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TOPOLOGICAL CATEGORIES CONTAINING ANY CATEGORY OF ALGEBRAS

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In [2], DE GROOT proved that each group is isomorphic to the group of all homeomorphisms of a topological space into itself.

A more general question, namely the representation of semigroups by means of continuous mappings of a given type, was considered in [6], [9], [11]. For example, it is proved in [6] that for each monoid M (= semigroup with a unit) there is a T_0-space X such that all open local homeomorphisms of X into X form a monoid isomorphic to M. By [9], the T_0-space in the preceding result cannot be replaced by a Hausdorff space. Nevertheless, we prove in the present paper that the space X can be always found to be T_1. Moreover, we show that every monoid is isomorphic to the monoid of all

- open continuous mappings of a suitable T_1-space into itself,
- locally one-to-one continuous mappings of a suitable T_1-space into itself,
- open uniformly continuous mappings of a suitable metric space into itself,
- open contractions of a suitable metric space into itself.

These results are formulated in a more general setting, namely in terms of representation of categories. We represent algebraic categories in various categories of topological spaces or in categories of presheaves. The theorems have been announced in [13]; propositions from [13] concerning semibinding categories are not explicitly formulated here, but they are clear from proofs. Theorem 4 from [13] (concerning presheaves in sets) is proved in [12].

I. Preliminaries and description of main results

1. We recall that a functor $\Phi : K \to H$ is called an embedding if it is one-to-one. It is called full if it is onto a full subcategory.

A category K is called *algebraically universal**) (abr. *alg-universal*) if the category

*) The older name is "binding categories". Here it is replaced by a more adequate name "alg-universal categories".

of graphs can be fully embedded into it [7]. An equivalent form of the definition is that each category of universal algebras can be fully embedded into it, see [5]. We recall the following properties of alg-universal categories.

(1) Every monoid (= semigroup with unity) can be represented as the endomorphism monoid of an object of any alg-universal category. In other words, each one-object category can be fully embedded into it.

(2) More generally, every small category (= a category objects of which form a set) can be fully embedded into any alg-universal category, [10].

(3) Under the set-theoretical assumption that there is only a set of measurable cardinals, each concrete category can be fully embedded into any alg-universal category (communicated in [3], for the proof see [8]).

The results (1)–(3) show that the algebraic universality is one of the important properties of categories. The question whether a given category is alg-universal generalizes and strengthens the problems of representations of groups or monoids as groups or monoids of all mappings of a given type (= morphisms of the given category).

2. In the present paper, we show that some topological categories are alg-universal. Denote by

$\mathbf{T}$ the category of topological spaces and continuous mappings,

$\mathbf{P}$ the category of proximity spaces and proximally continuous mappings,

$\mathbf{U}$ the category of uniform spaces and uniformly continuous mappings.

Clearly, none of them is alg-universal. Indeed, the monoid of all endomorphisms of any object either consists of the unit or contains non-trivial idempotents, namely constant mappings. Thus e.g. no non-trivial group can be represented as the endomorphism monoid of an object of any of them.

Consequently, we must consider other mappings as morphisms. This is done in Sec. II and III of the present paper. We prove that each category K, satisfying one of the following conditions, is alg-universal.

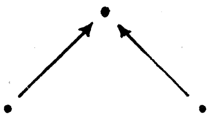
$$\mathbf{T}^1_{o,lh} \subset K \subset \mathbf{T}_o\,, \quad \mathbf{T}^1_{o,lh} \subset K \subset \mathbf{T}_{l1-1}\,, \quad \mathbf{M}_{o,c} \subset K \subset \mathbf{M}_{o,u}$$

where $\mathbf{T}$ (or $\mathbf{T}^1$ or $\mathbf{M}$) means that objects are topological spaces (or T_1-spaces or metric spaces); morphisms are always continuous mappings with the property described by the following abbreviation: o = open, lh = local homeomorphisms, l_{1-1} = locally one-to-one, c = contractions, u = uniformly continuous.

