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DIAGONALS OF CONVEX SETS

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In the present paper the authors introduce a new notion important in the theory of convexity, that of a diagonal of a convex set. This notion forms a natural counterpart of the notion of a face of a convex set.

Let us state now the definition of the *diagonal*. If M is a convex set in a linear space, the convex set D will be called diagonal of M if the following conditions are satisfied:

1° Every extreme point of D is also an extreme point of M;

2° a point $x \in D$ is a relative interior point of D if and only if it is a relative interior point of M.

Instead of setting up a number of superficial generalities about the notion of a diagonal the authors prefer to investigate an important particular case in order to demonstrate the usefulness of the notion by means of deeper results.

In the present paper we restrict ourselves to finite – dimensional spaces; the convex sets to be investigated will be polyhedral cones. Suppose that the extreme rays of a polyhedral cone K are generated by the vectors $p_1, p_2, \ldots, p_s$; these vectors may satisfy relations of the form

$$\sum_{j=1}^{s} \alpha_j p_j = 0 .$$

Connections are first discussed between diagonals of the cone K and relations for the vectors $p_1, \ldots, p_s$. It turns out that an indecomposable cone*) with at least two linearly independent relations has at least three diagonals. Of particular interest are cones with $n + 1$ extreme rays (n being the dimension of the cone) for which there is exactly one nontrivial relation. Such cones are called minimal.

Using the notion of a minimal cone, the authors have shown, for instance, that the cone of all linear operators which transform a given minimal cone into itself may

*) That is, roughly speaking, a cone whose generating vectors p_i cannot be split into two non-void subsets lying in two subspaces which form a direct decomposition of the space.

have extreme rays generated by operators of an arbitrary rank (up to a certain bound depending in a natural manner on the cones) with the exception of rank two. These results are contained in the authors' paper [1].

1. FACES, POLYHEDRAL CONES, DECOMPOSABILITY

Let E be a vector space over the real field. A cone K in E is a set such that $x \in K$ implies $\lambda x \in K$ for all $\lambda \geqq 0$. A cone K is said to be pointed if $K \cap (-K) = (0)$. Under a proper cone we shall always understand one which is convex, pointed and different from the one vector set zero.

Given a cone K, we shall denote by span K the linear space of all vectors of the form $k_1 - k_2$, with $k_i \in K$. We define the dimension of K as the dimension of the linear span K.

Given a pointed cone K in E, it is possible to define a relation $\leqq$ on E as follows

$$\text{we write} \quad x \leqq y \quad \text{if and only if} \quad y - x \in K\,.$$

Since K is pointed, $x \leqq y$ and $y \leqq x$ imply $x = y$. If $x \leqq y$ and $\lambda \geqq 0$ then $\lambda x \leqq \lambda y$. If K is convex, this relation is transitive. If K has an inner point or, more generally, if $K - K = E$, every $x \in E$ may be written in the form $x = x_1 - x_2$ with $x_1 \geqq 0$, $x_2 \geqq 0$.

For sake of completeness, we prove the following proposition.

(1,1) *Let K be a cone, F a subcone of K. Then the following conditions are equivalent*:

1° *if $y \in F$ and $y = x_1 + x_2$ with $x_1 \in K$, $x_2 \in K$ then both $x_1 \in F$, $x_2 \in F$;*

2° *if $y \in F$ and $y = x_1 + \ldots + x_n$ with $x_j \in K$ then all $x_j \in F$;*

3° *if $x \in F$, $z \in K$ and $x - z \in K$ then $z \in F$;*

4° *if $x \in F$, $z \in K$ and $z \leqq x$ then $z \in F$;*

5° *if $x \in K$ may be expressed in the form $x = f - y$ with $f \in F$ and $y \in K$ then $x \in F$.*

If one of these conditions is satisfied, F will be called a face of K.

Proof. The implication $2° \to 1°$ is immediate and $1° \to 2°$ may be proved easily by induction. Now assume 1° and let us prove 3°. If $x \in F$, $z \in K$ and $x - z \in K$, we have the decomposition $x = z + (x - z)$ with $z \in K$, $(x - z) \in K$. It follows from 1° that $z \in F$. The equivalence of 3° and 4° is obvious. Now assume 3° and let us prove 5°. Suppose that $x \in K$ and $x = f - y$ for some $f \in F$ and $y \in K$. Then $x + y \in F$, $x \in K$ and $(x + y) - x \in K$; it follows from 3° that $x \in F$; this establishes 5°. Now suppose that condition 5° is satisfied and consider a $y \in F$ which has a decomposition $y = x_1 + x_2$ with $x_1, x_2 \in K$. Then $x_1 = y - x_2$ with $y \in F$, $x_2 \in K$ whence

$x_1 \in F$. In an analogous manner, the equation $x_2 = y - x_1$ implies $x_2 \in F$. The proof is complete.

(1,2) Definition. Let K be a cone. Suppose that F is a one-dimensional face of K. Then F will be called *an extreme ray* of K. We shall denote by ext K the set of all extreme rays of K. Any nonzero vector in F will be called *an extreme vector of* K.

(1,3) Definition. Let K be a cone. Then K will be called *a polyhedral cone* if it is a proper cone and the set of all extreme rays of K is finite.

(1,4) Definition. A set of vectors $p_1, \ldots, p_s$ is said to be *convex irreducible* if no vector p_i lies in the cone generated by the remaining vectors.

In the sequel we shall adopt the following convention. Given a natural number s and vectors $p_1, \ldots, p_s \in E$, we shall consider the s-dimensional real affine space R_s and the following linear mapping V of R_S into E. If $\alpha^T = (\alpha_1, \ldots, \alpha_s)$ is a vector in R_s, we set

$$V\alpha = \sum_{j \in S} \alpha_j p_j .$$

Here S stands for the set 1, 2, ... s. A relation for the vectors $p_1, \ldots, p_s$ is a vector α such that $V\alpha = 0$. *A full relation* for $p_1, \ldots, p_s$ is a relation α such that all α_j are different from zero.

It is not difficult to prove the following lemma.

(1,5) *Let $p_1, \ldots, p_s$ be given nonzero vectors. Denote by K the set of all vectors of the form $\sum_{j \in S} \alpha_j p_j$ with nonnegative α_j. Then K is a pointed cone if and only if the only nonnegative relation for the vectors p_j is the zero relation. Suppose that K is a pointed cone. Then the following conditions are equivalent:*

1° *the vectors $p_1, \ldots, p_s$ form a convex irreducible set;*

2° *the positive multiples of the vectors $p_1, \ldots, p_s$ are exactly the extreme rays of the cone K;*

3° *any non-zero relation $\sum_{j \in S} \alpha_j p_j = 0$ contains of least two positive and at least two negative coefficients.*

In the present paper we shall frequently make no distinction between the extreme rays of a cone and vectors by which they are generated. There does not seem to be any danger of misunderstanding.

