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THE HULLS OF SEMIPRIME RINGS

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1. INTRODUCTION

Let G be a semiprime ring and for $a, b \in G$ define $a \geqq b$ if $agb = bgb$ for all $g \in G$. This is equivalent to the fact that a agrees with b on the support of b in each representation of G as a subdirect product of prime rings. Thus $\geqq$ is a partial order for G with smallest element 0 and for $a, b, x \in G$

$$a \geqq b \quad \text{implies} \quad ax \geqq bx\,, \quad xa \geqq xb \quad \text{and} \quad ab = ba\,.$$

We say that a is *disjoint* from b or that a is *orthogonal* to b if $aGb = 0$ (notation $\perp$). This is equivalent to the fact that a and b have disjoint support in each representation of G as a subdirect product of prime rings. Thus $a \perp b$ iff $b \perp a$ and in this case $0 = ab = ba$. Also note that $a \geqq b$ iff $a - b \perp b$, and $a + b \geqq b$ iff $a \perp b$. If X is a subset of G then

$$X' = \{g \in G \mid g \perp x \text{ for each } x \in X\}$$

is the *annihilator ideal* of X. LAMBECK [11] has shown that these ideals form a complete Boolean algebra which we shall denote by $P(G)$. G will be called

a P-*ring* if $G = g'' \oplus g'$ for each $g \in G$ (projectable)
an SP-*ring* if $G = X'' \oplus X'$ for each subset X of G (strongly projectable)
an L-*ring* if each pairwise disjoint subset of G has a l.u.b. (laterally complete)
an 0-*ring* if G is both an L-ring and an SP-ring (orthocomplete).

An overring H is a *left essential extension* of G if this is the case when H is considered as a left G-module. We prove the following theorems for X = P, SP, L or 0.

Theorem A. *Let G be a semiprime ring and let H be a left essential extension of G that is an X-ring. Then the intersection K of all the subrings of H that contain G*

[1]) These results were announced in: *The hulls of semiprime rings,* Bull. Australian Math. Soc. *12* (1975) 311–314.

and are X-rings is a minimal left essential extension of G that is an X-ring; called an X-hull of G.

Theorem B. *Each semiprime ring admits a unique X-hull G^{X}. G^{X} is semiprime and G^{X} is reduced (commutative) iff G is reduced (commutative). If G has an identity 1, then 1 is also the identity for G^{X}. Finally G^{X} is the minimal right essential extension of G that is an X-ring.*

If G is reduced then the proofs of these theorems are almost identical with the proofs of the corresponding theorems for lattice-ordered groups in [5]; one simply replace $a \wedge b$ by ab. For semiprime rings the proofs in [5] can be adapted. We show that

$$G \subseteq G^{P} \subseteq G^{SP} \subseteq (G^{SP})^{L} = (G^{P})^{L} = G^{0}$$

and $(G^{L})^{P} = (G^{L})^{SP} \subseteq G^{0}$, but here we need not have equality.

In order to prove Theorems A and B we show that if H is a left essential extension of the semiprime ring G then H is semiprime and there is a natural isomorphism of $P(H)$ onto $P(G)$. If H is laterally complete then G is an $\mathscr{L}$-subring of H (i.e., for each disjoint subset $\{g_\lambda \mid \lambda \in \Lambda\}$ of G for which $\bigvee_G g_\lambda$ exists, it follows that $\bigvee_H g_\lambda = \bigvee_G g_\lambda$).

If G is a Boolean ring then so is G^{X} and $G^{L} = G^{0}$. Also G^{0} is the Dedekind-MacNeille completion of G iff G has an identity. If G is regular then so is G^{P}, G^{SP} and G^{0}. We show that the ring G^{X} is determined by the addition and the partial order.

Theorem. *Suppose that G is a semiprime ring and consider the system $(G^{X}, +, \geqq)$ for X = P, SP or 0. Then there is a unique multiplication on G^{X} so that*

a) *G^{X} is a semiprime ring,*
b) *G is a subring of G^{X}, and*
c) *the multiplication on G^{X} induces the given partial order $\geqq$.*

Almost all of the theory for the X-hulls of latticeordered groups in [5] has a counterpart for semiprime rings. In particular, this is true for the annihilator preserving endomorphisms of G and for the theory of semiprime rings with a basis.

$P(G)$ is atomic iff G^{0} is a product of prime rings. From this it is easy to derive necessary and sufficient conditions for a reduced ring to be a product of integral domains; in particular, those in the literature for commutative rings (see for example [7] Theorem 4.3).

Abian [1] proved that a commutative semiprime ring G is a product of fields iff G is hyperatomic and laterally complete. A student of mine Otis Kenny has shown that a reduced ring H is a product of division rings iff H is hyperatomic and laterally complete. Thus H^{L} is a product of division rings if H is hyperatomic.

If G is a commutative semiprime ring with 1, then G^{P} is the Baer extension of G that was introduced by Kist [9] and G^{SP} is the Baer extension of G that was introduced by MEWBORN[12]. Thus for an arbitrary semiprime ring G with 1 we have the unique Baer hulls G^{P} and G^{SP}.

In [14] Speed using the technique developed in [4] (which is somewhat cruder than that used in [5]) constructed G^{P} and G^{SP} and some hulls in between for commutative semiprime rings with 1. His description of these hulls is categorical, but somewhat complicated.

If G is a semiprime ring then the complete ring of left (right) quotients of G is an 0-ring that contains G^0.

2. THE BOOLEAN ALGEBRA $P(R)$ OF ALL ANNIHILATOR IDEALS OF A SEMIPRIME RING R

We shall assume throughout this section that R is a subdirect product of prime rings T_i; $R \subseteq \Pi_I T_i$. Note that R is prime iff it contains no disjoint elements. Also an ideal A of R is a semiprime ring. For if $0 \neq a \in A$ then $ara \neq 0$ for some $r \in R$ and hence $arasara \neq 0$ for some $s \in R$. Thus $aAa \neq 0$. If R is reduced then $a \perp b$ iff $ab = 0$ and in this case we shall assume that the T_i are integral domains (see [2]).

Proposition 2.1. *If $\{A_\lambda \mid \lambda \in \Lambda\}$ is a set of subrings of R such that $a \perp b$ for each $a \in A_\alpha$ and $b \in A_\beta$ with $\alpha \neq \beta$ then the subring $[\bigcup A_\lambda]$ of R that is generated by the A_λ is the direct sum ΣA_λ of the ideals A_λ.*

Proof. Suppose that $0 = a_1 + a_2 + \ldots + a_r$, where the a_i belong to distinct A_{λ_i}. Then $0 = (a_1 + \ldots + a_r)ga_1 = a_1ga_1$ for all $g \in R$ so $a_1 = 0$ and similarly $a_2 = a_3 = \ldots = a_r = 0$. Thus $[\bigcup A_\lambda] = \Sigma A_\lambda$ as an additive group, but clearly the A_λ are ideals in $[\bigcup A_\alpha]$.

Corollary. *If $\{A_\lambda \mid \lambda \in \Lambda\}$ is a set of ideals of R such that $A_\alpha \cap A_\beta = 0$ for $\alpha \neq \beta$ then $[\bigcup A_\lambda] = \Sigma A_\lambda$.*

Proposition 2.2. *If $R = \Sigma K_i$ and K is an ideal of R such that R/K is semiprime, then $K = \Sigma(K_i \cap K)$.*

Proof. Suppose that $k = k_1 + \ldots + k_n \in K$, where the k_i belong to distinct K_{λ_i}. Then $kRk_1 = k_1Rk_1 \in K$ and hence $(K + k_1) R/K(K + k_1) = K$. Thus $K + k_1 = K$ and hence $k_1 \in K \cap K_{\lambda_1}$. Similarly $k_i \in K \cap K_{\lambda_i}$ for $i = 2, \ldots, n$ and hence $K \subseteq \Sigma(K_i \cap K)$.

Recall that for a subset A of R

$$A' = \{r \in R \mid r \perp a \quad \text{for all} \quad a \in A\}\ .$$

If A is an ideal or if R is reduced then

$$A' = \{r \in R \mid rA = 0\} = \{r \in R \mid Ar = 0\}\ .$$

1) *For subsets* A *and* B *of* R; $A \subseteq B$ *implies* $A' \supseteq B'$. $A \subseteq A''$ *and* $A' = A'''$. *In particular, each annihilator in the annihilator of an ideal.*

2) *For* $a \in R$, $a' = \langle a \rangle'$, *where* $\langle a \rangle$ *is the ideal generated by* a.

Proof. Since $a \in \langle a \rangle$, $a' \supseteq \langle a \rangle'$ and if $x \in a'$ then $xRa = aRx = 0$ so $x \in \langle a \rangle'$.

3) *If* A *is a subset of* R *then* R/A' *is semiprime and if* R *is reduced then so is* R/A'.

Proof. Here we use the representation of R as a subdirect sum of the prime rings T_i. Suppose that $A' = (A' + a)\, R/A'(A' + a)$. Then $aRa \subseteq A'$ so $arasx = 0$ for all $r, s \in R$ and $x \in A$. If $x_i \neq 0$ and $(ara)_i \neq 0$ for some $r \in R$ then $(ara)_i s_i x_i \neq 0$ for some $s \in R$ since T_i is a prime ring. Thus $x_i \neq 0$ implies $(ara)_i = 0$ for all $r \in R$ and hence since T_i is prime $a_i = 0$. Thus $a \perp x$ and hence $a \in A'$. Therefore R/A' is semiprime.

If R is reduced and $(A' + a)^2 = A'$ then $a^2 \in A'$ and hence $a^2 x = 0$ for all $x \in A$. But $a_i^2 x_i = 0$ implies that $a_i = 0$ or $x_i = 0$ since T_i is an integral domain. Thus $ax = 0$ and hence $a \in A'$. Therefore R/A' is reduced.

4) *If* A *and* B *are ideals then* $(A \cap B)'' = A'' \cap B''$. *Thus if* $A \cap B = 0$ *then* $A'' \cap B'' = 0$ *and if* $A, B \in P(G)$ *then* $A \cap B = (A \cap B)'' \in P(G)$.

Proof. Note that if $n \in A' \cap A''$ then $nRn = 0$ and so $n = 0$. $A \cap B \subseteq A$ and B so $(A \cap B)'' \subseteq A'' \cap B''$. Now consider $x \in A'' \cap B''$ and $y \in (A \cap B)'$ and show $x \perp y$. If $a \in A$ and $b \in B$ then $aRb \subseteq A \cap B$ so $yRaRb = 0$. Thus $xRyRaRb = 0$ so $xRyRa \subseteq B' \cap B'' = 0$ and hence $xRy \subseteq A' \cap A'' = 0$.

5) *If* $a, b \in R$ *then* $a'' \cap b'' = (aRb)''$ *so if* $a \perp b$ *then* $a'' \cap b'' = 0$. *Also if* R *is reduced then* $(aRb)'' = ab''$.

Proof. $aRb \subseteq a'' \cap b''$ so $(aRb)'' \subseteq a'' \cap b''$. Now suppose $x \in a'' \cap b''$ and $y \in (aRb)'$ and show $x \perp y$. Since $yRaRb = 0$, $xRyRaRb = 0$. Thus $xRyRa \subseteq b'' \cap b' = 0$ hence $xRy \subseteq a'' \cap a' = 0$.

Now we assume that R is reduced and show $(aRb)' = (ab)'$. If $x \in (ab)'$ the $xab = 0$ and hence $xagb = 0$ for all $g \in R$. Thus $(ab)' \subseteq (aRb)'$. If $x \in (aRb)'$ then $xa^2 b = 0$ and so $xab = 0$. Thus $(aRb)' \subseteq (ab)'$.

6) *Each annihilator ideal* B *is the intersection of all the minimal prime ideals that do not contain* B'.

This is well known (see [11]).