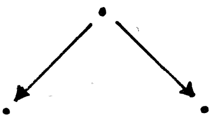
3. As we have seen, the basic topological categories $\mathbf{T}$, $\mathbf{P}$, $\mathbf{U}$ are not alg-universal. How far are they from the alg-universality? We formulate one of the possible criteria for it. Let k be a poset (= a partially ordered set)· As usual, k is also regarded as a thin category (its objects are precisely the elements of k and there is a morphism from a to b iff $a \leqq b$). The category of all functors from a poset k to a category K

is denoted by K^k (morphisms of K^k are transformations of these functors). Objects of K^k are called, as usual, presheaves in K over k. Denote by $\mathbf{P}$ the class of all non-empty posets. Under very natural assumptions about K, K is alg-universal iff K^k is alg-universal for each $k \in \mathbf{P}$ (see IV.1). Now, define $\mathbf{P}(K)$ as the class of all $k \in \mathbf{P}$ such that K^k is alg-universal. Thus K is alg-universal iff $\mathbf{P}(K) = \mathbf{P}$; so, the bigger $\mathbf{P}(K)$, the nearer K is to being alg-universal.

In Sec. IV–VI of the present paper, we describe fully the classes $\mathbf{P}(K)$ for many topological categories. We prove, for example, that $\mathbf{P}(\boldsymbol{T}) = \mathbf{P}(\boldsymbol{P}) = \mathbf{P}(\boldsymbol{U})$ and this is class of all posets containing one of the two posets below.



$\mathbf{P}(\boldsymbol{Comp})$, where $\boldsymbol{Comp}$ is the category of compact Hausdorff spaces and continuous mappings, is the class of all posets containing the first one (this result is proved under the assumption of nonexistence of any measurable cardinal). $\mathbf{P}(\boldsymbol{H_o})$, where $\boldsymbol{H_o}$ is the category of Hausdorff spaces and open continuous mappings, is the class of all posets containing



Notice that the class $\mathbf{P}(\boldsymbol{Sets})$ is described in [12].

II. Conventions and the basic construction

1. Let $\boldsymbol{G}$ be the category of all assymetric strongly connected graphs and compatible mappings. More in detail: objects of $\boldsymbol{G}$ are couples (X, R), where X is a set with at least two elements, $R \subset X \times X$ and

(a) if $(x, y) \in R$, then $(y, x) \notin R$,

(b) for each x, y in X there are $x_0 = x, x_1, \ldots, x_n = y$ in X such that $(x_i, x_{i+1}) \in R$ for $i = 0, 1, \ldots, n - 1$.

Morphisms from (X, R) to (X', R') in $\boldsymbol{G}$ are mappings $f : X \to X'$ such that $(f(x), f(y)) \in R'$ provided that $(x, y) \in R$.

The category $\boldsymbol{G}$ is alg-universal, see [4]. Thus, for a category to be alg-universal it is sufficient that $\boldsymbol{G}$ can be fully embedded into it. In the present paper, to prove that a category is alg-universal, we always construct a full embedding of $\boldsymbol{G}$ into it.

If (X, R) is an object of $\boldsymbol{G}$, $r = (x, y) \in R$, denote $\pi_1(r) = x$, $\pi_2(r) = y$. π_1 and π_2 are called, as usual, the first or the second projections, respectively.

2. Let X, X' be sets. If $f : X \to X'$ is a mapping, denote by $\bar{f} : X \times X \to X' \times X'$ the mapping such that $\bar{f}(x, y) = (f(x), f(y))$. Denote by $\mathscr{L}\mathbf{G}$ the following category: objects are all triples (X, R, d), where (X, R) is an object of $\mathbf{G}$, d is a mapping of R into the set Z of all integers; $f : (X, R, d) \to (X', R', d)$ is a morphism of $\mathscr{L}\mathbf{G}$ iff $f : (X, R) \to (X', R')$ is a morphism of $\mathbf{G}$ and $d(r) = d'(\bar{f}(r))$ for all $r \in R$. $\mathbf{G}$ will be considered as a full subcategory of $\mathscr{L}\mathbf{G}$: we shall identify each (X, R) with (X, R, c_1), where $c_1(r) = 1$ for all $r \in R$.

3. In [1], a compact metric continuum K is constructed such that

> if L is a subcontinuum of K, $f : L \to K$ is a continuous mapping then either f is constant or $f(x) = x$ for all $x \in L$.

Choose a collection $\{K_n \mid n \in Z\}$ (Z is the set of all integers) of disjoint non-degenerate (= having more than one point) subcontinua of K. Clearly,

> if $f : K_n \to K_m$ is a non-constant continuous mapping, then $n = m$ and f is the identity.