Let K be a polyhedral cone in E with extreme rays generated by the vectors $p_1, \ldots, p_S$. Let S be the set of natural numbers $1, 2, \ldots, s$. For the sake of brevity we shall write $K =$ cone S to describe this situation. If $T \subset S$ we denote by span T

the linear space generated by the vectors p_j, $j \in T$ and by cone T the set of all vectors of the form

$$\sum_{j \in T} \alpha_j p_j$$

with $\alpha_j \geqq 0$.

We shall need the following observation.

(1,6) *Suppose that* $K = \text{cone}\, S$. *Let* M_1 *and* M_2 *be two nonvoid subsets of* S. *If* $\text{cone}\, M_1 = \text{cone}\, M_2$ *then* $M_1 = M_2$. *In particular, the abbreviation* $K = \text{cone}\, S$ *and the notation cone* T *are consistent.*

Proof. According to (1,5) and to the convention regarding the notation cone S there exists a convex irreducible set of vectors p_j, $j \in S$ such that the rays generated by the p_j are exactly the extreme rays of K. Suppose now that $i \in M_1$, it follows that $p_i \in \text{cone}\, M_1 = \text{cone}\, M_2$ so that $p_i = \sum_{j \in M_2} \alpha_j p_j$ for suitable $\alpha_j \geqq 0$. If i does not belong to M_2 the above equation expresses p_i as a convex combination of the remaining p_j which is impossible. This proves $M_1 \subset M_2$. The inclusion $M_2 \subset M_1$ may be proved in the same manner.

We give next a characterization of faces of polyhedral cones in terms of the corresponding index sets.

(1,7) *Let* K *be a polyhedral cone,* $K = \text{cone}\, S$ *and let* $M \subset S$, $F = \text{cone}\, M$. *Then the folowing conditions are equivalent*

1° *F is a face of K;*

2° *if* $\sum_{j \in S} \gamma_j p_j = 0$ *with* $\gamma_j \geqq 0$ *for* $j \in S \setminus M$ *then* $\gamma_j = 0$ *for* $j \in S \setminus M$.

Proof. Suppose first that F is a face of K and that $\sum_{j \in S} \gamma_j p_j = 0$ for some coefficients γ such that $\gamma_j \geqq 0$ for $j \in S \setminus M$. Set $x = \sum_{j \in S \setminus M} \gamma_j p_j$ so that $x \in K$. Also, let $u = \{-\sum \gamma_j p_j;\, j \in M,\, \gamma_j \leqq 0\}$, $v = \{\sum \gamma_j p_j;\, j \in M,\, \gamma_j \geqq 0\}$ so that $x = u - v$. Since both u and v belong to F and F is a face of K, it follows from condition 5° of proposition (1,1), that $x \in F$. According to 2° of the same proposition, $\gamma_j p_j \in F$ for each $j \in S \setminus M$. If $\gamma_j > 0$ for some $j \in S \setminus M$ it follows that the corresponding p_j belongs to F which is a contradiction. Consequently all $\gamma_j = 0$ for $j \in S \setminus M$.

To prove that 2° implies 1°, suppose that an $f \in F$ may be written in the form $f = a + b$ with $a, b \in K$. Thus

$$f = \sum_{j \in M} \varphi_j p_j\,, \quad a = \sum_{j \in S} \alpha_j p_j\,, \quad b = \sum_{j \in S} \beta_j p_j\,,$$

where all coefficients are nonnegative. We have the following relation

$$\sum_{j \in S \setminus M} (\alpha_j + \beta_j)\, p_j + \sum_{j \in M} (\alpha_j + \beta_j - \varphi_j)\, p_j = 0$$

with $\alpha_j + \beta_j \geqq 0$ for $j \in S \setminus M$. If 2° is assumed, it follows that $\alpha_j + \beta_j = 0$ for $j \in S \setminus M$ so that $\alpha_j = \beta_j = 0$ for $j \in S \setminus M$. Hence both $u \in F$, $v \in F$ and 1° is established.

In the remainder of this section we collect some material concerning the notion of indecomposability for polyhedral cones which will be needed in the sequel.

The notion of indecomposability is analogous to that used in the theory of non-negative matrices and has been introduced for cones independently in [2] and [3].

(1,8) Definition. Let E be a linear space over the real field. Let S be a finite set of indices. For each $s \in S$ we are given a vector $p_s \in E$. A non-void subset $M \subset S$ is said *to be minimal* if the vectors p_m, $m \in M$ are linearly dependent but each proper subset is linearly independent. We define a relation R on S as follows: $[i, i] \in R$ for all $i \in S$; if $i \neq j$ then the pair $[i, j] \in R$ if and only if there exists a minimal subset $M \subset S$ such that both i and j belong to S.

(1,9) *If a non-void set $S' \subset S$ has the property that the p_i, $i \in S'$, are linearly dependent then S' contains a minimal subset.*

Proof. The proof is immediate if we consider among all linearly dependent subsets of S' one having the least number of elements.

The system of vectors $\{p_s; s \in S\}$ is said to be *decomposable* if there exists a subset $M_0 \subset S$ different from $\emptyset$ and S such that

$$RM_0 \subset M_0 .$$

Otherwise the system $\{p_s; s \in S\}$ is said to be *indecomposable.*

(1,10) Theorem. *These are equivalent*:

1° *the system $\{p_s; s \in S\}$ is decomposable*;

2° *there exists a non-trivial decomposition $S = M_1 \cup M_2$ such that* span $\{p_s, s \in S\}$ = span $\{p_s, s \in M_1\} \oplus$ span $\{p_s, s \in M_2\}$;

3° *there exists a non-void proper subset $S_0 \subset S$ such that any relation $\sum_{j \in S} \alpha_j p_j = 0$ implies $\sum_{j \in S_0} \alpha_j p_j = 0$*;

4° *there exist numbers λ_s, $s \in S$, not all equal to each other such that*

$$\sum_{j \in S} \alpha_j p_j = 0 \quad \textit{implies} \quad \sum_{j \in S} \alpha_j \lambda_j p_j = 0 .$$

Proof. 1° → 2°: By 1°, there exists a subset $M_0 \subset S$, $\emptyset \neq M_0 \neq S$, such that

$$RM_0 \subset M_0 .$$

Put $M_1 = M_0$, $M_2 = S \setminus M_0$. Assume there exists a vector $z \neq 0$, $z \in$ span $\{p_s, s \in M_1\} \cap$ span $\{p_s, s \in M_2\}$. For $k = 1, 2$ there exists a subset $\tilde{M}_k \subset M_k$ such that the vectors p_i, $i \in \tilde{M}_k$, form a basis for span $\{p_s, s \in M_k\}$. Thus

$$z = \sum_{i \in \tilde{M}_1} \alpha_i p_i ,$$

as well as

$$z = \sum_{i \in \tilde{M}_2} \alpha_i p_i$$

so that the vectors p_i, $i \in \tilde{M}_1 \cup \tilde{M}_2$, are linearly dependent. By (1, 9), there exists a minimal subset $T \subset \tilde{M}_1 \cup \tilde{M}_2$. Clearly $T \not\subset \tilde{M}_1$, $T \not\subset \tilde{M}_2$. Consequently, there exists an index $i \in \tilde{M}_1$ and an index $j \in \tilde{M}_2$ such that $[i, j] \in R$, a contradiction with $RM_0 \subset M_0$. This proves that span $\{p_s, s \in M_1\} \cap$ span $\{p_s, s \in M_2\} = 0$.