7) *If* A *is an ideal in* R *and* α *is an automorphism of* R *then* $A'\alpha = (A\alpha)'$ *and so* $A''\alpha = (A'\alpha)' = (A\alpha)''$. *Thus if* $A = A'' \in P(R)$ *then* $A\alpha = (A\alpha)''$ *and if* $A = A\alpha$ *then* $A'\alpha = A'$.

Proposition 2.3. *The set* $P(R)$ *of all annihilator ideals of a semiprime ring* R *form a complete Boolean algebra with respect to* $\subseteq$ *and with complement map*

$A \to A'$. Moreover

$$\sqcap B_\lambda = (\bigcup B'_\lambda)' = \bigcap B_\lambda , \quad \sqcup B_\lambda = (\bigcap B'_\lambda)' = (\bigcup B_\lambda)'' ,$$

$A \sqcap (\sqcup B_\lambda) = \sqcup (A \sqcap B_\lambda)$ and dually where A and the B_λ are elements from $P(R)$ and $\sqcap$ and $\sqcup$ are the join and meet operators in $P(R)$. In particular if $R = A \oplus B$ then $B = A'$ is uniquely determined by A.

This is well known (see [11]).

There is a converse to the last Proposition. Let S be an arbitrary ring and for each ideal A of S let

$$A^* = \{x \in S \mid xA = Ax = 0\}$$

and let

$$K(S) = \{A^* \mid A \text{ is an ideal of } S\} .$$

Proposition 2.4. *The following are equivalent for a ring S.*

1) *S is semiprime*

2) *$K(S)$ is a Boolean algebra with respect to $\subseteq$ and with complement map $X \to X^*$ and zero element 0.*

Proof. (Otis Kenny). If S is semiprime then for each ideal A of S, $A^* = A'$ and hence $K(S) = P(S)$. Then by the last Proposition (1) implies (2).

In (2) holds and $A^2 = 0$ for some ideal A of S then $A \subseteq A^* \cap A^{**} = 0$ so S is semiprime.

Note also that if for each $a \in S$

$$S = \langle a \rangle^* \oplus \langle a \rangle^{**}$$

then S is a semiprime ring. Thus P-rings are necessarily semiprime. For if $aSa = 0$ then $a \in \langle a \rangle \subseteq \langle a \rangle^{**}$ and hence if $a^2 = 0$ then $a \in \langle a \rangle^{**} \cap \langle a \rangle^* = 0$ and so S is semiprime. But we know that $a^3 = 0$ and $a^2Sa = aSa^2 = 0$ so $a^2 \in \langle a \rangle^{**} \cap \cap \langle a \rangle^* = 0$.

3. PROOF OF THEOREM A.

Throughout this section let G be a subring of H.

Lemma 3.1. *If G is semiprime and left large in H then H is semiprime, and if, in addition, $a, b \in G$ and $aGb = 0$ then $aHb = 0$. Thus a and b are disjoint in G iff they are disjoint in H.*

Proof. (Phil Montgomery). If $0 \neq h \in \text{rad } H$ then $0 \neq gh \in G$ for some $g \in G$ and since $gh \in \text{rad } H$ it is strongly nilpotent in H and hence in G. But G is semiprime and thus $gh = 0$, a contradiction. Therefore H is semiprime.

Now if $aGb = 0$ and $aHb \neq 0$ then $ahb \neq 0$ for some $h \in H$ and so $0 \neq xahb \in G$ for some $x \in G$. Since $bGa = 0$, $xahbGxahb = 0$, but this contradicts the fact that G is semiprime.

A similar argument shows that if $a, b \in A$ an ideal of G, then a and b are disjoint in A iff they are disjoint in G.

Corollary. *If G is semiprime and left large in H then $a \leqq b$ in G iff $a \leqq b$ in H.*

Proof. $a \leqq b$ in G if $a - b \perp b$ in G iff $a - b \perp b$ in H iff $a \leqq b$ in H.

Note that H is a subdirect product of prime rings $\{T_i \mid i \in I\}$ and so we have shown that $a \geqq b$ in G if $a_i = b_i$ for all $b_i \neq 0$.

Denote the annihilator operation in the semiprime ring $G(H)$ by $'(*)$. For $B \in P(G)$ and $C \in P(H)$ define

$$B\mu = (B')^* \quad \text{and} \quad C\gamma = C \cap G .$$

Proposition 3.2. *If G is semiprime and left large in H then μ is an isomorphism of $P(G)$ onto $P(H)$ and γ is the inverse of μ. Moreover, $B\mu = B^{**}$.*

Proof. If $a \in B'$ then $aGb = 0$ for all $b \in B$ and so by the last Lemma $aHb = 0$. Thus $B' \subseteq B^*$ and $(B')^* \supseteq B^{**} \supseteq B$. Therefore $(B')^* \cap G \supseteq B^{**} \cap G \supseteq B$. If $x \in (B')^* \cap G$ then $x \in G$ and $xB' = 0$ so $x \in B'' = B$. Thus

$$B\mu\gamma = (B')^* \cap G = B^{**} \cap G = B^{**}\gamma = B .$$

We next prove that $C \cap G = (C^* \cap G)' \in P(G)$. If $x \in C \cap G$ and $y \in C^* \cap G$ then $xy = 0$ and so $0 = x(C^* \cap G)$. Thus since $C^* \cap G$ is an ideal in G, $C \cap G \subseteq$ $\subseteq (C^* \cap G)'$. Now suppose (by way of contradiction) that $0 \neq x \in (C^* \cap G) \setminus C$. Then $xa \neq 0$ for some $a \in C^*$; otherwise $x \in C^{**} = C$. Thus $0 \neq yxa \in G$ for some $y \in G$ and hence $0 \neq yxa \in C^* \cap G$ and $yx \in (C^* \cap G)'$. Therefore $yxaGyxa = 0$, but this contradicts the fact that G is semiprime.

$$C\gamma\mu = (C \cap G)\,\mu = (C^* \cap G)'\mu = (C^* \cap G)''^* = (C^* \cap G)^* \supseteq C .$$

Here we use the fact that $C^* \cap G \in P(G)$ by the above.

Now suppose (by way of contradiction) that $0 \neq z \in (C^* \cap G)^* \setminus C$. Then $0 \neq za$ for some $a \in C^*$ and so $0 \neq yza \in G$ for some $y \in G$. Therefore $0 \neq yza \in C^* \cap G$ and $z \in (C^* \cap G)^*$. Thus $yza \in (C^* \cap G)^*$ and hence $yzaHyza = 0$, but this contradicts the fact that H is semiprime.

Corollary. *If G is semiprime and left large in H and X is a subset of G then*

(i) $(X'')^{**} = X^{**}$ *and* $X^{**} \cap G = X''$, *and*

(ii) $(X')^{**} = X^*$ *and* $X^* \cap G = X'$.

Proof. Since $X \subseteq X''$ we have $X^{**} \subseteq (X'')^{**}$. Also $X^{**} \cap G = X^{**}\gamma \supseteq X''$ since $X^{**}\gamma \in P(G)$ and contains X. Thus $X'' \subseteq X^{**}$ and hence $(X'')^{**} \subseteq X^{**}$

$$X^{**} \cap G = (X'')^{**} \cap G = X''\mu\gamma = X'' .$$

From (i) and the Proposition we have

$$X^* = (X'')^* = (X')'^* = (X')^{**} .$$

Finally from Lemma 3.1 we have

$$X^* \cap G = \{g \in G \mid gHx = 0 \text{ for all } x \in X\}$$
$$= \{g \in G \mid gGx = 0 \text{ for all } x \in X\} = X' .$$

G is an *$\mathscr{L}$-subring* of a semiprime ring H if for each disjoint subset $\{g_\lambda \mid \lambda \in \Lambda\}$ of G for which $\bigvee_G g_\lambda$ exists it follows that $\bigvee_H g_\lambda$ exists an dequals $\bigvee_G g_\lambda$.

Proposition 3.3. *If G is semiprime, left large in H and H is laterally complete, then G is an $\mathscr{L}$-subring of H. In particular, the intersection of all the laterally complete subrings of H that contain G is laterally complete.*

Proof. We may assume that H is a subdirect product of prime rings $\{T_i \mid i \in I\}$. Suppose that $\{a_\lambda \mid \lambda \in \Lambda\}$ is a disjoint subset of G and $g = \bigvee_G g_\lambda$ exists. Then by Lemma 3.1 $\{a_\lambda \mid \lambda \in \Lambda\}$ is also a disjoint subset of H. Let $h = \bigvee_H g_\lambda$. Then we must show show that $h \geqq g$. For each i such that $(a_\lambda)_i \neq 0$ for some λ we have $h_i = (a_\lambda)_i = g_i$[2]). Suppose that $g_i \neq 0$ and $(a_\lambda)_i = 0$ for all λ. To complete the proof we must show that $h_i = g_i$. If not then $h - g \neq 0$ and is disjoint from all the a_λ. Now $0 \neq t(h - g) \in G$ for some $t \in G$. Pick j so that $(t(h - g))_j \neq 0$. Then $g_j \neq 0$ or $h_j \neq 0$.

If $g_j \neq 0$ then $g + t(h - g)$ is an upper bound for the a_λ in G that does not exceed g, a contradiction. If $h_j \neq 0$ than $h + t(h - g)$ is an upper bound for the a_λ in H which does not exceed h, a contradiction.

Let K be the intersection of the set $\{H_\delta \mid \delta \in \Delta\}$ of all the laterally complete subrings of H that contain G and let $\{k_\lambda \mid \lambda \in \Lambda\}$ be a disjoint subset of K. Then for each δ $\bigvee_{H_\delta} a_\lambda = \bigvee_H a_\lambda$ since H_δ is left large in H. Thus $\bigvee_H a_\lambda$ is the least upper bound of the a_λ in K and hence K is laterally complete.

If H is not laterally complete then can we conclude that G is an $\mathscr{L}$-subring of H?

We are now ready to prove Theorem A. The last Proposition takes care of the case when X = L. Suppose that H is an SP-ring and consider $Y \subseteq K$ where K is the intersection of all the SP-subrings H_λ of H that contain G. Let the annihilator operations in H, K and H_λ be $*$, $\#$ and λ. We wish to prove $K = Y\# \oplus Y\#\#$. If $0 \neq x \in K \subseteq H_\lambda = Y^\lambda \oplus Y^{\lambda\lambda}$ then $x = x_1 + x_2$, where $x_1 \in Y^\lambda$ and $x_2 \in Y^{\lambda\lambda}$. Since H_λ is left large in H we have by the Corollary to Proposition 3.2.

$$Y^\lambda = Y^* \cap H_\lambda \quad \text{and} \quad Y^{\lambda\lambda} = Y^{**} \cap H_\lambda .$$

Thus $x = x_1 + x_2$ is the decomposition of x in $H = Y^* \oplus Y^{**}$ and this holds for *all* λ. Therefore $x_1, x_2 \in \bigcap H_\lambda = K$ and so $x_1 \in K \cap Y^* = Y\#$ and $x_2 \in K \cap Y^{**} = Y\#\#$. Thus $x \in Y\# \oplus Y\#\#$ and hence $K = Y\# \oplus Y\#\#$.

[2]) Here we use the corollary to Lemma 3.1.

A similar proof works for X = P and if K is both an SP-ring and an L-ring then it is an 0-ring.

Lemma 3.4. *If G is reduced and large in H then H is reduced.*

Proof. Suppose (by way of contradiction) that $0 \neq h \in H$ and $h^2 = 0$. There exist elements $a, b \in G$ so that $0 \neq ah$, $hb \in G$. Now $0 = ah^2b = (ah)(hb) = (ah)\, G(hb)$. Thus $0 = (ahb)(ahb)$ and hence $ahb = 0$. But $M = \{x \in G \mid xh \in G\}$ is a large left ideal of G and we have shown that $Mhb = 0$. Now G is a subdirect sum of integral domains and it follows that (the support of M) $\cap$ (the support of hb) is the null set. Therefore $M \cap Ghb = 0$ but this contradicts the fact that M is left large in G.