For each $n \in Z$, choose a metric ϱ_n on K_n such that ϱ_n defines the topology of K_n and $\operatorname{diam} K_n = 1$. Further, choose ${}^1a^n, {}^2a^n \in K_n$ with $\varrho_n({}^1a^n, {}^2a^n) = 1$. The spaces K_n and their points ${}^ia^n$ will be kept in what follows. K_n will be regarded as topological spaces, sometimes also as uniform or proximity or metric spaces (with respect to ϱ_n).

4. Construction. Let $o = (X, R, d)$ be an object of $\mathscr{L}\mathbf{G}$. Denote by $\mathscr{M}(o)$ the space formed from (X, R) by replacing each arrow $r \in R$ by a copy of $K_{d(r)}$ in the following way: Put $\mathscr{M}'(o) = \{(x, r) \mid r \in R, x \in K_{d(r)}\}$. Let $\mathscr{M}(o)$ be the metric space (determined uniquely up to an isometry) such that

(i) there is a surjective map $\varepsilon : \mathscr{M}'(o) \to \mathscr{M}(o)$;

(ii) if we denote $x_r = \varepsilon(x, r)$ for $r \in R$, $x \in K_{d(r)}$, then $x_r = y_s$ iff either $x = {}^ia^{d(r)}$, $y = {}^ja^{d(s)}$, $\pi_i(r) = \pi_j(s)$ or $(x, r) = (y, s)$;

(iii) the metric σ of $\mathscr{M}(o)$ is defined by

a) $\sigma(x_r, y_r) = \varrho_{d(r)}(x, y)$;

b) $\sigma(x_r, y_s) = \inf \sum_{i=0}^{n=1} \sigma(c_i, c_{i+1})$, the infimum being taken over all sequences $c_0 = x_r, c_1, \ldots, c_n = y_s$ such that $\sigma(c_i, c_{i+1})$ are defined by a).

The space $\mathscr{M}(o)$ will be viewed as a topological or uniform or proximity or metric space with respect to the above metric. For each $r \in R$, denote by $e_r : K_{d(r)} \to \mathscr{M}(o)$ the isometry, defined by $e_r(x) = x_r$. Put $\mathscr{K}_r = e_r(K_{d(r)})$. Clearly, $\mathscr{M}(o) = \bigcup_{r \in R} \mathscr{K}_r$, $\mathscr{K}_r$ meets $\mathscr{K}_s$ iff the arrows r and s have a common vertex; if $r \neq s$, then $\mathscr{K}_r$ and $\mathscr{K}_s$ have at most one common point (we recall that if $r = (x, y) \in R$, then $(y, x) \notin R$).

Let $o = (X, R, d)$, $o' = (X', R', d')$ be objects, $f : o \to o'$ a morphism of $\mathscr{L}\mathbf{G}$. Define a mapping

$$\mathscr{M}(f) : \mathscr{M}(o) \to \mathscr{M}(o')$$

by $[\mathscr{M}(f)](x_r) = x_{\bar{f}(r)}$, i.e. $\mathscr{M}(f) \circ e_r = e_{\bar{f}(r)}$ for all $r \in R$. Then $\mathscr{M}(f)$ is a contraction, which is an isometry on each $\mathscr{K}_r$.

5. Lemma. *Let $o = (X, R, d)$ be an object of $\mathscr{L}\mathbf{G}$, $f : K_n \to \mathscr{M}(o)$ a non-constant continuous mapping. Then there exists $r \in R$ such that $K_n = K_{d(r)}$ and $f = e_r$.*

This lemma is proved in [11] for d being a constant map. But this is not used in the proof. The proof goes through for d general, too, and therefore it is omitted here.

The basic lemma. *Let $o = (X, R, d)$, $o' = (X', R', d')$ be objects of $\mathscr{L}\mathbf{G}$. Let $g : \mathscr{M}(o) \to \mathscr{M}(o')$ be a continuous mapping such that g is non-constant o neach $\mathscr{K}_r$. Then $g = \mathscr{M}(f)$ for a morphism $f : o \to o'$ of $\mathscr{L}\mathbf{G}$.*

Proof. By the previous lemma, for each $r \in R$ find $r' \in R'$ such that $g \circ e_r = e_{r'}$. For each $x \in X$, choose $r \in R$ such that $\pi_1(r) = x$ (this is possible by the definition of $\mathbf{G}$) and put $f(x) = \pi_1(r')$. One can verify that $f : X \to X'$ is well-defined and $g = \mathscr{M}(f)$.