$2° \to 3°$. Put $S_0 = M_1$. If $\sum_{j \in S} \alpha_j p_j = 0$ then $\sum_{i \in M_1} \alpha_i p_i + \sum_{i \in M_2} \alpha_i p_i = 0$ implies

$$\sum_{i \in M_1} \alpha_i p_i = \sum_{i \in M_2} \alpha_i p_i = 0 \quad \text{by } 2° .$$

$3° \to 4°$. It suffices to put $\lambda_i = 1$ for $i \in S_0$, $\lambda_i = 0$ otherwise.

$4° \to 1°$. Define $M_0 = \{i; \lambda_i = \lambda_1\}$. Assume there exists an index $i \in M_0$ and an index $j \notin M_0$ such that $[i, j] \in R$. Let T be a minimal subset containing both i and j. Thus

$$\sum_{k \in T} \alpha_k p_k = 0$$

where at least one of the α_k is different from zero. Then

$$\sum_{k \in T} \alpha_k \lambda_k p_k = 0$$

as well. Consequently,

$$\sum_{k \in T \setminus \{i\}} \alpha_k(\lambda_k - \lambda_i)\, p_k = 0 ,$$

which implies, by the minimality of T, that $\lambda_j - \lambda_i = 0$, a contradiction with $j \notin M_0$.

2. DIAGONALS

In this section we introduce the notion of a diagonal. For the sake of simplicity and brevity we shall limit ourselves – throughout this section – to closed cones in finite dimensional spaces. By making obvious changes in the conditions the notion (and some of the results) may be extended to more general situations.

(2,1) Definition. If K is a proper cone we shall denote by rint K (relative interior of K) the set
$\{x \in K;$ for each pair $y_1 \in K$, $y_2 \in K$ there exists an $\varepsilon > 0$, $x - \varepsilon(y_1 - y_2) \in K\}$
and by rb K (relative boundary of K) the set

$$\operatorname{rb} K = K \setminus \operatorname{rint} K .$$

(2,2) Definition. Let K and D be two proper cones in the linear space E. The cone D will be called a *diagonal of* K if the following three conditions are satisfied

1° $\operatorname{ext} D \subset \operatorname{ext} K$;

2° $\operatorname{rint} D \subset \operatorname{rint} K$;

3° $\operatorname{rb} D \subset \operatorname{rb} K$.

It is not difficult to see that condition 3° may be replaced by the equivalent condition

4° $\operatorname{rint} K \cap D \subset \operatorname{rint} D$.

In other words, the extreme rays of D are also extreme rays of K and a vector $x \in D$ is a relative interior vector of D if and only if it is an interior vector of K.

In the rest of this section we shall clear up – for the case of polyhedral cones – the meaning of the notion of a diagonal. To help build up a geometric intuition we include here the following proposition although a part of it cannot be proved until later.

(2.3) *Let K be a proper cone in a linear space E. Then*

1° *K is a diagonal of K;*

2° *if D is a diagonal of K then $D \subset K$;*

3° *if D is a diagonal of K which is different from K then*

$$1 < \dim D < \dim K\,.$$

Proof. The first two assertions are obvious. Now let D be a diagonal of K. Suppose that $\dim D = 1$ so that D is the ray generated by a vector $p \neq 0$. According to condition 1° of the definition, this ray is also an extreme ray of K. Suppose that K has at least one extreme ray different from D; it follows that D is not contained in $\operatorname{rint} K$ so that condition 2° cannot be satisfied. Hence $\dim D = 1$ implies $\dim K = 1$ and, indeed, $D = K$. This proves the inequality $1 < \dim D$ if $D \neq K$. The second inequality is a consequence of 32° in theorem (2,11) to be proved later.

(2,4) *Let K be a proper cone in E. If D is a diagonal of K and H is a diagonal of D then H is a diagonal of K.*

Proof. Obvious.

(2,5) Definition. Let D be a diagonal of the proper cone K. Then D will be called a *proper diagonal of* K if $D \neq K$; it will be called a *minimal diagonal* of K if there exists no diagonal D' of K properly contained in D.

Our first observation consists in showing that we may limit ourselves to the investigation of diagonals of indecomposable cones only.

Indeed, we shall show that the diagonals of decomposable cones can be found easily from the knowledge of the diagonals of the indecomposable components. Let us recall that a cone K is called *decomposable* if the identity operator I of the linear space spanned by K can be written as a sum $I = P_1 + P_2$ of nontrivial mutually orthogonal projectors P_1, P_2 in such a way that

$$K = K_1 \oplus K_2$$

where $K_i = P_i K$, $i = 1, 2$.

Otherwise, K is called *indecomposable*. The following proposition has been proved in [2]:

(2,6) *Any two-dimensional proper cone is decomposable. For any cone K, there exist indecomposable cones $K_1, \ldots, K_r$ such that*

$$K = K_1 \oplus K_2 \oplus \ldots \oplus K_r .$$

This decomposition of K is unique (except for a possible renumbering).

The following lemma is easily checked.

(2,7) Lemma. *Let K be a decomposable cone, $K = K_1 \oplus K_2 \oplus \ldots \oplus K_r$ its decomposition as a sum of indecomposable cones K_i $i = 1, \ldots, r - 1$.*

Then

1° $\operatorname{rint} K = \operatorname{rint} K_1 \oplus \operatorname{rint} K_2 \oplus \ldots \oplus \operatorname{rint} K_r$,

2° $\operatorname{ext} K = \operatorname{ext} K_1 \cup \operatorname{ext} K_2 \cup \ldots \cup \operatorname{ext} K_r$,

3° $\operatorname{rb} K = \operatorname{rb} K_1 \oplus K_2 \oplus \ldots \oplus K_r \cup K_1 \oplus \operatorname{rb} K_2 \oplus \ldots \oplus K_r \cup \ldots$
$\ldots \cup K_1 \oplus K_2 \oplus \ldots \oplus \operatorname{rb} K_r$.

(2,8) Theorem. *Let K be a decomposable cone, $K = K_1 \oplus K_2 \oplus \ldots \oplus K_r$ its decomposition as a sum of indecomposable cones K_i, $i = 1, \ldots, r$. Then D is a diagonal of K if and only if*

$$D = D_1 \oplus D_2 \oplus \ldots \oplus D_r$$

where all D_i, $i = 1, \ldots, r$ are diagonals of K_i.