Another proof. Since G is reduced the singular ideals of G are zero. Thus [6] H is a quotient ring of G and so H is reduced [15].

4. PROOF OF THEOREM B.

Throughout let G be a semiprime ring. A *partition* of $P(G)$ is a maximal pairwise disjoint set of non-zero annihilator ideals of G. Let $D(G)$ be the set of all partitions of $P(G)$ and for $\mathscr{A}, \mathscr{C} \in D(G)$ define $\mathscr{A} \leqq \mathscr{C}$ if each $A \in \mathscr{A}$ is contained in some $C \in \mathscr{C}$. This is a lower directed partial order for $D(G)$. In fact, if $\mathscr{C}, \mathscr{D} \in D(G)$ then

$$\mathscr{C} \cap \mathscr{D} = \{C \cap D \mid C \in \mathscr{C},\ D \in \mathscr{D} \text{ and } C \cap D \neq 0\}$$

is the greatest lower bound of $\mathscr{C}$ and $\mathscr{D}$ in $D(G)$.

If $\{A_\lambda \mid \lambda \in \Lambda\} \subseteq P(G)$ and $C = \bigsqcup A_\lambda = (\bigcap A'_\lambda)'$ then $C' = \bigcap A'_\lambda$ and so there is a natural isomorphism

$$C' + g \to (\text{—} A'_\lambda + g \text{—})$$

of G/C' into $\Pi G/A'_\lambda$. Now if $\mathscr{A} \leqq \mathscr{C}$ in $D(G)$ then for each $C \in \mathscr{C}$ we have $C = \bigsqcup A_\lambda$, where the $A_\lambda \in \mathscr{A}$, so there is a natural isomorphism

$$G_{\mathscr{C}} = \Pi\, G/C' \to^{\Pi_{\mathscr{C}\mathscr{A}}} \Pi_{\mathscr{A}}\, G/A' = G_{\mathscr{A}}\,.$$

Let $\mathscr{O}(G)$ be the direct limit of these rings $G_{\mathscr{C}}$. Then $\mathscr{O}(G)$ consists of all vectors $l = (\text{—} l_{\mathscr{C}} \text{—})$ such that for $\mathscr{A} \geqq \mathscr{B}$ in $D(G)$ we have

$$l_{\mathscr{A}} \neq 0 \quad \text{or} \quad l_{\mathscr{B}} = 0 \quad \text{implies} \quad l_{\mathscr{A}}\Pi_{\mathscr{A}\mathscr{B}} = l_{\mathscr{B}}\,, \quad \text{and}$$
$$l_{\mathscr{A}} = 0 \quad \text{and} \quad l_{\mathscr{B}} \neq 0 \quad \text{implies} \quad l_{\mathscr{B}} \notin G_{\mathscr{A}}\Pi_{\mathscr{A}\mathscr{B}}\,.$$

Note that each non-zero component $l_{\mathscr{A}}$ of l completely determines l. Also if G is commutative so is $\mathscr{O}(G)$.

The map $\sigma_{\mathscr{C}}$ of $x \in G_{\mathscr{C}}$ onto the element $l \in \mathscr{O}(G)$ with $l_{\mathscr{C}} = x$ is an l-isomorphism of $G_{\mathscr{C}}$ into $\mathscr{O}(G)$. $\mathscr{O}(G)$ is the join of the directed w.r.t. inclusion set of subgroups $G_{\mathscr{C}}\sigma_{\mathscr{C}}$.

1) *The map* $g \to \tilde{g}$ *is an isomorphism of* G *into* $\mathcal{O}(C)$, *where*

$$\tilde{g}_C = (—C' + g—) \quad \textit{for all} \quad C \in \mathscr{C} .$$

If G *has an identity* 1 *then* $\tilde{1}$ *is the identity for* $\mathcal{O}(C)$.

2) *If* $0 \neq l,\ k \in \mathcal{O}(G)$ *then* $0 \neq \tilde{c}l \in \tilde{G}$ *and* $\tilde{c}k \in \tilde{G}$ *for some* $c \in G$. *Thus* $\mathcal{O}(G)$ *is a ring of left quotients of* $\tilde{G}$ *and also a ring of right quotients. In particular,* $\tilde{G}$ *is large in* $\mathcal{O}(G)$.

Proof. Pick $\mathscr{C} \in D(G)$ so that $l_{\mathscr{C}} \neq 0 \neq k_{\mathscr{C}}$. Then $l_{\mathscr{C}} = (—C' + x—)$ with say $C' + x \neq C'$ and hence $cx \neq 0$ for some $c \in C$. Now $cD = 0$ for all $D \in \mathscr{C}$, $D \neq C$, so $D' + cx = D'$ for all such D. Thus $C' + cx$ is the only non-zero component of $\tilde{c}\tilde{x}_{\mathscr{C}}$. For if $cx \in C'$ then $cx \in C' \cap C = 0$, a contradiction. Thus $0 \neq \tilde{c}_{\mathscr{C}}\ l_{\mathscr{C}} = \tilde{c}\tilde{x}_{\mathscr{C}}$ and hence $0 \neq \tilde{c}l = \tilde{c}\tilde{x} \in \tilde{G}$.

Now $k_{\mathscr{C}} = (—C' + y—) \neq 0$. If $cy \neq 0$ then as above $0 \neq \tilde{c}k = \tilde{c}\tilde{y} \in \tilde{G}$ and if $cy = 0$ then $\tilde{c}_{\mathscr{C}} \neq 0 \neq k_{\mathscr{C}}$ and $\tilde{c}_{\mathscr{C}}k_{\mathscr{C}} = 0$, but then $\tilde{c}k = 0 \in \tilde{G}$.

Corollary. $\mathcal{O}(G)$ *is semiprime and if* G *is reduced then os is* $\mathcal{O}(G)$.

Proof. This follows from (2) and Lemmas 3.1 and 3.4. One can, of course, prove this directly from the construction of $\mathcal{O}(G)$ since $\mathcal{O}(G)$ is the set theoretical join of a directed set of copies of the $G_{\mathscr{C}}$.

3) $\mathcal{O}(G)$ *is laterally complete.*

Proof. Let S be a disjoint subset of $\mathcal{O}(G)$. It suffices to find a partition $\mathscr{E}$ of $P(G)$ so that the elements $l \in S$ have non-zero disjoint support in $G_{\mathscr{E}}$. For then $\bigvee l_{\mathscr{E}}$ exists in $G_{\mathscr{E}}$ and hence $\bigvee l$ exists in $\mathcal{O}(G)$. Suppose that $l, k \in S$ and have non-zero components $l_{\mathscr{C}}$ and $k_{\mathscr{D}}$. Then $l_{\mathscr{C}} = (\ldots, C' + l(C), \ldots)$, where $l(C) \in G$, and $C' + l(C) \neq C'$ iff $l(C)\, C \neq 0$ iff $\langle l(C) \rangle\, C \neq 0$ iff $\langle l(C) \rangle \cap C \neq 0$. Let $\mathscr{A}$ be a partition of $P(G)$ so that $\mathscr{A} \leqq \mathscr{C}$ and $\mathscr{A}$ contains all the $(\langle l(C) \rangle \cap C)'' \neq 0$. Then $(\langle l(C) \rangle \cap C') + l(C)$ are the only non-zero components of $l_{\mathscr{A}}$. For suppose that $A \in \mathscr{A}$, $A \subseteq C \in \mathscr{C}$ and $A \cap (\langle l(C) \rangle \cap C)'' = 0$. Then

$$\langle l(C) \rangle\, A \subseteq \langle l(C) \rangle \cap A \subseteq \langle l(C) \rangle \cap C \cap A \subseteq (\langle l(C) \rangle \cap C)'' \cap A = 0$$

so $A' + l(C) = A'$.

We next show that $(D \cap \langle k(D) \rangle)'' \cap (C \cap \langle l(C) \rangle)'' = 0$. First

$$l_{\mathscr{C} \cap \mathscr{D}} = (\ldots, (C \cap D)' + l(C), \ldots) \quad \text{and} \quad k_{\mathscr{D} \cap \mathscr{C}} = (\ldots, (C \cap D)' + k(D), \ldots)$$

and since $l \perp k$ it follows that $\langle l(C) \rangle \langle k(D) \rangle \subseteq (C \cap D)'$. Now $G/D \cap C)'$ is semiprime and since the product $\langle k(D) \rangle \langle l(C) \rangle$ is zero modulo $(D \cap C)'$ so is the intersection. Thus $\langle k(D) \rangle \cap \langle l(C) \rangle \subseteq (D \cap C)'$ and so

$$(D \cap \langle k(D) \rangle)'' \cap (C \cap \langle l(C) \rangle)'' =$$
$$= (D \cap C \cap \langle k(D) \rangle \cap \langle l(C) \rangle)'' \subseteq (D \cap C \cap (D \cap C)')'' = 0'' = 0 .$$

Now choose a partition $\mathscr{E}$ of $P(G)$ that contains all of the $(C \cap \langle l(C)\rangle)'' \neq 0$ for all the $l \in S$. Note that $\mathscr{E}$ need not be $\leqq \mathscr{C}$. For a fixed $l \in S$ choose $\mathscr{C}$ and $\mathscr{A}$ as above

$$\begin{array}{ccc} G_{\mathscr{C}} & & \\ \downarrow & & \\ G_{\mathscr{A}} & & \\ \downarrow & & \\ G_{\mathscr{A}\cap\mathscr{E}} & \longleftarrow & G_{\mathscr{E}} \end{array}$$

Pick the element $t \in \mathcal{O}(G)$ will non-zero $\mathscr{E}$ compinents $(\langle l(C)\rangle \cap C)' + l(C)$ for this fixed $l \in S$ where, of course, $\langle l(C)\rangle \cap C \neq 0$. Then

$$l_{\mathscr{A}\cap\mathscr{E}} = l_{\mathscr{C}}\Pi_{\mathscr{C}\mathscr{A}}\Pi_{\mathscr{A},\mathscr{A}\cap\mathscr{E}} = t_{\mathscr{E}}\Pi_{\mathscr{E},\mathscr{A}\cap\mathscr{E}} \, .$$

Thus $0 \neq t_{\mathscr{E}} = l_{\mathscr{E}}$ and so each $l \in S$ has non-zero support in $G_{\mathscr{E}}$ and these supports are disjoint.

4) $\mathcal{O}(G)$ *is a* P-*ring*.

Proof. We need to show that for $0 \neq l \in \mathcal{O}(G)$

$$\mathcal{O}(G) = l^{**} \oplus l^{*} \, .$$

Consider $0 \neq k \in \mathcal{O}(G)$ and pick $\mathscr{C} \in D(G)$ such that $l_{\mathscr{C}} \neq 0 \neq k_{\mathscr{C}}$. Then $l_{\mathscr{C}} = (-C' + l(C)-)$. Pick $\mathscr{C} \geqq \mathscr{A} \in D(G)$ so that each $(C \cap \langle l(C)\rangle)'' \neq 0$ belongs to $\mathscr{A}$. Then

$$\begin{array}{rcccc} G_{\mathscr{A}} & = & \Pi G/(C \cap \langle l(c)\rangle)' & \oplus & \Pi\, G/A'_{\lambda} \\ k_{\mathscr{A}} & = & x_{\mathscr{A}} & + & y_{\mathscr{A}} \, . \end{array}$$

Let $x(y)$ be the element in $\mathcal{O}(G)$ with $\mathscr{A}$-th component $x_{\mathscr{A}}$ if $x_{\mathscr{A}} \neq 0$ $(y_{\mathscr{A}}$ if $y_{\mathscr{A}} \neq 0)$ and zero otherwise. Then $k = x + y$. Now we have shown that the only non-zero components of l are of the form $(C \cap \langle l(C)\rangle)' + l(C)$. Thus $l_{\mathscr{A}} \perp y_{\mathscr{A}}$ and so $y \in l^{*}$ and hence it suffices to show that $x \in l^{**}$. Consider $0 \neq t \in \mathcal{O}(G)$ such that $l \perp t$. To complete the proof we must show that $x \perp t = 0$.

