_j, u_j)\}$ is a given collection of non-negative integer pairs and G is a given graph. There is an orientation of G where either $r_j \leqq \mathrm{od}(p_j)$ or $u_j \leqq \mathrm{id}(p_j)$ for each point p_j in G if and only if there is a partitioning $\{G^r, G^u\}$ of the points of G such that $L(G_i) \geqq \sum_{p_e \in G_i} r_e$ for all $G_i \subset G^r$ and $L(G_i) \geqq \sum_{p_e \in G_i} u_e$ for all $G_i \subset G^u$.*

Note. G^r or G^u may be empty.

Proof. The conditions are easily seen to be necessary. To show sufficiency choose an orientation h so that

$$F(h) = \sum_{p_k \in G^r} \max\{0, r_k - \mathrm{od}(p_k)\} + \sum_{p_k \in G^u} \max\{0, u_k - \mathrm{id}(p_k)\}$$

is minimal. From this point on the proof continues in the same manner as the proof of the Main Theorem.

For the last example set $G_i^x = G^x \cap G_i$ and let S_i^{xy} be the set of lines with one end point in G_i^x and the other in G_i^y.

Theorem. *Let G be a given graph and $\{(r_j, v_j)\}$ be a given collection of non-negative integer pairs. There is an orientation of G such that either $r_j \leqq \mathrm{od}(p_j)$ or $\mathrm{id}(p_j) \leqq v_j$ for each point p_j in G if and only if there is a partitioning $\{G^r, G^v\}$ of the points of G for which*

$$\sum_{p_k \in G_i{}^v} v_k - K(G_i^v) + L(G_i^r) - \sum_{p_k \in G_i{}^r} r_k \geqq |S_i^{rv}| \quad \textit{for all } G_i \textit{ in } \mathscr{G}.$$

Proof. To show necessity, suppose we are given such an orientation and any G_i in $\mathscr{G}$; we see that $K(G_i^v) = \mathrm{id}\, G_i^v - \alpha \leqq \sum_{p_k \in G_i{}^v} v_k - \alpha$ and $L(G_i^r) = \mathrm{od}\,(G_i^r) + \beta \geqq \geqq \sum_{p_k \in G_i{}^r} r_k + \beta$ where α is the number of lines in S_i^{rv} oriented to a point in G_i^v and β the number oriented into G_i^r. Consequently

$$\sum_{p_k \in G_i{}^v} v_k - K(G_i^v) + L(G_i^r) - \sum_{p_k \in G_i{}^r} r_k \geqq \alpha + \beta = |S_i^{rv}|.$$

To show sufficiency choose an orientation h so that

$$F(h) = \sum_{p_e \in G^r} \max\{0, r_e - \mathrm{od}(p_e)\} + \sum_{p_e \in G^v} \max\{0, \mathrm{id}(p_e) - v_e\}$$

is minimal and proceed as in the proof of the Main Theorem.

5. CONCLUSION

The proofs of the results of the last section were constructive in that the desired orientation was obtainable by first giving a graph G any orientation and then, in a systematic way, replacing paths by their converses until the required orientation was obtained. This suggests the following simple algorithm for obtaining an orientation satisfying given degree conditions if such exists.

Given an undirected graph G and a set S of indegree and outdegree restrictions such as specified in any of the preceding theorems or corollaries, an orientation D of G

satisfying the restrictions of S may be found, if such exists, using the following algorithm.

Algorithm. i) Assign an arbitrary orientation h to G.

ii) Evaluate the appropriate function $F(h)$ as defined in Section 4.

iii) If $F(h) = 0$ the required orientation D has been obtained.

iv) If $F(h) > 0$ choose a $p_i -, p_j$ path such that when this path is replaced by its converse to form a new orientation h', $F(h') < F(h)$. (If no such path exists the desired orientation does not exist.)

v) Repeat ii).

Since F is never negative, the algorithm must terminate.

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