Proof. First let D_i be a diagonal of K_i for $i = 1, \ldots, r$, and let

$$D = D_1 \oplus D_2 \oplus \ldots \oplus D_r .$$

By 1°,

$$\operatorname{rint} D = \sum_{i=1}^{r} \operatorname{rint} D_i \subset \sum_{i=1}^{r} \operatorname{rint} K_i = \operatorname{rint} K .$$

By 2°,

$$\operatorname{ext} D = \bigcup_i \operatorname{ext} D_i \subset \bigcup_i \operatorname{ext} K_i = \operatorname{ext} K .$$

By 3°,

$$\operatorname{rb} D = \bigcup_{j=1}^{r} (\operatorname{rb} D_j + \sum_{k \neq j} D_k) \subset \bigcup_{j=1}^{r} (\operatorname{rb} K_j + \sum_{k \neq j} K_k) = \operatorname{rb} K .$$

Consequently, D is a diagonal of K.

To prove the converse part, suppose D is a diagonal of K. Let P_i, $i = 1, \ldots, r$, be projectors of the space spanned by K on the subspaces spanned by K_i so that P_i form an orthogonal complete system of projectors and

$$\sum_{i=1}^{r} P_i = I ,$$

the identity.

Define $D_i = P_i D$, $i = 1, \ldots, r$. We have then clearly

$$D = D_1 \oplus D_2 \oplus \ldots \oplus D_r .$$

Let us first show that $D_i \neq 0$ for $i = 1, \ldots, r$. Assume, say, $D_1 = 0$. Then

$$D = (I - P_1)\, D \subset (I - P_1)\, K = K_2 + \ldots + K_r \subset \operatorname{rb} K .$$

Since

$$\operatorname{rint} D \subset \operatorname{rint} K ;$$

we have

$$\operatorname{rint} D = \operatorname{rint} D \cap D \subset \operatorname{rint} K \cap \operatorname{rb} K = \emptyset ,$$

a contradiction with $D \neq 0$.

Let us show now that for $i = 1, \ldots, r$, D_i is a diagonal of K_i.

By 1° of lemma (2,7)

$$\operatorname{rint} D = \sum_{i=1}^{r} \operatorname{rint} D_i .$$

Since

$$\operatorname{rint} D \subset \operatorname{rint} K = \sum_{i=1}^{r} \operatorname{rint} K_i ,$$

it follows that

$$(*) \qquad \operatorname{rint} D_i \subset \operatorname{rint} K_i , \quad i = 1, \ldots, r .$$

Let now $x \in \operatorname{ext} D_i$ for a fixed i so that $x \neq 0$. By 2°, $x \in \operatorname{ext} D$ so that $x \in \operatorname{ext} K$. Since $x = P_i x$, it follows that $x \in P_i \operatorname{ext} K = \operatorname{ext} K_i \cup 0$. Consequently, $x \in \operatorname{ext} K_i$ so that

$$(**) \qquad \operatorname{ext} D_i \subset \operatorname{ext} K_i .$$

Assume that $0 \neq x_i \in \operatorname{rb} D_i$ for a fixed i. Since $\operatorname{rint} D_k \neq \emptyset$ for all k, there exist vectors x_j, $j = 1, \ldots, r$, $j \neq i$, such that

$$x_j \in \operatorname{rint} D_j .$$

The vector

$$x = \sum_{t=1}^{r} x_t \in \operatorname{rb} D$$

by (**); therefore, $x \in \operatorname{rb} K$. By (*),

$$x_j \in \operatorname{rint} K_j \quad \text{for} \quad j \neq i .$$

Consequently,

$$x \notin \bigcup_{j \neq i} (\operatorname{rb} K_j + \sum_{s \neq i} K_s) ;$$

by (*),

$$x \in \operatorname{rb} K_i + \sum_{j \neq i} K_j$$

so that

$$x_i = P_i x \in \operatorname{rb} K_i$$

and

$$\operatorname{rb} D_i \subset \operatorname{rb} K_i .$$

The proof is complete.

In the rest of this section we intend to describe diagonals of polyhedral cones in terms of the index sets of their extreme rays and discuss the connection between diagonals and relations. We shall need several lemmas.

(2,9). *Let K be a polyhedral cone, $K = \operatorname{cone} S$. Let $M \subset S$. Then these are equivalent*:

1° $\operatorname{cone} S \cap \operatorname{span} M \subset \operatorname{cone} M$;

2° $\operatorname{cone} (S \setminus M) \cap \operatorname{span} M \subset \operatorname{cone} M$;

3° *whenever $\sum_{j \in S} \gamma_j p_j = 0$ is a relation such that $\gamma_j \geqq 0$ for $j \in S \setminus M$ then there exists a relation*

$$\sum_{j \in S \setminus M} \gamma_j p_j + \sum_{j \in M} \omega_j p_j = 0$$

with

$$\omega_j \leqq 0 \quad \text{for} \quad j \in M .$$

4° *The order relation defined on* span M *by* cone M *coincides with that induced on* span M *by the order relation defined by* cone S.

Proof. May be left to the reader.

(2,10) *Let K be a polyhedral cone, $K = \text{cone } S$. Let $M \subset S$ and set $D = \text{cone } M$. Then the following two conditions are equivalent*:

1° $\text{rint } D \subset \text{rint } K$;

2° *there exists a relation* $\sum_{j \in S} \gamma_j p_j = 0$ *such that* $\gamma_j > 0$ *for* $j \in S \setminus M$.

Proof. To prove that 1° implies 2°, set $x = \sum_{j \in M} p_j$ so that $x \in \text{rint } D$. Since $\text{rint } D \subset \text{rint } K$, there exists a representation of x

$$x = \sum_{j \in S} \beta_j p_j$$

with all $\beta_j > 0$. It follows that

$$\sum_{j \in S \setminus M} \beta_j p_j + \sum_{j \in M} (\beta_j - 1) p_i = 0$$

is a relation with positive coefficients β_j for $j \in S \setminus M$.

Suppose now that 2° holds. If $x \in \text{rint } D$ then $x = \sum_{i \in M} \beta_i p$ with all β_i positive. It follows that there exists an $\varepsilon > 0$ such that $\beta_i + \varepsilon \gamma_i > 0$ for all $i \in M$. Then

$$x = \sum_{j \in M} \beta_j p_j + \varepsilon \sum_{j \in S} \gamma_j p_j$$

shows that $x \in \text{rint } K$ as well. The proof is complete.

(2,11) Theorem. *Let K be a polyhedral cone, $K = \text{cone } S$. Let M be a non-void subset of S and let $D = \text{cone } M$. Then the following conditions are equivalent*:

1° *D is a proper diagonal of K*;

2° *the following three conditions are satisfied*:

21° $\text{rint } D \subset \text{rint } K$,

22° $\text{rint } K \cap D \subset \text{rint } D$;

23° $M \neq S$;

3° *the following three conditions are satisfied*:

31° $\text{rint } D \subset \text{rint } K$;

32° $K \cap \text{span } M \subset D$;

33° $M \neq S$;

4° *the following three conditions are satisfied*

41° *there exists a relation* $\sum_{i \in S} \alpha_i p_i = 0$ *such that* $\alpha_i > 0$ *for all* $i \in S \setminus M$;

42° *whenever* $\sum_{i \in S} \alpha_i p_i = 0$ *with* $\alpha_i > 0$ *for all* $i \in S \setminus M$ *then there exists a relation* $\sum_{i \in S} \alpha'_i p_i = 0$ *such that* $\alpha_i = \alpha'_i$ *for* $i \in S \setminus M$ *and* $\alpha'_i \leqq 0$ *for* $i \in M$;

43° $M \neq S$.