Pick $\mathscr{D} \in D(C)$ so that $0 \neq t_{\mathscr{D}} = (-D' + t(D)-)$. We know that $(C \cap \langle l(C)\rangle)'' \cap \cap (D \cap \langle t(D)\rangle)'' = 0$ so we may choose $\mathscr{D} \geqq \mathscr{B} \in D(C)$ that contains the $(C \cap \langle l(C)\rangle)'' \neq 0$ and the $(D \cap \langle t(D)\rangle)'' \neq 0$.

$$\begin{array}{ccc} G_{\mathscr{A}} & & \\ \downarrow & & \\ G_{\mathscr{A}\cap\mathscr{B}} & \longleftarrow & G_{\mathscr{B}} \end{array}$$

Now x_a has non-zero components of the form $(C \cap \langle l(C) \rangle)' + z$ and these are also the non-zero component of $x_{\mathscr{A} \cap \mathscr{B}}$. Also t has non-zero components of the form $(D \cap \langle t(D) \rangle)' + t(D)$. It follows that $x_{\mathscr{A} \cap \mathscr{B}} \perp t_{\mathscr{A} \cap \mathscr{B}}$ and hence $x \perp t = 0$.

Lemma 4.1. *If G is a semiprime ring and also a* P-*ring and an* L-*ring then G is an* 0-*ring.*

Proof. Consider $C \in P(G)$ and let $\{a_\lambda \mid \lambda \in \Lambda\}$ be a maximal disjoint subset of C. Then $a = \bigvee a_\lambda$ exists and it suffices to show that $C = a''$, for then $G = a'' \oplus a' = C \oplus a'$. Now G is a subdirect sum of prime rings $\{T_i \mid i \in I\}$. If $a \notin C$ then $0 \neq ax$ for some $x \in C'$ and since $x \perp a_\lambda$ we have that ax is disjoint from the support of each of the a_λ. Then $ax + a$ is an upper bound for the a_λ that is not comparable with a, a contradiction. Thus $a \in C$ and so $a'' \subseteq C'' = C$.

Now it suffices to show that $a' \subseteq C'$. If $0 \neq y \in a'$ then $yGa = 0$ and so $yGa_\lambda = 0$ for all λ. Now if $y \notin C'$ then $0 \neq cy$ for some $c \in C$. Thus $\{cy\} \cup \{a_\lambda \mid \lambda \in \Lambda\}$ is a disjoint subset of C, but this contradicts the maximality of $\{a_\lambda \mid \lambda \in \Lambda\}$.

For an arbitrary semiprime ring G we have the following corollaries.

Corollary I. *$\mathcal{O}(G)$ is an* 0-*ring.*

Corollary II. *If $C \in P(G)$, $\{a_\lambda \mid \lambda \in \Lambda\}$ is a disjoint subset of C and $a = \bigvee a_\lambda$ exists then $a \in C$.*

Thus we have proven the existence of an X-hull for a semiprime ring G, where X = P, SP, L or 0. We next prove the uniqueness.

First suppose only that G is semiprime and left large in H.

Lemma 4.2. *There is a natural isomorphism τ of $\mathcal{O}(G)$ into $\mathcal{O}(H)$ and $\tilde{G}\tau$ is left large in $\mathcal{O}(H)$.*

Proof. Since G is left large in H for each $C \in P(G)$ we have $C = G \cap C^{**}$ and $C' = G \cap C^*$. Thus $C' + g \to C^* + g$ is an isomorphism of G/C' into H/C^*. For each $\mathscr{C} \in D(G)$ let

$$\mathscr{C}^- = \{C^{**} \mid C \in \mathscr{C}\}\,.$$

Then $\mathscr{C}^- \in D(H)$ and there is a natural isomorphism $\tau_{\mathscr{C}}$ of $G_{\mathscr{C}}$ into $H_{\mathscr{C}^-}$. Moreover if $\mathscr{A} \leqq \mathscr{C}$ in $D(G)$ then

$$\begin{array}{ccc} G_{\mathscr{C}} & \xrightarrow{\tau_{\mathscr{C}}} & H_{\mathscr{C}^-} \\ \Big\downarrow \Pi_{\mathscr{C}\mathscr{A}} & & \Big\downarrow \Pi_{\mathscr{C}^-\mathscr{A}^-} \\ G_{\mathscr{A}} & \xrightarrow{\tau_{\mathscr{A}}} & H_{\mathscr{A}^-} \end{array}$$

commutes. Then the $\tau_{\mathscr{C}}$ determines an isomorphism τ of $\mathcal{O}(G)$ into $\mathcal{O}(H)$.

Let α (β) be the natural isomorphism of G (H) into $\mathcal{O}(G)$ $(\mathcal{O}(H))$

$$\begin{array}{ccc} H & \xrightarrow{\beta} & \mathcal{O}(H) \\ & & \uparrow \tau \\ G & \xrightarrow{\alpha} G\alpha = \tilde{G} \subseteq & \mathcal{O}(G) \end{array}$$

If $g \in G$ and $\mathscr{C}^- \in D(H)$ then $(g\alpha\tau)_{\mathscr{C}^-} = (g\alpha)_{\mathscr{C}}\, \tau_{\mathscr{C}} = (—C' + g—)\, \tau_{\mathscr{C}} = (—C^* + g—) = (g\beta)_{\mathscr{C}^-}$. Thus $g\alpha\tau = g\beta$.

Consider $0 \neq l \in \mathcal{O}(H)$ with $l_{\mathscr{C}^-} = (—C^* + x—)$ where say

$$C^* + x \neq C^* = \{y \in H \mid yHC = 0\} = \{y \in H \mid CHy = 0\}\,.$$

We first show that $Cx \neq 0$. For suppose that $Cx = 0$ and hence $CGx = 0$. We know $xHC \neq 0$ and so $0 \neq xhc$ for some $h \in H$ and $x \in C$. Thus there exists an element $g \in G$ such that $0 \neq gxhc = k \in G \cap C^{**} = C$. Thus $kGx = 0$ and hence $kGgx = 0$. But then $kGk = 0$ and hence $k = 0$, a contradiction.

Thus $0 \neq cx$ for some $c \in C = G \cap C^{**}$. Now find $D \neq C^{**}$ in $\mathscr{C}^-$, then $cD = Dc = 0$ so $D^* + cx = D^*$. Therefore $C^* + cx$ is the only non-zero component of $(cx)\,\beta_{\mathscr{C}^-}$. Therefore $(cx)\,\beta_{\mathscr{C}^-} = (c\beta)_{\mathscr{C}^-}\, l_{\mathscr{C}^-}$ and hence $(cx)\,\beta = c\beta l$ in $H\beta$.

Now $G\beta$ is left large in $H\beta$ so there exists $g \in G$ such that $0 \neq g\beta(cx)\,\beta \in G\beta$. But

$$g\beta(cx)\,\beta = g\beta\, c\beta l = (gc)\,\beta l\,.$$

Thus $G\beta = \tilde{G}\tau$ is left large in $\mathcal{O}(H)$.

Now suppose that H is an X-hull of G. We show that H is unique by showing that α can be extended to an isomorphism ϱ of H onto the X-hull K of $G\alpha$ in $\mathcal{O}(G)$.

$$\begin{array}{ccc} G & \subseteq & H \\ \big| & & \downarrow \varrho \\ \lfloor & \xrightarrow{\alpha} & G\alpha \subseteq K \subseteq \mathcal{O}(G) \end{array}$$

Now $G\beta = G\alpha\tau \subseteq \mathcal{O}(G)\,\tau$ which is an X-group. By Lemma 4.2 $G\beta$ is left large in $\mathcal{O}(H)$. Thus $H\beta \cap \mathcal{O}(G)\,\tau$ is an X-group that contains $G\beta$ and since $H\beta$ is an X-hull of $G\beta$ we have

$$G\alpha\tau = G\beta \subseteq H\beta \subseteq \mathcal{O}(G)\,\tau \subseteq \mathcal{O}(H)\,.$$

Thus $H\beta\tau^{-1}$ is an X-group that contains $G\alpha$ and so

$$G\alpha = G\beta\tau^{-1} \subseteq K \subseteq H\beta\tau^{-1} \subseteq \mathcal{O}(G)$$

and since $H\beta\tau^{-1}$ is an X-hull of $G\beta\tau^{-1}$ we have $K = H\beta\tau^{-1}$.

Thus if H_1 and H_2 are X-hulls of G then there is an isomorphism of H_1 onto H_2 that induce the identity on G. Actually it follows from Theorem 5.4 that the isomorphism is unique. This completes the proof of Theorem B.

5. PROPERTIES OF X-HULLS

Throughout this section let G be a semiprime ring. If $\mathscr{C} \in D(G)$ then there exists a partition $\mathscr{A} \leqq \mathscr{C}$ that consists of principal annihilators of G. For in each $C \in \mathscr{C}$ pick a maximal disjoint subset $\{a_\alpha \mid \alpha \in A_C\}$. Then $C = (\bigcap a'_\alpha)' = (\bigcup a''_\alpha)'' = \bigsqcup a''_\alpha$. For $a_\alpha \in C$ and so $a'_\alpha \supseteq C'$ and hence $\bigcap a'_\alpha \supseteq C'$. Suppose that $x \in \bigcap a'_\alpha$ then $xGa_\alpha = 0$ for all α. If $x \notin C'$ then $0 \neq xc \in C$ for some $c \in C$ and hence $a_\alpha Gxc = 0$ for all α, but this contradicts the fact that $\{a_\alpha \mid \alpha \in A_C\}$ is a maximal disjoint subset of C.

Theorem 5.1. *If G is a* P*-ring then each $0 \neq l \in \mathcal{O}(G)$ is the join of a disjoint subset of $\tilde{G}$. In particular $\tilde{G}^{\mathrm{L}} = \mathcal{O}(G)$ and hence G^{L} is an* SP*-group.*

Proof. Consider $0 \neq l \in \mathcal{O}(G)$ and suppose that $l_{\mathscr{C}} \neq 0$. Then their exists a partition $\mathscr{A} \leqq \mathscr{C}$ that consists of principal annihilators of G.

$$\mathscr{A} = \{a''_\lambda \mid \lambda \in \Lambda\}\,.$$

Now $0 \neq l_{\mathscr{A}} = (\text{—}a'_\lambda + l(\lambda)\text{—})$ and since $G = a''_\lambda \oplus a'_\lambda$ we may assume that each $l(\lambda)$ belong to a''_λ. In particular the $l(\lambda)$ are disjoint in G and

$$\widetilde{l(\lambda)}_{\mathscr{A}} = (0 - 0,\ a'_\lambda + l(\lambda),\ 0 - 0)\,.$$

Thus $l_{\mathscr{A}} = \bigvee \widetilde{l(\lambda)}_{\mathscr{A}}$ and hence $l = \bigvee l(\lambda)$.

Corollary I. *If G is an* 0*-ring then $G \cong \tilde{G} = \mathcal{O}(G)$.*

Corollary II. *$\tilde{G} \subseteq \tilde{G}^{\mathrm{P}} \subseteq \tilde{G}^{\mathrm{SP}} \subseteq (\tilde{G}^{\mathrm{SP}})^{\mathrm{L}} = (\tilde{G}^{\mathrm{P}})^{\mathrm{L}} = \tilde{G}^{0} = \mathcal{O}(G)$ when the indicated X-hulls are in $\mathcal{O}(G)$. In particular $\mathcal{O}(G)$ is the orthocompletion of G.*

Proof. It is clear that $\tilde{G} \subseteq \tilde{G}^{\mathrm{P}} \subseteq \tilde{G}^{\mathrm{SP}}$ and $(\tilde{G}^{\mathrm{P}})^{\mathrm{L}} \subseteq (\tilde{G}^{\mathrm{SP}})^{\mathrm{L}} \subseteq \tilde{G}^{0} \subseteq \mathcal{O}(G)$ so it suffices to show that $(\tilde{G}^{\mathrm{P}})^{\mathrm{L}} = \mathcal{O}(G)$.