III. Algebraically universal topological categories

Let K be a category of some topological spaces and all their continuous mappings. Then K is not alg-universal, see I.2. Nevertheless, if we consider only some types of continuous mappings, we can obtain an alg-universal category.

1. We use the symbol $\subset$ also for categories. $K \subset H$ means that K is a subcategory of H. Let us denote by $\mathbf{T}_o^1$ (or $\mathbf{T}_{l1-1}^1$ or $\mathbf{T}_{o,lh}^1$) the category of all T_1-spaces and all their open continuous mappings (or all locally one-to-one continuous mappings or all open local homeomorphisms, respectively).

Theorem. *Each category K such that either*

α) $\mathbf{T}_{o,lh}^1 \subset K \subset \mathbf{T}_o^1$ *or*
β) $\mathbf{T}_{o,lh}^1 \subset K \subset \mathbf{T}_{l1-1}^1$

is alg-universal.

Proof. Let K be a category satisfying either α) or β). We construct a full embedding $\Phi : \mathbf{G} \to K$. Let us recall that the collection $\{K_n \mid n \in Z\}$ is introduced in II,3. We shall need K_1 and K_2 only.

a) Given an object $G = (X, R)$ of $\mathbf{G}$, let ΦG be a topological space defined as follows: Its underlying set is

$$H = (K_1 \times R) \cup ((K_2 \setminus \{{}^2a^2\}) \times X).$$

The topology of ΦG is inductively generated by the collection of mappings $\varphi_r : K_1 \to \to H$ and $\varphi_{r,x} : K_2 \to H$, where r runs over R and x over $\{\pi_1(r), \pi_2(r)\}$, and

$$\varphi_r(c) = (c, r) \quad \text{for all} \quad c \in K_1 ,$$
$$\varphi_{r,x}(c) = (c, x) \quad \text{for all} \quad c \in K_2 \setminus \{{}^2a^2\} ,$$
$$\varphi_{r,x}({}^2a^2) = ({}^ia^1, r) \quad \text{for} \quad x = \pi_i(r) .$$

Clearly, each $\varphi_r : K_1 \to \Phi G$ and each $\varphi_{r,x} : K_2 \to \Phi G$ are homeomorphisms into the space ΦG.

b) ΦG is a T_1-space but not a Hausdorff space. Two points $a, b \in \Phi G$ are not separated if and only if $a = \varphi_{r,x}({}^2a^2)$, $b = \varphi_{s,x}({}^2a^2)$ for some $x \in X$, $r, s \in R$, i.e. $a = ({}^ia^1, r)$, $b = ({}^ja^1, s)$ where $\pi_i(r) = \pi_j(s)$.

c) Further, given a compatible mapping $f : G \to G'$, where G is as above, put

$$[\Phi(f)]\,(c, r) = (c, \bar{f}(r)) \quad \text{for all} \quad c \in K_1 , \quad r \in R ,$$
$$[\Phi(f)]\,(c, x) = (c, f(x)) \quad \text{for all} \quad c \in K_2 \setminus \{{}^2a^2\} , \quad x \in X .$$

Evidently, $\Phi(f)$ is an open local homeomorphism; in particular, $\Phi(f)$ is a morphism of K.

d) Clearly, Φ is an embedding of G into K. It remains to show that Φ is full. First, define a functor

$$\Psi : \mathbf{G} \to \mathscr{L}\mathbf{G}$$

as follows. If $G = (X, R)$, put $\Psi G = (\tilde{X}, \tilde{R}, d)$, where

$$\tilde{X} = X \cup (X \times \{0\}) \quad (\text{where we suppose } X \cap (X \times \{0\}) = \emptyset) ,$$
$$\tilde{R} = R \cup \{((x, 0), x) \mid x \in X\} ,$$