Another set of four equivalent conditions may be obtained if we leave out the word proper in 1° and the third condition in each of 2°, 3°, 4°.

Proof. By the definition of a proper diagonal 1° and 2° are easily seen to be equivalent.

Let us show now that 2° implies 3°. Since 21° implies 31° and 23° implies 33°, it remains to prove 32°. Let $x \in K \cap \operatorname{span} M$ and suppose $x \notin D$. By 23°, there exists a $y \in \operatorname{rint} D$. Since $x \in \operatorname{span} M$ and $x \notin D$, there is a vector z of the form $z = \alpha x + \beta y$, $\alpha > 0$, $\beta > 0$, $\alpha + \beta = 1$ such that $z \notin \operatorname{rb} D$. However, $y \in \operatorname{rint} K$ by 21°, $x \in K$ so that $z \in \operatorname{rint} K$. By 22°, $z \in \operatorname{rint} D$, a contradiction.

To prove that 3° implies 4°, we observe that 33° implies 43° trivially and that 31° implies 41° by (2,10). Hence it remains to prove 42°. Let $\sum_{i \in S} \alpha_i p_i = 0$ and suppose that $\alpha_i > 0$ for all $i \in S \setminus M$. The vector $x = \sum_{j \in S \setminus M} \alpha_j p_j$ belongs to K and, since $x = -\sum_{k \in M} \alpha_k p_k$, $x \in \operatorname{span} M$ as well. It follows from 32° that $x \in D$ so that $x = \sum_{i \in M} \gamma_i p_i$ with $\gamma_i \geqq 0$ for $i \in M$. Hence

$$\sum_{j \in S \setminus M} \alpha_j p_j - \sum_{j \in M} \gamma_j p_j = 0$$

is a relation which satisfies the requirement of 42°.

To complete the proof, we shall show that 4° implies 2°. By (2,10), 41° implies 21°. Now let 42° be satisfied and suppose that $x \in \operatorname{rint} K \cap D$. Thus, $x = \sum_{j \in S} \beta_j p_j$, $\beta_j > 0$ as well as $x = \sum_{j \in M} \gamma_j p_j$. Consequently,

$$\sum_{j \in S} \beta_j p_j - \sum_{j \in M} \gamma_j p_j = 0$$

is a relation which is positive on $S \setminus M$. By 42°, there exists a relation

$$\sum_{j \in S} \alpha'_j p_j = 0$$

such that $\alpha'_i = \beta_i$ for $i \in S \setminus M$ and $\alpha'_i \leqq 0$ for $i \in M$. Therefore,

$$x = \sum_{j \in S} \beta_j p_j - \sum_{i \in S} \alpha'_i p_j = \sum_{i \in M} (\beta_i - \alpha'_i)\, p_i$$

shows that $x \in \operatorname{rint} D$ since $\beta_i - \alpha'_i > 0$ for $i \in M$. This proves 22°. Since 23° follows from 43°, the proof is complete.

The rest is proved analogously.

(2.12) *Let K be an indecomposable polyhedral cone of dimension greater than one. Let $D = \operatorname{cone} M$ be a proper diagonal of K. Then $S \setminus M$ contains at least two indices.*

Proof. According to (2,11), condition 41° and 42°, there exists a relation the set of positive coefficients of which corresponds to indices in $S \setminus M$. By 3° of (1,5), the number of these coefficients is at least two.

(2,13) *Suppose that K = cone S is indecomposable. Then there exists, for each $k \in S$, a relation $\sum_{i\in S} \alpha_i p_i = 0$ for the vectors $p_1, \ldots, p_s$ such that*

$$\alpha_k > 0 .$$

Proof. Suppose, on the contrary, that every relation $\sum_{i\in S} \alpha_i p_i = 0$ for the vectors $p_1, \ldots, p_s$ has $\alpha_k = 0$ and let us prove that span $\{k\}$ and span $(S \setminus \{k\})$ form a direct decomposition of the space E for which both projections are positive. First of all, let us show that

$$\operatorname{span}\{k\} \cap \operatorname{span}(S \setminus \{k\}) = (0) .$$

If x belongs to this intersection, we have $x = \lambda p_k$ and $x = \sum_{j\in S\setminus\{k\}} \gamma_j p_j$. Hence $\sum_{j\in S\setminus\{k\}} \gamma_j p_j - \lambda p_k = 0$ is a relation for the vectors p_j, $j \in S$. According to our assumption we have $\lambda = 0$ so that $x = 0$.

Denote by P_1 and P_2 the decomposition projections on span $\{k\}$ and span $(S \setminus \{k\})$ respectively. We show first that P_2 is nonnegative. To see that, it suffices to prove that if $z \in K$, $z = \sum_{j\in S} \beta_j p_j$ then $\sum_{\substack{j\in S\\ j\neq k}} \beta_j p_j$ again belongs to K. Hence suppose that $\sum_{j\in S} \beta_j p_j = \sum_{j\in S} \alpha_j p_j$ with $\alpha_j \geqq 0$. It follows from assumption that $\beta_k = \alpha_k$ whence

$$\sum_{\substack{j\in S\\ j\neq k}} \beta_j p_j = \sum_{\substack{j\in S\\ j\neq k}} \alpha_j p_j \in K .$$

This proves that P_2 is nonnegative.

To prove that P_1 is nonnegative consider a point $z \in K$; if z is expressed in the form $z = \lambda_k p_k + \sum_{j\in S\setminus k} \lambda_j p_j$ we are to show that $\lambda_k \geqq 0$. Since $z \in K$ it also has a representation of the form

$$z = \alpha_k p_k + \sum_{j\in S\setminus k} \alpha_j p_j$$

with all coefficients nonnegative. Now $\lambda_k p_k = P_1 z = P_1(\sum \alpha_r p_r) = \alpha_k p_k$. Thus $\lambda_k \geqq 0$.

The proof is complete.

(2,14) *Let K be an indecomposable polyhedral cone of dimension* >1 *generated by the vectors $p_1, p_2, \ldots, p_s$. Then there exists at least one full relation for the vectors p_i.*

Proof. Since K is indecomposable, K is not simplicial. Accordingly, there exists at least one relation

$$\alpha_1 p_1 + \ldots + \alpha_r p_r = 0$$

such that at least one of the α_j is different from zero. Suppose that the number of nonzero coefficients in this relation is maximal and that at least one of them, α_k say, equals zero. It follows from lemma (2,13) that there exists at least one relation

$$\beta_1 p_1 + \ldots + \beta_r p_r = 0$$

with $\beta_k \neq 0$. If ε is chosen small enough, the relation $\sum(\alpha_i + \varepsilon\beta_i)\, p_i = 0$ will have nonzero coefficients for all i where $\alpha_i \neq 0$ and for the index k as well. This is a contradiction with the maximality of the α's. It follows that the relation $\sum \alpha_i p_i = 0$ is full.