Let H be the P-hull of G and α, β, τ be as in the proof of Theorem B. Then by Theorem 5.1 $(H\beta)^{\mathrm{L}} = \mathcal{O}(H)$ and

$$\begin{array}{ccc} H \xrightarrow{\ \beta\ } H \subseteq (H\beta)^{\mathrm{L}} & = & \mathcal{O}(H) \\ & & \uparrow \tau \\ G \xrightarrow{\ \alpha\ } \tilde{G} \subseteq (\tilde{G}^{\mathrm{P}})^{\mathrm{L}} & \subseteq & \mathcal{O}(G) \end{array}$$

Thus $H\beta = \tilde{G}^{\mathrm{P}}\tau \subseteq (\tilde{G}^{\mathrm{P}})^{\mathrm{L}}\tau \subseteq \mathcal{O}(H)$ and $(\tilde{G}^{\mathrm{P}})^{\mathrm{L}}\tau$ is an L-ring that contains $H\beta$. Thus $(\tilde{G}^{\mathrm{P}})^{\mathrm{L}}\tau = \mathcal{O}(H)$ and so $(\tilde{G}^{\mathrm{P}})^{\mathrm{L}} = \mathcal{O}(G)$.

Proposition 5.2. $(G^{\mathrm{L}})^{\mathrm{P}} = (G^{\mathrm{L}})^{\mathrm{SP}} \subseteq G^0$, *but we need not have equality. Thus the operators* SP *and* L *need not commute.*

Proof. If $C \in P((G^{\mathrm{L}})^{\mathrm{P}}$ then $C \cap G^{\mathrm{L}} = C\gamma \in P(G^{\mathrm{L}})$ so as in Lemma 4.1 $C\gamma = a''$ for some $a \in C\gamma$. Thus

$$C = C\gamma\mu = a''\mu = (a'')^{**} = a^{**}$$

and hence $(G^{\mathrm{L}})^{\mathrm{P}} = a^{**} \oplus a^* = C \oplus a^*$ and so $(G^{\mathrm{L}})^{\mathrm{P}}$ is an SP-ring.

We now give an example to show that $(G^{\mathrm{L}})^{\mathrm{SP}}$ need not equal G^0. Let $D = Z[x]$ be the ring of polynomials with integral coefficients and let $V = \Pi_{i=1}^{\infty} D_i$. Then V is a ring with identity $e = (1, 1, \ldots)$. Let

$$G = \{v \in V \mid \text{the constant term in each } v_i \text{ is the same}\},$$
$$H = \{v \in V \mid \text{the } v_i \text{ have only a finite number of distinct constant turns}\}.$$

It is reasonably clear that:

1) *G is laterally complete but not a* P-*ring,*

and

2) *H is an* SP-*ring that is not laterally complete and since* $H \supseteq \Sigma_{i=1}^{\infty} D_i$, $H^{\mathrm{L}} = V = H^0$.

Thus it suffices to show that

3) $H = G^{\mathrm{SP}} = G^{\mathrm{P}}$.

Now G is large in the SP-ring H and $e \in G$. Suppose that $G \subseteq K \subseteq H$, where K is a P-ring and let $'(*)$ be the annihilator operator in $K(H)$. Let Y be a subset of $\{1, 2, \ldots\}$ and define $s \in G$ by

$$s_i = \begin{cases} x & \text{if } i \in Y, \\ 0 & \text{otherwise}. \end{cases}$$

Then $K = s'' \oplus s'$, $H = s^{**} \oplus s^*$, $s^{**} \cap K = s''$ and $s^* \cap K = s'$. Now $e = a + b$ in $s'' \oplus s'$ and this is also the decomposition of e in H. Thus $a \in K$ and a is the characteristic function of Y. But these characteristic functions together with G clearly generate H and hence $K = H$. Therefore $H = G^{\mathrm{P}}$.

Proposition 5.3. *The complete ring $Q(G)$ of left (or right) quotients of G is an* 0-*ring and* $G \subseteq G^0 \subseteq Q(G)$.

Proof. G is left large in $Q(G)$ and hence $Q(G)$ is semiprime. Now as we have seen $Q(G)^0$ is a ring of left quotients of $Q(G)$ so $Q(G) = Q(G)^0$.

Theorem 5.4. *If α is an isomorphism of G_1 onto G_2, where the G_i are semiprime rings, then there exists a unique extension of α to an isomorphism of G_1^{X} onto G_2^{X}, where* $\mathrm{X} = \mathrm{P}$, SP, L *or* 0.

Proof. The proof of Theorem 2.7 in [5] establishes that α can be extended to an isomorphism of G_1^{X} onto G_2^{X}.

For the uniqueness is suffice to show that an automorphism α of G^X that induces the identity on G is the identity, where G is a semiprime ring. Now α induces the identity of $P(G)$ and hence on $P(G^X)$ and by the above we may assume that $X = 0$. Thus we may assume that α is an automorphism of $\mathcal{O}(G)$ that induces the identity on $\tilde{G}$. Consider $l \in \mathcal{O}(C)$ with $l_{\mathscr{C}} = (-C' + y—)$ and suppose that $(l\alpha)_{\mathscr{C}} = (—C' + x—)$ where $C' + x \neq C' + y$. Then

$$(\tilde{g} - l)_{\mathscr{C}} \quad \text{and } (0 - 0,\ C' + y - x,\ 0 - 0) \text{ are disjoint in } G_{\mathscr{C}},$$

and

$$((\tilde{g} - l)\,\alpha)_{\mathscr{C}} \text{ and } (0 - 0,\ C' + y - x,\ 0 - 0) \text{ are not}.$$

Thus it follows that α does not induce the identity on $P(\mathcal{O}(G))$, a contradiction.

Proposition 5.5. *If G is a semiprime ring, α is an automorphism of G^0 and $X =$ = P, SP, L or 0 then*

(i) *$G^X\alpha = (G\alpha)^X$ and so if $G\alpha = G$ then $G^X\alpha = G^X$, and*

(ii) *if $G\alpha \subseteq G$ then $G^X\alpha \subseteq G^X$.*

Corollary. *If α is an endomorphism of G^X that induces an automorphism on G then α is an automorphism of G^X.*

The proof is entirely similar to the proof of Proposition 2.8 in [5] and so we omit it.

Proposition 5.6. *If G is a regular ring then so are G^P, G^{SP} and G^0.*

Proof. Since homomorphic images and products of regular rings are regular, each $G_{\mathscr{C}}$ used in the construction of $\mathcal{O}(G)$ is regular and hence $G^0 \cap \mathcal{O}(G)$ is regular. Now CHAMBLESS [3] has shown that G^P and G^{SP} are (isomorphic to) direct limits of certain of the $G_{\mathscr{C}}$ and hence they too are regular.

Question. *If G is regular then is G^L regular?*

HUIJSMANS [7] shows that many of the theorems about commutative regular rings hold for hyperarchimedean lattice-ordered groups and conversely. In particular, each principal ideal of such a ring R is a summand. Therefore $R = R^P$ and so $R^L = R^0$. Now the principal l-ideals of a hyperarchimedean l-group A are summands and so A is a P-group. However A^0 need not be hyperarchimedean. For if A is the cardinal sum of a countable number of copies of the group of reals then $A^L = A^0$ is the cardinal product which is not hyperarchimedean. So the analogy between commutative regular rings and hyperarchimedean l-groups is far from complete.

Suppose that G is a Boolean ring. Then the partial order that we have introduced is the natural lattice ordering of G. For $x \geqq y$ iff $xy = x \wedge y = y = y^2$. Also

$x \geqq xy$, and $x \geqq z$ and $y \geqq z$ imply $xy \geqq z$. Since G is regular $G = G^{\mathrm{P}}$ and so $G^{\mathrm{L}} = G^{0}$. Clearly $G^0 \cong \mathcal{O}(G)$ is Boolean and hence so is G^{SP}.

1) *The map* $a \to^{\gamma} a''$ *is an isomorphism* G *into* $P(G)$.

This is well known and easy to prove.

2) $P(G) = (G\gamma)^{\mathrm{L}} = (G\gamma)^{0}$.

Proof. Consider $C \in P(G)$ and pick $0 \neq g \in C$. Then $g\gamma = g'' \subseteq C'' = C$ and hence $g\gamma C = g\gamma \cap C = g\gamma \in G\gamma$. Then $G\gamma$ is large in $P(G)$ and $G\gamma$ is a P-ring. Therefore since $P(G)$ is an L-ring $P(G) \supseteq (G\gamma)^{\mathrm{L}} = (G\gamma)^{0}$. But if $\{a_\lambda \mid \lambda \in \Lambda\}$ is a maximal disjoint subset of C then $C = \bigsqcup a''_\lambda$. Therefore $P(G)$ is the L-hull of $G\gamma$.

3) *$P(G)$ is the Dedekind-MacNeille completion of $G\gamma$ iff G has an identity.*

Proof. We have shown that each element in $P(G)$ is the join of disjoint elements from $G\gamma$. Thus if $C \in P(G)$ then $C' = \bigsqcup a''_\lambda = (\bigcap a'_\lambda)'$ and so $C = \bigcap a'_\lambda$. But $G = a'_\lambda \oplus a''_\lambda$ and so since $e = (e + a_\lambda) + a_\lambda$ we have $a'_\lambda = (e + a_\lambda)''$. Therefore $C = \bigcap (e + a_\lambda)''$.

4) *$P(G) \cong \mathcal{O}(G)$ and this is also the complete ring of quotient of G.*

Proof. We know $P(G) \cong G^{\mathrm{L}} \cong \mathcal{O}(G)$ and (see [11]) $P(G)$ is its own ring of quotients. Thus $\mathcal{O}(G)$ is the complete ring of quotient of G.

Remark. The fact that $P(G)$ is the ring of quotients of G is established in [11].
Now let α be the natural isomorphism of G into $\mathcal{O}(G)$.

$$\begin{array}{ccc} G & \xrightarrow{\ \gamma\ } & P(G) \\ {\scriptstyle\alpha}\downarrow & & \\ \mathcal{O}(G) & & \end{array}$$

It follows from Theorem 5.4 that there is a unique extension ϱ of $\gamma^{-1}\alpha$ to an isomorphism of $P(G)$ onto $\mathcal{O}(G)$.

For $K \in P(G)$ let $\{\alpha_\lambda \mid \lambda \in \Lambda\}$ be a maximal disjoint subset of K and pick a partition $\mathscr{C}$ of $P(G)$ that contains the a''_λ. Now $K = \bigsqcup a''_\lambda$ and so ϱ must map take this onto $\bigvee a_\lambda \alpha$. Therefore $K\varrho$ is the element in $\mathcal{O}(G)$ with $\mathscr{C}$-th component

$$(-C' + l(C)-)$$

where $l(C) = a_\lambda$ if $C = a''_\lambda$ and $l(C) = 0$ if C is not one of the a''_λ

5) *Thus we have factored the natural embedding α of G into $\mathcal{O}(G)$ through $P(G)$*

$$g \overset{\gamma}{—} g'' \overset{\varrho}{—} g\alpha = \tilde{g}\ .$$

Suppose that G is a commutative semiprime ring Abian [1] calls an element $0 \neq a \in G$ a *hyperatom* if for each element $x \in G$

$$x \leqq a \quad \text{implies} \quad x = 0 \quad \text{or} \quad a\,,$$

and

$$x \neq 0 \quad \text{implies} \quad axs = a \quad \text{for some} \quad s \in G\,.$$

G is *hyperatomic* if for each $0 \neq g \in G$ there is a hyperatom $a \leqq g$. Abian shows that G is (isomorphic to) a direct product of fields iff G is hyperatomic and laterally complete.