$d : \tilde{R} \to Z$ is defined by $d(r) = 1$ for all $r \in R$, $d((x, 0), x) = 2$ for all $x \in X$. For any compatible mapping $f : G \to G'$ define $\Psi(f) = \tilde{f}$ by $\tilde{f}(x) = f(x)$, $\tilde{f}(x, 0) = = (f(x), 0)$ for each $x \in X$. One can verify that $\Psi : \mathbf{G} \to \mathscr{L}\mathbf{G}$ is a full embedding. Put

$$\tilde{\Phi} = \mathscr{M} \circ \Psi .$$

e) If $G = (X, R)$ is an object of $\mathbf{G}$, define $\varepsilon_G : \Phi G \to \tilde{\Phi} G$ as follows: $\varepsilon_G(c, r) = c_r$ for all $r \in R$, $c \in K_1$, $\varepsilon_G(c, x) = c_{((x,0),x)}$ for all $x \in X$, $c \in K_2 \setminus \{{}^2a^2\}$. One can see that $\varepsilon = \{\varepsilon_G\} : \Phi \to \tilde{\Phi}$ is a transformation of these functors.

f) Now, we are prepared to show that Φ is full. Let $g : \Phi G \to \Phi G'$ be a morphism in K. In particular, it is a continuous mapping which is either locally one-to-one or open. One can verify that there exists a continuous mapping $\tilde{g} : \tilde{\Phi} G \to \tilde{\Phi} G'$ such that $\tilde{g} \circ \varepsilon_G = \varepsilon_{G'} \circ g$, $\tilde{g}$ is non-constant on any $\mathscr{K}_r$ $(r \in R)$ and on any $\mathscr{K}_{((x,0),x)}$ $(x \in X)$. Thus, $\tilde{g} = \mathscr{M}(h)$ for a morphism $h : \Psi(G) \to \Psi(G')$ of $\mathscr{L}\mathbf{G}$. Since Ψ is full, $h = \Psi(f)$ for a compatible mapping $f : G \to G'$. One can prove that $g = \Phi f$.

2. Let us recall that a map $m : (P, \varrho) \to (P', \varrho')$ is called a contraction iff for every $p, q \in P$, $\varrho'(m(p), m(q)) \leqq \varrho(p, q)$. Denote by $\mathbf{M}_{o,c}$ (or $\mathbf{M}_{o,u}$) the category of all metric spaces and all their open contractions (or all open uniformly continuous mappings, respectively).

Theorem. *Each category K such that*

$$\mathbf{M}_{o,c} \subset K \subset \mathbf{M}_{o,u}$$

is alg-universal.

Proof. Let us construct a full embedding $\Phi : \mathbf{G} \to K$. For any object $G = (X, R)$ of $\mathbf{G}$, put $\Phi G = \mathscr{M}(G) \setminus \{{}^1a_r^i \mid r \in R, i = 1, 2\}$. Given a compatible mapping $f : G \to G'$, let $\Phi f : \Phi G \to \Phi G'$ be the domain-range restriction of $\mathscr{M}(f)$. Clearly, Φf is an open contraction. In particular, it is a morphism of K. Thus, Φ is an embedding. It remains to show that Φ is full. Let $g : \Phi G \to \Phi G'$ be a morphism of K. As g is uniformly continuous and $\mathscr{M}(G)$ (or $\mathscr{M}(G')$) is a completion of ΦG (or $\Phi G'$, respectively), g can be uniquely extended to a uniformly continuous mapping $\tilde{g} : \mathscr{M}(G) \to \mathscr{M}(G')$. As g is open, g is non-constant on any $\mathscr{K}_r \subset \mathscr{M}(G)$. By Basic Lemma (II.5), $\tilde{g} = \mathscr{M}(f)$ for a compatible mapping $f : G \to G'$ and g is a domain-range restriction of $\tilde{g}$, i.e. $g = \Phi f$. This completes the proof.

IV. Presheaves in basic topological categories

As noted in I, if the class of morphisms of a category K includes all constant mappings, then K is not alg-universal. This is the case of the basic topological categories

- $\mathbf{T}$ of all topological categories and continuous mappings,
- $\mathbf{P}$ of all proximity spaces and proximally continuous mappings,
- $\mathbf{U}$ of all uniform spaces and uniformly continuous mappings,
- $\mathbf{M}$ of all metric spaces and contractions.

In this part, we characterize those k for which the category $\mathbf{T}^k$ of presheaves over k is alg-universal. It appears that $\mathbf{T}^k$ is alg-universal iff $\mathbf{P}^k$ (or $\mathbf{U}^k$, or $\mathbf{M}^k$) is.

1. We recall that an object o of a category K is called its *initial* (or *terminal*) object if the set of all morphism from o into any object of K (or from any object of K into o, respectively) contains precisely one morphism.