(2,15) Notation. Suppose that p_s, $s \in S$ is a convex irreducible set of vectors. Let r be a relation for the vectors p_s. Let us denote by α_j, $j \in S$ the coefficients of the relation r so that $\sum_{j \in S} \alpha_j p_j = 0$. We shall denote by $p(r)$ the set of all indices $i \in S$ for which $\alpha_i > 0$. Clearly $p(r)$ is nonvoid and different from S.

(2,16) *Let K be an indecomposable polyhedral cone of dimension greater than one. Then there exists at least one proper diagonal D of K.*

More precisely: Let $K =$ cone S and let r be any nontrivial relation for the vectors p_s, $s \in S$ (such a relation exists since K is indecomposable and its dimension is greater than one). Then there exists a diagonal $D =$ cone M such that $M \subset \subset S - p(r)$.

Proof. Denote by $\mathscr{T}$ the family of all sets of the form $p(r')$ such that r' is a relation and $p(r') \supset p(r)$. Let $p(r'')$ be a maximal element of $\mathscr{T}$. Let r''' be a relation such that

$$p(r''') = p(r'')$$

and that the number of negative coefficients of r''' is maximal. Let us show that the coefficients of r''' are all different from zero. Suppose on the contrary that there exists an index k such that $r'''_k = 0$. By lemma (2,14) there exists a relation $r^{(0)}$ such that $r^{(0)}_k > 0$. Consider now the relation $r''' - \varepsilon r^{(0)}$ for a small $\varepsilon > 0$. If ε is small enough, $r''' - \varepsilon r^{(0)}$ will stay positive on $p(r''')$ and will stay negative where $p(r''')$ was negative. Now $p(r''' - r^{(0)}) = p(r''') = p(r'')$ because of the maximality of $p(r'')$. The remaining coefficients will be nonpositive; they will be negative where r''' was negative and for the index k, we shall have another negative coefficient. This is a contradiction. Hence $p(-r''') = S \setminus p(r''')$. Set $M = S \setminus p(r''')$ so that $M = S \setminus p(r''') \subset S \setminus p(r)$. Let $D =$ cone M and let us show that D is a diagonal of K. We shall do that using condition 4° of (2,11). Since $S \setminus M = p(r''')$ condition 41° is satisfied trivially. To prove 42°, consider a relation $\tilde{r}$ such that $p(\tilde{r}) \supset S \setminus M = p(r''')$. It follows from the maximality of $p(r''')$ that $p(\tilde{r}) = p(r''')$ so that 42° holds. Since $p(r)$ is nonvoid, we have also $M \neq S$. Hence D is a proper diagonal of K.

(2,17) Corollary. *A cone K has no proper diagonal if and only if it is simplicial; or, in other words, if and only if there exist n linearly independent vectors $p_1, \ldots, p_n$ such that the cone consists of all elements of the form $\sum_{j=1}^{n} \alpha_j p_j$ with nonnegative α_j.*

Proof. According to (2.6) the cone K may be written in the form of a direct sum

$$K = K_1 \oplus \ldots \oplus K_r$$

with indecomposable cones $K_i = P_i K$ where P_i are the projection operators of the direct sum. Suppose first that K is simplicial; then $r = n$ and each K_j is one dimensional. If D is a diagonal of K then the projections $P_i D$ are diagonals of K_i hence $P_j D = K_j$ for all j so that $D = K$.

Conversely, suppose that K has no proper diagonal. It follows from (2,8) that none of the K_j can have a proper diagonal. Since the K_j are indecomposable, it follows from the preceding lemma (2,16) that $\dim K_j = 1$ for all j. This proves that K is simplicial.

(2,18) Theorem. *An indecomposable cone of dimension greater than one has at least two proper diagonals.*

The same conclusion holds for a decomposable cone provided at least one of its indecomposable components has dimension greater than one.

Proof. Let $K = \text{cone } S$ be an indecomposable cone of dimension greater than one. By lemma (2,14) there exists a full relation for the vectors p_i, $i \in S$. If we apply lemma (2,16) to the relations r and $-r$, we obtain two different proper diagonals of K. The rest follows from theorem (2,8).

(2,19) Theorem. *Let K be polyhedral cone, $K =$ cone S. Let $M' \subset M \subset S$, let* cone M *be a diagonal of K. Then the following are equivalent*:

1° cone M' *is a diagonal of* cone M,

2° cone M' *is a diagonal of K.*

Proof. The implication 1° → 2° follows from the definition. Suppose now that 2° is fulfilled. If $M = M'$ or $M = S$, the assertion is trivial. Let thus $K \neq \text{cone } M \neq \text{cone } M'$. By 21° and 22° of (2,11)

$$\text{rint cone } M' \subset \text{rint } K\,,$$

$$\text{rint } K \cap \text{cone } M = \text{rint cone } M\,.$$

Since also

$$\text{rint cone } M' \subset \text{rint } K\,,$$

and

$$\text{rint cone } M' \subset \text{cone } M\,,$$

we have

$$\text{rint cone } M' \subset \text{rint } K \cap \text{cone } M \subset \text{rint cone } M\,.$$

Furthermore

$$\text{rint cone } M \cap \text{cone } M' \subset \text{rint } K \cap \text{span } M' \subset \text{rint cone } M' .$$

By (2,11), cone M' is a proper diagonal of cone M and the proof is complete.

Remark. It follows from theorem (2,19) that the set of all diagonals of cone S possesses the following property analogous to a property of the set of all faces:

if both cone M_1 *and* cone M_2 *are diagonals of* cone S *then* cone M_1 *is a diagonal of* cone M_2 *if and only if* $M_1 \subset M_2$.

(2,20) *Let K be a polyhedral cone, $K =$ cone S, let M be a non-void subset of S, $D =$ cone M. Then the following conditions are equivalent*:

1° *D is a minimal diagonal of K*;

2° *D is a diagonal of K which is a simplicial cone*;

3° *the following two conditions are satisfied*:

31° *there exists a relation* $\sum_{i \in S} \alpha_i p_i = 0$ *such that* $\alpha_i > 0$ *for all* $i \in S \setminus M$;

32° *whenever* $\sum_{i \in S} \alpha_i p_i = 0$ *is a relation such that* $\alpha_i > 0$ *for all* $i \in S \setminus M$ *then* $\alpha_k \leqq 0$ *for all* $k \in M$.

If K is indecomposable, then these conditions are also equivalent to the following condition:

4° *the following two conditions are satisfied*:

41° *there exists a relation* $\sum_{i \in S} \alpha_i p_i = 0$ *such that* $\alpha_i > 0$ *for all* $i \in S \setminus M$;

42° *whenever* $\sum_{i \in S} \alpha_i p_i = 0$ *is a relation such that* $\alpha_i > 0$ *for all* $i \in S \setminus M$ *then* $\alpha_k < 0$ *for all* $k \in M$.