Proposition 5.8. *Let G be a commutative semiprime ring. Then G^{L} is a product of fields iff G is hyperatomic.*

Proof. ($\leftarrow$) Using Abian's results we may assume that $\Sigma F_i \subseteq G \subseteq \Pi F_i$, where the F_i are fields. Now clearly $\Sigma(F_i)^{\mathrm{L}} = \Pi F_i$ and hence $G^{\mathrm{L}} = \Pi F_i$.

($\rightarrow$). We are given that $G \subseteq G^{\mathrm{L}} = \Pi F_i$. Since G is large in G^{L}, for each i there is an element of the form $(0—0, a_i, 0—0) \in G$, where $0 \neq a_i \in F_i$. Let P_i be the projection of G onto the i-th coordinate. Then

$$G \subseteq \Pi P_i \subseteq \Pi F_i$$

and since ΠP_i is an L-ring it follows that $\Pi P_i = \Pi F_i$. Thus there is an element of the form $(— a_i^{-1} —)$ in G and hence $(0 - 0, 1, 0 - 0) \in G$. In particular $F_i \subseteq G$ and so $\Sigma F_i \subseteq G$. Thus G is hyperatomic.

Remark. Otis Kenny has shown this Abian's results can be extended to non-commutation reduced rings. Thus if R is a reduced ring then R^{L} is a product of division rings iff R is hyperatomic.

6. SEMIPRIME RINGS R FOR WHICH $P(R)$ IS ATOMIC

In the next few proofs we will use the fact that if $a, b \in A$ an ideal of R then $a \perp b$ iff a and b are disjoint in A. Thus $a \leqq b$ in A iff $a \leqq b$ in R. For if $arb \neq 0$ for some $r \in R$ then $arbsarb \neq 0$ for some $s \in R$ and since $rbsar \in A$ we have $aAb \neq 0$.

Theorem 6.1. *For an ideal $A \neq 0$ in a semiprime ring R the following are equivalent.*

a) *A is a prime ring.*
b) *$a' = A'$ for each $0 \neq a \in A$.*
c) *A' is a prime ideal.*
d) *A' is a minimal prime ideal.*
e) *A'' is the largest ideal containing A that is a prime ring.*
f) *A'' is an ideal that is maximal w. r. t. being a prime ring.*
g) *A'' is a prime ring.*
h) *A'' is an atom in $P(R)$.*
i) *A' is maximal in $P(R)$.*

Remarks. (a) If R is reduced then "prime ring" becomes "integral domain" and a minimal prime ring is completely prime [2]). Thus A' is completely prime.

(b) If $A = \langle s \rangle$ is principal then $A' = s'$ and $A'' = s''$. We shall call s *basic* provided the above conditions are satisfied. In particular, if s and t are basic then by (h), $s'' \cap t'' = 0$ or $s'' = t''$.

Corollary I. *If R is reduced, then $0 \neq s \in R$ is basic iff Rs is an integral domain.*

Proof. ($\rightarrow$) $Rs \subseteq \langle s \rangle$ which is an integral domain.

($\leftarrow$) Suppose (by way of contradiction) that $0 \neq x$, $y \in s''$ and $xy = 0$. Then Then $xsys = 0$. Now $xs = 0$ implies $x \in s' \cap s'' = 0$ and so $xs \neq 0 \neq ys$ and $xsys = 0$. Then Rs is not an integral domain, a contradiction.

Corollary II. *If $C \in P(R)$ and A is an ideal in R and a prime ring then $C \supseteq A$ or $C \cap A = 0$.*

Proof. If $0 \neq a \in C \cap A$ then $A' = a' \supseteq C'$ so $A \subseteq A'' \subseteq C'' = C$.

Proof of the theorem. (a $\rightarrow$ b). Consider $a, b \in A$ with $a \neq 0$. If $x \in a'$ then $xRa = 0$ so $xsbRa = 0$ for all $s \in R$. Thus since A is a prime ring and xsb, $a \in A$ we have $xsb = 0$ for all $s \in R$ and so $x \in A'$. Thus $a' \subseteq A'$ and since $a \in A$, $a' \supseteq A'$.

(b $\rightarrow$ c). If (c) is false then there exists $x, y \in R \setminus A'$ such that $xRy \subseteq A'$. Thus for $0 \neq a \in A$ we have $xtaRysa = 0$ for all $s, t \in R$. If $xta \neq 0$ for some t then $ysa \in (xta)' = A'$ so $ysa \in A' \cap A = 0$ for all $s \in R$ and so $y \in a' = A'$, a contradiction. If $xta = 0$ for all $t \in R$ then $x \in a' = A'$, a contradiction.

(c $\rightarrow$ d). We know that A' is the intersection of minimal prime ideals.

(d $\rightarrow$ e). If $0 = a, b \in A''$ then $a, b \in R \setminus A'$ and since R/A' is a prime ring $axb \notin A'$ for some $x \in R$. Then $aRb \neq 0$ and so $aA''b \neq 0$ and hence A'' is a prime ring. Suppose that B is an ideal of R and a prime ring that contains A''. We use the fact that (a) implies (b). If $0 \neq a \in A$ the $A' = A''' = a' = B'$ and hence $A'' = B'' \supseteq B$.

(e $\rightarrow b \rightarrow g$). Clear.

(g $\rightarrow$ h). Suppose that $0 \neq B \subseteq A''$ and $B \in P(G)$. Then since B is an ideal in the prime ring A'', B is also a prime ring. Then for $0 \neq b \in B$ we have $B' = b' = A''' = A'$ and hence $B = B'' = A''$.

(h $\rightarrow$ i). The map $X \rightarrow X'$ is an antiautomorphism of $P(G)$.

(i $\rightarrow$ a). Suppose (by way of contradiction) $0 \neq a$, $b \in A$ and $a \perp b$. Then $a' \supseteq A'$ and since $b \in a' \setminus A'$ we have $a' \supset A'$ which contradicts the maximality of A'.

A subset S of a semiprime ring R is a *basis* if

(a) S is a maximal disjoint set,

and

(b) each $s \in S$ is basic.

The following properties of a basis $S = \{s_\lambda \mid \lambda \in \Lambda\}$ are clear.

I. *If τ is an automorphism of G the $S\tau$ is a basis.*

II. $\{s_\lambda'' \mid \lambda \in \Lambda\}$ *is the set of all ideals of* R *that all maximal with respect to being prime rings.*

III. $B = \Sigma s_\lambda''$ *is the basic ideal.* B *is independent of the choice of* S *and invariant under all automorphism of* R.

IV. *A basis for* R *contains one and only one (non-zero) element from each* s_λ''.

Theorem 6.2. *For a semiprime ring* R *the following are equivalent:*

1) R *has a basis.*

2) *If* $0 \neq g \in R$ *then* $gRs \neq 0$ *for some basic element* s.

3) $P(R)$ *is atomic.*

4) $0 = \bigcap$ *all annihilator ideals that are also prime ideals.*

5) $X' = 0$ *where* X *is the set join of all the ideals of* R *that are also prime rings.*

Proof. $(1 \to 2)$. This follows from the fact that a basis for R is a maximal disjoint set.

$(2 \to 3)$ If $0 \neq g \in B \in P(G)$ then $0 \neq grs$ for some basic element s and some $r \in R$. In particular $grs \in \langle s \rangle$ and so it is basic. Therefore $B \supseteq (grs)''$ an atom.

$(3 \to 4)$ If $0 \neq g \in R$ then $g'' \supseteq A$ an atom in $P(R)$. Thus A' is a prime ideal and if $g \in A'$ then $A \subseteq g'' \subseteq A''' = A'$, a contradiction.

$(4 \to 5)$ Let $\{C_\lambda \mid \lambda \in \Lambda\}$ be a set of annihilator ideals that are also prime ideals and such that $\bigcap C_\lambda = 0$. By Theorem 6.1 each C_λ' is a prime ring and so $X \supseteq C'$. Then

$$X' \subseteq (\bigcup C')' = \bigcap C = 0 .$$

$(5 \to 1)$ Let $\{A_\lambda \mid \lambda \in \Lambda\}$ be the set of all ideals of G that are maximal with respect to being prime rings. Then

$$X = \bigcup A_\lambda \subseteq \Sigma A_\lambda .$$

For each $\lambda \in \Lambda$ pick $0 \neq a_\lambda \in A_\lambda$. Then $\{a_\lambda \mid \lambda \in \Lambda\}$ is a disjoint set of basic elements. If $x \in R$ and $xRa_\lambda = 0$ for all λ then $x \in a_\lambda' = A_\lambda'$ so $x \in (\bigcup A_\lambda)' = X' = 0$. Therefore $\{a_\lambda \mid \lambda \in \Lambda\}$ is a maximal disjoint set and hence a basis.

Remark. Since $P(G)$ is a Boolean algebra it is atomic iff each proper annihilator ideal is contained in a maximal annihilator ideal. Also, of course, R has a finite basis iff $P(G)$ is finite.

Lemma 6.3. *If* R *is a semiprime ring and* $0 \neq s \in C$ *an ideal of* R *then* s *is basic in* C *iff* s *is basic in* R.

Proof $(\to)$. If s is not basic in R then there exists $0 \neq x, y \in \langle s \rangle \subseteq C$ such that $xRy = 0$. Since C is semiprime $xc_1x \neq 0 \neq yc_2y$ for $c_1, c_2 \in C$. Now $xc_1x, yc_2y \in$

$\in \langle s \rangle_c$, the ideal of C generated by s, and $xc_1xRyc_2y = 0$. Then $xc_1x\langle s \rangle_i yc_2y = 0$ but this contradicts the fact that $\langle s \rangle_c$ is a prime ring.

($\leftarrow$). If s is not basic in C then there exist $0 \neq x, y \in \langle s \rangle_c$ such that $x\langle s \rangle_c y = 0$ and since $\langle s \rangle_c$ is an ideal in the semiprime ring C, $xCy = 0$ and similarly $xRy = 0$ but the means that $\langle s \rangle$ is not a prime ring, a contradiction.

Corollary. *For a semiprime ring R following are equivalent*:

a) *R has a basis.*
b) *$\langle a \rangle$ has a basis for each $0 \neq a \in R$.*
c) *Each proper ideal of R has a basis.*

Proof. (c $\to$ b & c) Consider $0 \neq c \in C$ an ideal. Then $cRs \neq 0$ for some basic element s of R. $0 \neq crs$ is basic in R and belongs to C so it is basic in C.

$$crsCcrs \neq 0 \text{ since } C \text{ is semiprime .}$$

Therefore $cCcrs \neq 0$ so C has a basis.

(b $\to$ a) If $0 \neq a \in R$ then $a\langle a \rangle s \neq 0$ for some basic element s in $\langle a \rangle$. Thus $aRs \neq 0$ and s is basic in R.

(c $\to$ a) Consider $0 \neq g \in R$. If $\langle g \rangle$ is a prime ring then g is basic and $gRg \neq 0$. If $\langle g \rangle$ is not a prime ring the there exist $0 \neq a, b \in \langle g \rangle$ such that $aRb = 0$. Thus $0 \neq a'' \cap \langle g \rangle \subset \langle g \rangle$. Pick $0 \neq c \in C = a'' \cap \langle g \rangle$ then $gRc \neq 0$; otherwise $c \in \in g' \cap g'' = 0$. Thus $yRs \neq 0$ where s is basic in C and hence in R.

Proposition 6.4. *Suppose that R is a semiprime ring and a large left subring of S.*

a) *If $K = \{k_\lambda \mid \lambda \in \Lambda\}$ is a basis for R then it is also a basis for S.*
b) *If S has a basis then so does R.*

Proof. (a) If $0 \neq s \in S$ and $s \perp k_\lambda$ for all λ then pick $x \in R$ such that $0 \neq xs \in R$. Then $xs \perp k_\lambda$ for all λ but this contradicts the fact that K is a maximal disjoint subset of R. Thus K is also a maximal disjoint subset of S. Now k_λ'' is an atom in $P(R)$ and so $(k_\lambda'')^{**} = k_\lambda^{**}$ is an atom in $P(S)$. Thus each k_λ is basic in S and so K is a basic for S.