Proposition. *Let a category K have either an initial or a terminal object. Then K is alg-universal iff K^k is alg-universal for any non-void poset k.*

Proof. If K^k is alg-universal for each non-void poset k, then K is alg-universal, K being isomorphic to K^1, where 1 is the one-point poset. Conversely, let K be alg-universal and let a non-void poset k be given. Let t be a terminal object of K. Choose $a \in k$. For any object o of K define $\Omega(o)$ as the functor from k into K such that $[\Omega(o)](b) = o$ whenever $b \leqq a$, $[\Omega(o)](b) = t$ otherwise. One can verify that Ω defines a full embedding of K into K^k, so K^k is alg-universal. If K has an initial object instead of a terminal one, consider "the dual situation".

2. Let l_1, l_2 be the following posets.

$$l_1 = (\{a, b, c\}, <), \quad a < b > c,$$
$$l_2 = (\{o_n \mid n = 0, 1, 2, \ldots\}, <), \quad o_0 > o_1 > o_2 > \ldots.$$

Let us notice that both of them are sketched in I.

Theorem. *Let K be a full subcategory of T containing all metrizable spaces. Then the following conditions on a poset k are equivalent.*

(i) *K^k is alg-universal.*

(ii) *Either l_1 or l_2 can be fully embedded into k.*

Proof. a) Construction of a full embedding $\Phi : \mathbf{G} \to K^{l_1}$. Choose $p \in K_1 \setminus \{{}^1a^1, {}^2a^1\}$. For any object $G = (X, R)$ of $\mathbf{G}$, ΦG is the functor $\Lambda_G : l_1 \to K$ defined as follows. $\Lambda_G(b) = \mathscr{M}(G)$, considered as a topological space; $\Lambda_G(a)$ and $\Lambda_G(c)$ are subspaces of $\Lambda_G(b)$ with the underlying sets

$$|\Lambda_G(a)| = \{p_r \mid r \in R\}, \quad |\Lambda_G(c)| = \{{}^ia_r^1 \mid i = 1, 2, \ r \in R\};$$

$\Lambda_G\binom{a}{b}$ and $\Lambda_G\binom{c}{b}$ are inclusions. For any compatible mapping $f : G \to G'$, the transformation $\Phi f : \Lambda_G \to \Lambda_{G'}$ is defined as follows. $\Phi f = \{\Lambda_f(a), \Lambda_f(b), \Lambda_f(c)\}$, where $\Lambda_f(b) = \mathscr{M}(f)$, $\Lambda_f(a)$ and $\Lambda_f(c)$ are the domain-range restrictions of Λ_f. Obviously, Φ is an embedding of $\mathbf{G}$ into K^{l_1}. To show that Φ is full, let us consider a transformation $\tau : \Phi G \to \Phi G'$. Since $\Lambda_G\binom{a}{b}$, $\Lambda_G\binom{c}{b}$ are inclusions, τ_a and τ_c are domain-restrictions of τ_b. Each $\mathscr{K}_r \subset \Lambda_G(b)$ meets both $\Lambda_G(a)$ and $\Lambda_G(c)$ so that $\tau_b(\mathscr{K}_r)$ meets both $\Lambda_{G'}(a)$ and $\Lambda_{G'}(c)$. In particular, τ_b is non-constant on any $\mathscr{K}_r$ and so $\tau_b = \mathscr{M}(f)$ for a compatible mapping $f : G \to G'$ by Basic Lemma II.5. Hence $\tau = \Phi f$, which proves that Φ is a full embedding.

b) Construction of a full embedding $\Psi : \mathbf{G} \to K^{l_2}$. First, choose a decreasing sequence $A_1, A_2, A_3, \ldots$ of nonvoid subsets of $K_1 \setminus \{{}^1a^1, {}^2a^1\}$ such that $\bigcap A_i = \emptyset$. Given an object $G = (X, R)$ of $\mathbf{G}$, put $\Psi G = \Sigma_G : l_2 \to K$ such that $\Sigma_G(o_0) = \mathscr{M}(G)$, $\Sigma_G(o_n)$ is the subspace of $\Sigma_G(o_0)$ with the underlying set $|\Sigma_G(o_n)| = \{x_r \mid x \in A_n, r \in R\}$ for all $n = 1, 2, 3, \ldots$, $\Sigma_G\binom{o_n}{o_m}$ are inclusions $(m > n)$; given a compatible mapping $f : G \to G'$, let $\Psi f : \Sigma_G \to \Sigma_{G'}$ be the transformation Σ_f defined as follows.