Proof. The implication 1° → 2° follows from corollary (2,17) and from Theorem (2,19).

Now assume 2° and let us prove 3°. Condition 31° of the present theorem follows immediately from condition 41° of Theorem (2,11). To prove 32°, let us observe that according to 42° of (2,11) every relation $\sum_{i \in S} \alpha_i p_i = 0$ which is positive on $S \setminus M$ may be completed by nonpositive numbers on M. Since cone M is simplicial, the set of coefficients α_i on M is unique. This proves condition 3°.

Let us show now that 3° implies 1°. First of all, using 4° of (2,11) it is easy to see that D is a diagonal of K (for $M = S$ this is obvious, for $M \neq S$ we use condition 4°). Suppose now that $D' =$ cone $M' \subset D$ is a diagonal of K. By 41° of (2,11) there exists a relation $\sum_{j \in S} \beta_j p_j = 0$ such that $\beta_j > 0$ for $j \in S \setminus M'$. Since $S \setminus M' \supset S \setminus M$, we have $\beta_k \leqq 0$ for $k \in M$. Hence $M' = M$.

This establishes the equivalence of the first three conditions.

To complete the proof, observe that $4° \to 3°$ immediately. Let us now prove the implication $3° \to 4°$ under the assumption of indecomposability. Let $\sum_{i \in S} \sigma_i p_i = 0$ be a relation with $\sigma_j > 0$ for $j \in S \setminus M$. By 3°, we have $\sigma_j \leqq 0$ for $j \in M$. Now suppose that $\sigma_m = 0$ for some $m \in M$. According to (2,13) there exists a relation $\sum_{i \in S} \beta_i p_i = 0$ such that $\beta_m < 0$. Consider the relation $\tau = \sigma - \varepsilon\beta$. If $\varepsilon > 0$ is small enough, we shall have $\tau_j > 0$ for $j \in S \setminus M$. At the same time $\tau_m > 0$. This is a contradiction, since, again by 3°, $\tau_j > 0$ for $j \in S \setminus M$ implies $\tau_j \leqq 0$ for $j \in M$.

(2,21) Definition. Let F be a family of relations. Then $M \subset S$ is called *maximal with respect to F* if the following two conditions are satisfied:

1° $M = p(r_0)$ for some $r_0 \in F$;

2° whenever $r \in F$ and $p(r) \supset M$ then $p(r) = M$.

(2,22) *Let* $K = \text{cone}\, S$ *be an indecomposable polyhedral cone of dimension greater than one. Let* $S_1 \subset S$. *Denote by F the set of all relations such that* $p(r) \supset S_1$ *and by F′ the set of all full relations r such that* $p(r) \supset S_1$. *Then M is maximal with respect to F if and only if M is maximal with respect to F′.*

Proof. Let us show first that every subset M maximal with respect to F is also maximal with respect to F'. Let M be maximal with respect to F. There exists a relation $r_0 \in F$ such that $p(r_0) = M$. By lemma (2,14), there exists a full relation r'. If $\varepsilon > 0$ is small enough, the relation $r_1 = r_0 + \varepsilon r'$ belongs to F' and $p(r_1) \supset M$. Thus $r_1 \in F$ so that $p(r_1) = M$. We have found a relation $r_1 \in F'$ such that $p(r_1) = M$. Let now $r \in F'$ and $p(r) \supset M$. Then $r \in F$ so that $p(r) = M$ by the maximality of M with respect to F.

To prove the converse, let M be maximal with respect to F'. Since $M = p(r_0)$ for some $r_0 \in F' \subset F$, 1° is fulfilled. Let now $r \in F$ be such that $p(r) \supset M$. If r' is a full relation, there exists sufficiently small $\varepsilon > 0$ such that $r_1 = r + \varepsilon r'$ is in F' and satisfies $p(r_1) \supset p(r)$. By maximality, $p(r_1) = M$ so that $p(r) = M$ as well. The proof is complete.

(2,23) *Let* $K = \text{cone}\, S$ *be an indecomposable polyhedral cone of dimension greater than one. Let* r_0 *be a nontrivial relation,* $p(r_0) = S_1$.

Let F be the family of all relations r′ such that $p(r') \supset S_1$. *Let* $Q \subset S$ *be maximal with respect to F. Then* $\text{cone}\,(S \setminus Q)$ *is a minimal diagonal of K.*

Proof. Follows immediately from (2,20) and (2,4).

(2,24) Theorem. *Let K be an indecomposable polyhedral cone. Suppose that the vectors* p_j *satisfy at least two linearly independent relations. Then K has at least three different minimal diagonals.*

Proof. Since there are at least two linearly independent relations, $\dim K > 1$. Let r_1 be a relation for the vectors p_j such that the set $p(r_1)$ is maximal. According to the preceding observation (2,22) we may assume that r_1 is full. Set $M_1 = S \setminus p(r_1)$. It follows from (2,20) that $D_1 = \text{cone } M_1$ is a minimal diagonal of $K = \text{cone } S$. Denote by F the family of all full relations r such that $p(r) \supset M_1$. Since $-r_1 \in F$ there exists a full relation r_2 such that the set $p(r_2)$ is maximal in the family F. Set $M_2 = S \setminus p(r_2)$ and $D_2 = \text{cone } M_2$. Hence D_2 is a minimal diagonal according to (2,20). Since $M_1 \cap M_2 = M_1 \cap S \setminus p(r_2) \subset p(r_2) \cap (S - p(r_2)) = 0$ the diagonals D_1 and D_2 are different from each other by (1,6). Let us show now that it is possible to assume that r_1 and r_2 are linearly independent. Indeed, if r_2 is a multiple of r_1 we consider $r_2 + \varepsilon\tilde{r}$ where $\tilde{r}$ is linearly independent of r_1. If ε is small enough we shall have $p(r_2 + \varepsilon\tilde{r}) = p(r_2)$ and $p(-(r_2 + \varepsilon\tilde{r})) = p(-r_2)$ hence $r_2 + \varepsilon\tilde{r}$ is a full relation. Therefore we shall consider the following situation: r_1 and r_2 are two linearly independent full relations such that $M_1 = S \setminus p(r_1)$ and $M_2 = S \setminus p(r_2)$ are disjoint.

Denote by $r(\lambda)$ the relation

$$r(\lambda) = r_1 + \lambda r_2 \, .$$

Since r_2 is full there exists only a finite number of values of λ such that at least one coordinate of $r(\lambda)$ is zero. Since M_1 and M_2 are disjoint there exists a positive λ and an index $k \in M_1$ such that $r(\lambda)_k = 0$. Let λ_0 be the minimal positive λ such that at least one coordinate of $r(\lambda)$ is zero. Let Z be the set of all $j \in S$ such that $r(\lambda_0)_j = 0$. We have $Z \subset M_1 \cup M_2$ since both relations r_1 and r_2 are positive on $S \setminus (M_1 \cup M_2) = (S \setminus M_1) \cap (S \setminus M_2) = p(r_1) \cap p(r_2)$. Let us prove now that

$$(\alpha) \qquad Z \cap M_2 \quad \text{is nonvoid} \, .$$

Suppose, on the contrary that $Z \cap M_2 = 0$. Then $Z \subset M_1$. Since λ_0 is minimal we have $r(\lambda_0)_j > 0$ for $j \in S \setminus M_1$ and $r(\lambda_0)_j = 0$ for some $j \in M_1$. This is impossible according to 42° of (2,20).