(b) Suppose that $K = \{k_\lambda \mid \lambda \in \Lambda\}$ is a basis for S. For each λ pick an element $a_\lambda \in R$ such that $0 \neq a_\lambda k_\lambda \in R$. Now $(a_\lambda k_\lambda)^{**} = k_\lambda^{**}$ so $a_\lambda k_\lambda$ is basic in S and since $(a_\lambda k_\lambda)'' = (a_\lambda k_\lambda)^{**} \cap R$ we have that $(a_\lambda k_\lambda)''$ an atom in $P(R)$ and so $a_\lambda k_\lambda$ is basic in R. Since $\{a_\lambda k_\lambda \mid \lambda \in \Lambda\}$ is a basis for S it is a maximal disjoint subset of S hence of R. Then $\{a_\lambda k_\lambda \mid \lambda \in \Lambda\}$ is a basis for R.

Let $S = \{s_\lambda \mid \lambda \in \Lambda\}$ be a basis for the semiprime ring R. Then each R/s_λ' is a prime ring and $\bigcap s_\lambda' = 0$. Thus

$$g \to^{\sigma} (—s_\lambda' + g—)$$

is an isomorphism of R into $K = \Pi R/s_\lambda'$.

Theorem 6.5. $K = (R\sigma)^0$ *and if* S *is finite* $K = (R\sigma)^{\mathrm{P}}$. *In particular* $P(R)$ *is atomic iff* R^0 *is a product of prime rings.*

Proof. Consider $0 \neq x = (-s'_\lambda + x_\lambda -) \in K$ with say $s'_\alpha + x_\alpha \neq s'_\alpha$. Then $0 \neq a = s_\alpha g x_\alpha$ for some $g \in R$ and since $a \in (\bigcap_{\lambda \neq \alpha} s'_\lambda) \setminus s'_\alpha$ we have

$$a\sigma = (0 - 0, s'_\alpha + a, 0 - 0) = ((s_\alpha g)\,\sigma)\,x\,.$$

Thus $R\sigma$ is left large in K and so $R\sigma \subseteq (R\sigma)^{\mathrm{P}} \subseteq K$. We next show that $\overline{s'_\alpha + x_\alpha} = (0 - 0, s'_\alpha + x_\alpha, 0 - 0) \in (R\sigma)^{\mathrm{P}}$ and hence $(R\sigma)^{\mathrm{P}} \supseteq \Sigma R/s'_\lambda$.

Let $*(\#)$ be the annihilator operators in $(R\sigma)^{\mathrm{P}}\,(K)$.

$$(R\sigma)^{\mathrm{P}} = \overline{s'_\alpha + s_\alpha}\,{**}\oplus\,\overline{s'_\alpha + s^*_\alpha} = (s_\alpha\sigma)\,{**}\oplus\,(s_\alpha\sigma)^*\,,$$

$$x_\alpha\sigma = c + d$$

but this is also the decomposition of $x_\alpha\sigma$ in

$$K = \overline{s'_\alpha + s_\alpha}\,\#\#\oplus\,\overline{s'_\alpha + s_\alpha}\,\# \cong R/s'_\alpha \oplus \Pi_{\lambda \neq \alpha} R/s'_\lambda\,.$$

Therefore $C = \overline{s'_\alpha + x_\alpha}\,(R\sigma)^{\mathrm{P}}$.

Now clearly K is the lateral completion $\Sigma R/s'_\lambda$ and hence of $(R\sigma)^{\mathrm{P}}$. Therefore $K = (R\sigma)^0$. If S is finite then $K = \Sigma R/s'_\lambda$ and so $(A\sigma)^{\mathrm{P}} = K$.

Finally if $R \subseteq R^0 = \Pi T_i$ where the T_i are prime rings then R^0 has a basis and so by Proposition 6.4 R has a basis. Thus $P(R)$ is atomic.

Remark. If R is reduced then each s'_λ is completely prime [2] and so the R/s'_λ are integral domains.

Corollary. *If* R *is a semiprime* P*-ring with a basis* $\{s_\lambda \mid \lambda \in \Lambda\}$ *then* $R = s''_\lambda \oplus s'_\lambda$ *for each* $\lambda \in \Lambda$ *and hence there is a natural isomorphism* τ *such that* $\Sigma s''_\lambda \subseteq R\tau \subseteq \Pi s''_\lambda \cong K$. *In particular* $\Pi s''_\lambda$ *is the* 0*-hull of* $R\tau$ *and hence* R *is a* 0*-ring iff* $R\tau = \Pi s''_\lambda$.

We say that a disjoint subset $\{s_\lambda \mid \lambda \in \Lambda\}$ is *bounded* by $x \in R$ if $xRs_\lambda \neq 0$ for each $\lambda \in \Lambda$.

Theorem 6.6. *If* R *is a semiprime ring that satisfies* (F) *each bounded disjoint subset of* R *is finite, then* R *has a basis.*

Proof. It suffices to show that if $0 \neq g \in R$ then $gRs \neq 0$ for some basic element s. If g is basic then let $s = g$. Suppose that g is not basic and hence $\langle g \rangle$ is not a prime ring. Then there exist (non-zero) disjoint elements g_1 and g_2 in $\langle g \rangle$. Now $gRg_1 \neq 0$; otherwise $g_1 \in g' \cap \langle g \rangle = 0$. Thus if g_1 is basic we are done. If not there exist disjoint elements g_{11} and g_{12} in $\langle g_1 \rangle$. Note that $g_{12} \in g_1''$ and $g_1'' \cap g_2'' = 0$ so $g_{12} \perp g_2$. We proceed in the way

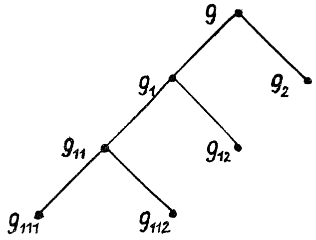

Fig. 1

Since g bounds the disjoint set $g_2, g_{12}, g_{112}, \ldots$ this process must halt.

Corollary I. *R has a basis of n-elements iff R contains n disjoint element but not* $n + 1$ *such elements.*

Proof. ($\rightarrow$). If $a_1, a_2, \ldots, a_{n+1}$ are disjoint then we can find basic elements $s_1, \ldots, s_{n+1}$ and elements $g_1, \ldots, g_{n+1} \in R$ such that $a_1 g_1 s_1, \ldots, a_{n+1}\, g_{n+1}\, s_{n+1}$ are disjoint and basic, a contradiction.

($\leftarrow$). R satisfies (F) and so has a basis that contains at most n-elements. Also we are given a disjoint set $a_1, \ldots, a_n$ so for a suitable choice of basic elements s_i we have

$$a_1 g_1 s_1, \ldots, a_n g_1 s_n$$

are basic and disjoint. So R has a basis of n-elements.

Corollary II. *R has a finite basis iff each disjoint subset of R is finite.*

Proof. ($\rightarrow$) If $a_1, a_2, \ldots$ is a disjoint subset of R then for suitable choices of s_i and g_i.

$$a_1 g_1 s_1,\ a_2 g_2 s_2, \ldots$$

is a set of disjoint basic elements. Thus $a_1, a_2, \ldots$ must be finite.

($\leftarrow$). Since R satisfies (F) it has a basis which must be finite.

Corollary III. *The following are equivalent.*

1) *R satisfies* (F).
2) *Each* $\langle g \rangle$ *has a finite basis.*

Proof. ($1 \rightarrow 2$) Let $a_1, a_2, \ldots$ be a disjoint subset of $\langle g \rangle$. Then $gRa_i \neq 0$ for all i and hence the set is finite. Thus by the last Corollary $\langle g \rangle$ has a finite basis.

(2 → 1) Suppose $s_1, s_2, \ldots$ is a disjoint subset of R and $gRs_i \neq 0$ for all i and a fixed $g \in R$. Then $gr_1s_1, gr_2s_2, \ldots$ is a disjoint subset in $\langle g \rangle$ and so must be finite. Thus the set $s_1, s_2, \ldots,$ is finite

Corollary IV. *For a ring R the following are equivalent.*

1) *R is semiprime and satisfies* (F).
2) *R is a subdirect sum of prime rings.*

Proof. (1 → 2) Let $\{s_\lambda \mid \lambda \in \Lambda\}$ be a basis for R and consider $0 \neq g \in R$. Then $gRs_\lambda = 0$ for all but a finite number of the s_λ and so $g \in s'_\lambda$ for all but a finite number of λ. Now each s'_λ is a prime ideal and

$$g \to (-s'_\lambda + g-)$$

is an isomorphism of R onto a subdirect sum of $\Sigma R/s'_\lambda$.

(2 → 1) Consider $A = \Sigma A_\lambda$ where A_λ are prime rings. Then clearly A satisfies (F). If R is a subdirect sum of ΣA_λ then R is semiprime and each bounded disjoint subset is finite.

Remark. *If R is reduced then each R/s'_λ is an integral domain so R is a subdirect sum of integral domains.*

Theorem 6.7. *A semiprime ring R satisfies* (F) *iff R^{P} is a direct sum of prime rings.*

Proof. (→) By the last Corollary $R \subseteq \Sigma A_i$ when the A_i are prime rings and since $A_i \cap R \neq 0$ for each i it follows that R is left large in ΣA_i. Therefore $R \subseteq R^{\mathrm{P}} \subseteq \Sigma A_i$, but as in the proof of Theorem 6.5 it follows that $R^{\mathrm{P}} \supseteq \Sigma A_i$.

(←) Clearly R^{P} satisfies (F) and hence so does R.

Corollary. *A semiprime ring is a direct sum of prime rings iff it is a* P*-ring that satisfies* (F).

Proposition 6.8. *Suppose that R is a semiprime ring and let*

$$X = \{x \in R \mid x \text{ bounds at most a finite number of disjoint elements}\}.$$

Then X is an ideal that satisfies (F) *and if T is an ideal that satisfies* (F) *then $T \subseteq X$. Let $\{A_\lambda \mid \lambda \in \Lambda\}$ be the set of all ideals of R that are maximal w.r.t being prime rings. Then $\Sigma A_\lambda \subseteq X \subseteq (\Sigma A_\lambda)''$ and ΣA_λ is the basic ideal of X.*

Proof. Consider $x, y \in X$ and suppose that $(x \pm y) Ra_i \neq 0$ for some infinite disjoint set $a_1, a_2, \ldots$. Then an infinite number of the $xRa_i \neq 0$ or an infinite number of the $yRa_i \neq 0$ a contradiction. Thus $(X, +)$ is a group.

If $ryRa_i \neq 0$ then $yRa_i \neq 0$ so $ry \in X$ are similarly $yr \in X$. Thus X is an ideal that satisfies (F).

Now suppose (by way of contradiction) that $x \in X = (\Sigma A_\lambda)''$. Thus $y = xz \neq 0$ for some $z \in (\Sigma A_\lambda)'$ and since $R \supseteq (\Sigma A_\lambda)'' \oplus (\Sigma A_\lambda)'$ it follows that $yA_\lambda = 0$ for all λ. Then $\langle y \rangle$ is not prime and hence there exist $y_1, y_2 \in \langle y \rangle$ such that $y_1 \perp y_2$. Thus we have

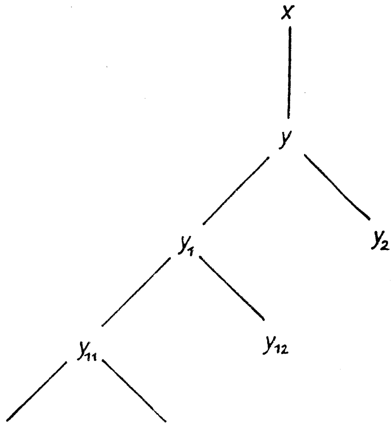

Fig. 2

And hence x bounds the disjoint elements $y_2, y_{12}, \ldots$, a contradiction. Note that if R has a basis then $\Sigma A_\lambda \subseteq X \subseteq (\Sigma A_\lambda)'' = R$ and ΣA_λ is the basis ideal of R.