$\Sigma_J(o_0) = \mathcal{M}(f)$, $\Sigma_f(o_n)$ is the domain-range restriction of $\Sigma_f(o_0)$ for any $n = 1, 2, \ldots$. Ψ is obviously an embedding. We show that it is full. In fact, given a transformation $\tau : \Sigma_G \to \Sigma_{G'}$, τ_{o_0} is non-constant on any $\mathcal{K}_r$ (if not, then the only value of τ_{o_0} on $\mathcal{K}_r$ is a point of $\Sigma_{G'}(o_n)$ for every $n = 1, 2, \ldots$, which is impossible because $\bigcap_{n=1}^{\infty} \Sigma_{G'}(o_n) = \emptyset$). Thus, $\tau_{o_0} = \mathcal{M}(f)$ for a compatible mapping $f : G \to G'$. As all $\Sigma_G\begin{pmatrix} o_n \\ o_m \end{pmatrix}$, $\Sigma_{G'}\begin{pmatrix} o_n \\ o_m \end{pmatrix}$ are inclusions, any τ_{o_n} is a domain-range restriction of τ_{o_0} and so $\tau = \Psi f$.

c) Now, we show that K^k is alg-universal whenever l_1 or l_2 can be fully embedded into k. Let us suppose that l_1 is a full subcategory of k. Each presheaf ΦG over l_1, described in a), can be extended to a presheaf $\Phi^* G$ over k and each $\Phi f : \Phi G \to \Phi G'$ can be extended to a $\Phi^* f : \Phi^* G \to \Phi^* G'$ such that Φ^* is a full embedding of $\mathbf{G}$ into K^k. It suffices to put

$$\begin{aligned}
(\Phi^* G)(o) &= \Lambda_G(b) \quad \text{whenever} \quad o > a \quad \text{and} \quad o > c\,,\\
(\Phi^* G)(o) &= \Lambda_G(a) \quad \text{whenever} \quad o > a \quad \text{but not} \quad o > c\,,\\
(\Phi^* G)(o) &= \Lambda_G(c) \quad \text{whenever} \quad o > c \quad \text{but not} \quad o > a\,,\\
(\Phi^* G)(o) &= \emptyset \quad \text{otherwise}\,;
\end{aligned}$$

all the mappings $(\Phi^* G)\begin{pmatrix} o \\ p \end{pmatrix}$ are to be defined as inclusions or identities.

The full embedding Ψ of $\mathbf{G}$ into K^{l_2}, described in b), can be quite analogously extended to a full embedding Ψ^* of $\mathbf{G}$ into K^k for any k containing a full subcategory isomorphic to l_2. Put

$$\begin{aligned}
(\Psi^* G)(o) &= \Sigma_G(o_n) \quad \text{where } n \text{ is the smallest integer such that} \quad o_n < o\,,\\
(\Psi^* G)(o) &= \emptyset \quad \text{provided that such } n \text{ does not exist}\,.
\end{aligned}$$

d) It remains to prove that K^k is not alg-universal provided that k contains neither l_1 nor l_2. Let Γ be a presheaf over k in K; denote $k' = \{o \mid \Gamma o \neq \emptyset\}$. Then neither l_1 nor l_2 can be fully embedded into k', so each component C of k' has a smallest element, say o_C. Choose $x_C \in \Gamma o_C$ for each component C of k' and define a transformation $\tau : \Gamma \to \Gamma$ by

$$\tau_o(x) = \left(\Gamma\begin{pmatrix} o_C \\ o \end{pmatrix}\right)(x_C) \quad \text{for any} \quad o \in C\,, \quad x \in \Gamma_o\,.$$

Thus, each τ_o is constant and so $\tau \circ \tau = \tau$. Consequently, the endomorphism monoid of any presheaf Γ over k either is trivial (i.e. $\tau = 1_\Gamma$ and 1_Γ is the only endotransformation of Γ) or contains non-trivial idempotents. Thus, any non-trivial monoid without non-trivial idempotents cannot be represented as the endomorphism monoid of some $\Gamma : k \to K$. The proof is complete.