Now we intend to prove that

$$(\beta) \qquad M_1 \setminus Z \quad \text{is nonvoid} \, .$$

Suppose, on the contrary, that $M_1 \subset Z$. Since $S \setminus Z \subset S \setminus M_1$ and $r(0) = r_1$ is positive on $S \setminus M_1$, it follows from the minimality of λ_0 that $r(\lambda_0)_j > 0$ for $j \in S \setminus Z$ and $r(\lambda_0)_j = 0$ for $j \in Z$ which is impossible. This proves (β).

Summing up (α) and (β) we see that there exists $j \in M_2$ with $r(\lambda_0)_j = 0$ and an index $k \in M_1$ such that $r(\lambda_0)_k < 0$. If $\varepsilon > 0$ is small enough the relation $r(\lambda_0 + \varepsilon)$ will satisfy

$$r(\lambda_0 + \varepsilon)_k < 0 \, , \quad r(\lambda_0 + \varepsilon)_j < 0 \, .$$

Let $\tilde{M}$ be the set of all indices i such that $r(\lambda_0 + \varepsilon)_i < 0$ and F'' the family of all relations r such that $p(r) \supset \tilde{M}$. Since $-r(\lambda_0 + \varepsilon) \in F''$, F'' is non-void. Let r_3 be a relation for which $p(r)$ is maximal in the class F''. Set $M_3 = S \setminus p(r_3)$ and $D_3 =$ $=$ cone M_3 so that D_3 is a minimal diagonal according to (2,23).

Since $k \in M_1$ and $k \in \tilde{M} \subset p(r_3) = S \setminus M_3$ it follows that D_3 is different from D_1. Since $j \in M_2$ and $j \in \tilde{M} \subset S \setminus M_3$ it follows that D_3 is different from D_2. The proof is complete.

(2,25) Theorem. *Let K be an n-dimensional cone, $K =$ cone S. Then these are equivalent*:

1° *K is indecomposable and has exactly $n + 1$ extreme vectors;*

2° *K has exactly $n + 1$ extreme vectors $p_1, \ldots, p_{n+1}$ which satisfy – up to a multiple – exactly one relation*

$$\alpha_1 p_1 + \ldots + \alpha_{n+1} p_{n+1} = 0$$

and in this relation all coefficients α_i are different from zero;

3° *K is indecomposable and has exactly two proper diagonals;*

4° *K is indecomposable and has exactly two minimal diagonals.*

Proof. Suppose that 1° is satisfied. Since the dimension of K is n, there exists exactly one relation

$$r\colon\ \alpha_1 p_1 + \ldots + \alpha_{n+1} p_{n+1} = 0$$

such that at least one $\alpha_j \neq 0$. Let us show that all α_j are different from zero. Denote by M the set of all j for which $\alpha_j = 0$. Suppose that M is nonvoid. Set $K_1 =$ cone M, $K_2 =$ cone $(S \setminus M)$ and let us show that $K = K_1 \oplus K_2$ is a direct decomposition. Suppose that $x = \sum_{j \in M} \beta_j p_j = \sum_{j \in S \setminus M} \gamma_j p_j$ and $x \neq 0$. We have then a nontrivial relation

$$\sum_{j \in M} \beta_j p_j - \sum_{j \in S \setminus M} \gamma_j p_j = 0 .$$

Consequently, this relation is a non-zero multiple of $\sum_{j \in S} \alpha_i p_i = 0$ which implies $\beta_j = 0$ for all $j \in M$. It follows that $x = 0$, a contradiction. This proves 2°.

Now suppose 2° satisfied. Let us show first that K is indecomposable. Suppose on the contrary that K is decomposable. According to (1,10) there exists a proper non-void subset M of S such that

$$\sum_{i \in S} \beta_j p_j = 0$$

implies $\sum_{j \in M} \beta_j p_j = 0$. This, however, is impossible since there is only one relation (up to a multiple) and this relation has all coefficients different from zero. Hence K is indecomposable.

According to (2,18), K has at least two proper diagonals D_1, D_2 such that $D_1 = $ $= \text{cone } p(r)$, $D_2 = \text{cone } p(-r) = \text{cone } (S \setminus p(r))$.

We observe first that these diagonals are necessarily minimal since any diagonal contained in D_1 or D_2 would yield another (linearly independent) relation. Suppose D_3 is a diagonal of K. We may assume that D_3 is minimal. If $D_3 = \text{cone } M_3$, there exists a relation r_3 such that $p(r_3) = S \setminus M_3$, $p(-r_3) = M_3$. However, $r_3 = \varrho r$ so that either $M_3 = M_1$ or $M_3 = M_2$. Thus $D_3 = D_1$ or $D_3 = D_2$. This proves 3°.

To prove that 3° implies 4°, we shall use the following simple observation:

Given a proper diagonal $D = \text{cone } M$ of an indecomposable cone $K = \text{cone } S$, there exist two minimal diagonals D_1, D_2 such that $D_1 \subset D$ and $D_2 = \text{cone } M_2$ where M_2 is contained in $S \setminus M$.

This observation follows immediately from (2,4).

By 3°, K has exactly two proper diagonals $\tilde{D}_1$, $\tilde{D}_2$. Using the observation just mentioned applied to $\tilde{D}_1$, we see that there exist two minimal diagonals D_1, D_2 such that $D_1 \subset \tilde{D}_1$ and $D_2 \subset \tilde{D}_2$. Since $\text{cone}^{-1} D_2$ is disjoint with $\text{cone}^{-1} \tilde{D}_1$, the equality $\tilde{D}_1 = D_2$ is impossible. Consequently, $\tilde{D}_1 = D_1$ and $\tilde{D}_2 = D_2$. Both proper diagonals are thus minimal; since no other minimal diagonals can exist, 4° is proved.

To complete the proof, we shall show that 4° implies 1°.

By 4°, K is indecomposable. One of the minimal diagonals yields one relation among the extreme vectors of K; if there were two linearly independent relations, K would have, by (2,24), at least three minimal diagonals, a contradiction. Thus the number of extreme vectors of K is $n + 1$ and the proof is complete.

Concluding remarks. We have observed already in the introduction that the notion of a diagonal and its properties are by no means restricted to the case of cones. The theory developed for polyhedral cones makes it possible to obtain analogous results for polyhedra using the classical technique of constructing convex polyhedra as intersections of polyhedral cones and suitable hyperplanes. Although the main difficulties have already been overcome in the theory of polyhedral cones, the case of convex polyhedra is sufficiently interesting to deserve a separate paper. This will form the subject matter of another publication of the authors.

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