7. THE RING $\mathscr{P}(G)$ OF ALL p-ENDOMORPHISMS OF A SEMIPRIME RING G

Throughout let G be a semiprime ring.

If G is reduced then $a \geqq b$ iff $ab = b^2$ so each ring endomorphism of G preserves order. In general $a \geqq b$ iff $agb = bgb$ for all $g \in G$, so if α is a ring endomorphism of G and $G\alpha$ is semiprime then $a\alpha \geqq b\beta$ in $G\alpha$ but perhaps not in G. Now

$$a + b \geqq b \quad \text{iff} \quad a \perp b .$$

Thus if α is an endomorphism of the group $(G, +)$ then α preserves order iff α preserves disjointness.

Definition. A *p-endomorphism* of G is an endomorphism α of $(G, +)$ such that for $a, b \in G$

$$a \perp b \quad \text{implies} \quad a\alpha \perp b$$

or equivalently

$$C \in P(G) \quad \text{implies} \quad C\alpha \subseteq C .$$

Proposition 7.1. *The set $\mathscr{P}(G)$ of all p-endomorphisms of G is a ring of order preserving endomorphism of $(G, +)$.*

Proof. Consider $\alpha, \beta \in \mathscr{P}(G)$, $a, b \in G$ and $C \in P(G)$. If $a \perp b$ then $a\alpha \perp \beta$ and hence $a\alpha \perp b\alpha$. Thus α preserves order. Next $C\alpha \subseteq C$ and so $C\alpha\beta \subseteq C\beta \subseteq C$ and hence $\alpha\beta \in \mathscr{P}(G)$. If $aGb = 0$ then $a(\alpha \pm \beta)\, Gb = (a\alpha \pm a\beta)\, Gb \subseteq a\alpha Gb \pm a\beta Gb = 0$. Thus $\alpha \pm b \in P(G)$.

Note that each right multiplication of G is a p-endomorphism

$$x \to xg \quad \text{for all} \quad x \in G \quad \text{and a fixed} \quad g \in C\,.$$

Now we may assume that $G \subseteq \Pi T_i$ where the T_i are prime rings. If $\alpha \in \mathscr{P}(G)$, $a \geqq b$ and $b_i \neq 0$ then $(a\alpha)_i = (b\alpha)_i$. For $a - b \mid b$ and hence $a\alpha - b\alpha \perp b$. Thus if $b_i \neq 0$ the $(a\alpha - b\alpha)_i = 0$.

Lemma 7.2. *If G is an* L*-ring, $\{a_\alpha \mid \alpha \in A\}$ is a disjoint subset of G and $\sigma \in \mathscr{P}(G)$ then*

$$(\bigvee a_\alpha)\, \sigma = \bigvee(a_\alpha \sigma)$$

Proof. Since σ preserve order $(\bigvee a_\alpha)\, \sigma \geqq a_\alpha \sigma$ for each α and hence $(\bigvee a_\alpha)\, \sigma \geqq \bigvee(a_\alpha \sigma)$. Also by the above

$$(a_\alpha)_i \neq 0 \quad \text{implies} \quad ((\bigvee a_\alpha)\, \sigma)_i = (a_\alpha \sigma)_i = (\bigvee(a_\alpha \sigma))_i\,.$$

Now $(\bigvee a_\alpha)\, \sigma + x = \bigvee(a_\alpha \sigma)$ for $x \in G$ and we shall show that $x = 0$. If $(a_\alpha)_i \neq 0$ the $x_i = 0$ so $x \perp a_\alpha$ for all α. Thus $\bigwedge a_\alpha + x \geqq a_\alpha$ for all α and so $\bigvee a_\alpha + x \geqq \bigvee a_\alpha$. But this means that $\bigvee a_\alpha \perp x$ and hence $(\bigvee a_\alpha)\, \sigma \perp x$. Thus it follows that $x = 0$.

Remarks. The proof only uses the existence of $\bigvee a_\alpha$ and $\bigvee(a_\alpha \sigma)$. Note that we have shown that if $x \perp a_\alpha$ for all α the $x \perp \bigvee a_\alpha$. Thus if $\{a_\alpha \mid \alpha \in A\} \subseteq C \in P(G)$ and $\bigvee a_\alpha$ exists then $\bigvee a_\alpha \in C$. Therefore C is closed with respect to joins of disjoint elements.

Corollary. *If $\{a_\alpha \mid \alpha \in A\}$ is a disjoint subset of an* L*-ring G then for each $g \in G$*

$$(\bigvee a_\alpha)\, g = \bigvee(a_\alpha g)\,.$$

Proof. This follows from the fact that $x \to xg$ is a p-endomorphism of G.

Actually one can prove a stronger result. If $\{a_\alpha \mid \alpha \in A\}$ is a subset of a semiprime ring G and $\bigvee a_\alpha$ exists then $\bigvee(a_\alpha g)$ exists and equals $(\bigvee a_\alpha)\, g$. Whether or not the corresponding result holds for any p-endomorphism of G is an open questions.

Theorem 7.3. *Let G be a semiprime ring and let* X = P, SP, L *or* 0.

1) *A p-endomorphism σ of G has a unique extension to a p-endomorphism σ^{X} of G^{X}.*
2) *If σ is $1 - 1$ so is σ^{X}. If σ is onto then so is σ^{X} for* X = P, SP *or* 0.
3) *If α is a p-endomorphism of G^{0} such that $G\alpha \subseteq G$ then $G^{X}\alpha \subseteq G^{X}$.*

The proof is almost identical with the proof of Theorem 4.4 in [5] and so we omit it.

Theorem 7.4. *Suppose that G is a semiprime ring and consider the system $(G^{X}, +, \leqq)$ for* X = P, SP *or* 0. *Then there exists a unique multiplication on G^{X} so that*

a) *G^{X} is a semiprime ring.*

b) *G is a subring of G^{X}, and*

c) *this multiplication on G^{X} induces the given partial order $\leqq$.*

Proof. Note that $a \perp b$ iff $a + b \geqq b$ so we have the concept of disjointness in $(G^{X}, +, \leqq)$. We first verify the result for X = 0. Suppose that $\circ$ is a multiplication of $\mathcal{O}(G)$ that satisfies a), b) and c). We wish to show that this is the natural multiplication in $\mathcal{O}(G)$. The right multiplication of the elements in $\tilde{G}$ by a fixed $\tilde{g} \in \tilde{G}$ is a p-endomorphism of $\tilde{G}$ and hence it has a unique extension to a p-endomorphism of $\mathcal{O}(G)$. Therefore

$$x \circ \tilde{g} = x\tilde{g} \quad \text{for all} \quad x \in \mathcal{O}(G)\,.$$

Thus $(-(x \circ \tilde{g})_{\mathscr{C}}-) = (-(x\tilde{g})_{\mathscr{C}}-)$. In particular if $x_{\mathscr{C}} \neq 0 \neq \tilde{g}_{\mathscr{C}}$ then

$$(x \circ \tilde{g})_{\mathscr{C}} = (x\tilde{g})_{\mathscr{C}} = x_{\mathscr{C}}\tilde{g}_{\mathscr{C}}\,.$$

Suppose that $x_{\mathscr{C}} = (0 - 0, C' + t, 0 - 0)$ where $C' + t \neq C' \neq C' + g$. Then

$$\tilde{g}_{\mathscr{C}} = (0 - 0,\ C' + g,\ 0 - 0) + (\text{the other } \mathscr{C}\text{-components of } g) = a_{\mathscr{C}} + b_{\mathscr{C}}\,.$$

Now let a and b be the element in $\mathcal{O}(G)$ with $\mathscr{C}$-th component $a_{\mathscr{C}}$ and $b_{\mathscr{C}}$. In particular, if $b_{\mathscr{C}} = 0$ then let $b = 0$. Now b and x are disjoint so $x \circ b = 0$. Thus $x \circ a = x \circ (a + b)$ and hence

$$(x \circ a)_{\mathscr{C}} = (x \circ (a + b))_{\mathscr{C}} = (x \circ \tilde{g})_{\mathscr{C}} = (x\tilde{g})_{\mathscr{C}} = $$
$$= (0 - 0,\ C' + tg,\ 0 - 0) = x_{\mathscr{C}}a_{\mathscr{C}}\,.$$

Now consider $x, y \in \mathcal{O}(G)$ with $x_{\mathscr{C}} \neq 0 \neq y_{\mathscr{C}}$. Then

$$x_{\mathscr{C}} = (-C' + x(C)-) = \bigvee x_C\,, \quad \text{where} \quad x_C = (0 - 0,\ C' + x(C),\ 0 - 0)\,,$$
$$y_{\mathscr{C}} = (-C' + y(C)-) = \bigvee y_C\,, \quad \text{where} \quad y_C = (0 - 0,\ C' + y(C),\ 0 - 0)\,.$$

Let $\bar{x}_C(\bar{y}_C)$ be the element in $\mathcal{O}(G)$ with $\mathscr{C}$-th coordinate $x_C(y_C)$ and, in particular, $\bar{x}_C = 0$ if $x_C = 0$ ($\bar{y}_C = 0$ if $y_C = 0$). Then $x = \bigvee \bar{x}_C$ and $y = \bigvee \bar{y}_C$ so

$$x \circ y = (\bigvee \bar{x}_C) \circ (\bigvee \bar{y}_C) = \bigvee (\bar{x}_C \circ \bar{y}_C) = \bigvee \bar{x}_C\bar{y}_C = (\bigvee \bar{x}_C)(\bigvee \bar{y}_C) = xy\,.$$

Therefore $\circ$ is the natural multiplication in $\mathcal{O}(G)$.

An entirely similar proof works for G^{P} and G^{SP} since they are both direct limits.

8. BAER RINGS

There are various definitions of Baer rings in the literature. Kist [9] defines a commutative ring R to be a Baer ring if for each $a \in R$

$$a^* = \{x \in R \mid xa = 0\} = Re$$

for some idempotent e. In particular $R = 0^* = Re$ so the ring has an identity. Also Kist shows that R is semiprime. For if $a^2 = 0$ then $a \in a^* = Re$ and hence $a = ae = 0$. In particular $a^* = a'$.

(1) *If R is a commutative semiprime ring with* 1 *then R is a Baer ring iff R is a* P-*ring.*

MEWBORN [12] defines a commutative ring R to be a Baer ring if for each subset A of R

$$A^* = \{x \in R \mid xA = 0\} = Re$$

for some idempotent e.

(2) *If R is a commutative semiprime ring with* 1 *then R is a Baer ring in the sense of Mewborn iff R is an* SP-*ring.*

KAPLANSKY [8] defines a ring R to be a Baer ring if it satisfies two and hence all three of the following conditions.

(a) If A is a subset of R then $r(A) = \{s \in R \mid As = 0\} = eR$ for some idempotent e.
(b) If A is a subset of R then $l(A) = \{s \in R \mid sA = 0\} = Re$ for some idempotent e.
(c) R has an identity 1.

Note that Mewborn's definition is the commutative version of Kaplansky's.

(3) *If R is a reduced ring with* 1 *then R is a Baer ring in the sence of Kaplansky iff R is an* SP-*ring.*

Proof. Since R is reduced $r(A) = l(A) = A'$ and each idempotent is central.

(4) *Let R be a commutative semiprime ring with* 1. *Then R^{P} is the Baer extension of R constructed by Kist and R^{SP} is the Baer extension of R constructed by Mewborn.*

Finally we note that SPEED [14] has used the direct limit construction of [4] to construct R^{P} and R^{SP} for a commutative semiprime ring with 1 and also various Baer hulls of R that lie between R^{P} and R^{SP}.

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