3. Theorem. *Let K (or L) be a full subcategory of $\mathbf{P}$ (or $\mathbf{U}$, respectively) containing all metrizable spaces. Then the following conditions on a poset k are equivalent.*

(i) *K^k is alg-universal.*

(ii) *L^k is alg-universal.*

(iii) *$\mathbf{M}^k$ is alg-universal.*

(iv) *Either l_1 or l_2 can be fully embedded into k.*

Proof. The proofs of the equivalences (i) $\Leftrightarrow$ (iv), (ii) $\Leftrightarrow$ (iv), (iii) $\Leftrightarrow$ (iv) follow the same lines as the proof of the previous theorem. Regard only the spaces $\mathcal{M}(G)$ as proximity or uniform or metric spaces, respectively.

V. Presheaves in categories with open mappings

As has been proved in [6], the category of all T_0-spaces and local homeomorphisms is alg-universal. We have shown in Sec. III that this can be strengthened to T_1-spaces with various local-homomorphism-type morphisms. By [9], T_0-spaces in the result of [6] cannot be replaced by Hausdorff ones. Nevertheless, we show that the category of Hausdorff spaces and local homeomorphisms is relatively near to being alg-universal in the sense that the class of all posets k, for which the corresponding presheaf category is alg-universal, is "large". The result is formulated in a more general setting.

1. A subcategory K of the category $\mathbf{T}$ of all topological spaces and all continuous mappings is said to be *regular* if it is closed under finite direct sums and direct summands. This means: If a space X (or X') is a topological sum of spaces A, B (or A', B') and if $f : X \to X'$ is a mapping such that $f(A) \subset A'$ and $f(B) \subset B'$, then the domain-range restrictions $f_A : A \to A'$ and $f_B : B \to B'$ of f are morphisms of K iff f is.

2. Let h be the following poset.

$$h = (\{a, b, c\}, <), \quad a > b < c,$$

i.e. h is dual to l_1. The categories

$\mathbf{H}_o$ of all Hausdorff spaces and all open continuous mappings and

$\mathbf{H}_{o,lh}^{\mathrm{metr}}$ of all metrizable topological spaces and all open local homeomorphisms

are considered in the following theorem.

3. Theorem. *Let K be a regular category such that*

$$\mathbf{H}_{o,lh}^{\mathrm{metr}} \subset K \subset \mathbf{H}_o .$$

Let k be a poset. Then the following conditions are equivalent.

(i) *K^k is alg-universal,*

(ii) *h can be fully embedded into k.*

Proof. a) Let us consider the graph (Z, S), where Z is the set of all integers and $S = \{(n, n+1) \mid n \in Z\}$. Define $d_i : S \to Z$, $i = 1, 2$, by $d_1(n, n+1) = 2n$, $d_2(n, n+1) = 2n + 1$ for all $n \in Z$. Thus (Z, S, d_i), $i = 1, 2$ are objects of $\mathscr{L}G$. Put $H_i = \mathscr{M}(Z, S, d_i)$, where H_i is considered as a topological space, denote $\mathscr{K}^i_r = e_r(K_{d_i(r)})$ for any $r \in S$, $i = 1, 2$ (we recall that the mappings e_r are introduced in II,4). Consider $H = H_1 \cup H_2$ (notice that H_1 and H_2 are disjoint) endowed with the topology of the sum of H_1 and H_2. Let L be the space formed from H by adding two (distinct) points s_1, s_2 such that sets of the form $\bigcup_{n \leqq n_0} (\mathscr{K}^1_{(n,n+1)} \cup \mathscr{K}^2_{(n,n+1)})$ form a local base at s_1 and sets of the form $\bigcup_{n \geqq n_0} (\mathscr{K}^1_{(n,n+1)} \cup \mathscr{K}^2_{(n,n+1)})$ form a loca base at s_2, n_0 running over Z. Let M (or N) be the subspace of L with the underlying set $L \setminus \{s_1\}$ (or $L \setminus \{s_2\}$, respectively).

b) By means of the spaces H, M, N, we construct an embedding $\Phi : \mathbf{G} \to K^h$ as follows. Let $G = (X, R)$ be an object of $\mathbf{G}$. Consider X and R as discrete spaces and define $\Phi G = \Lambda_G : h \to K$, where

$$\Lambda_G(a) = M \times X\,, \quad \Lambda_G(b) = H \times R\,, \quad \Lambda_G(c) = N \times R\,,$$

$$\left[\Lambda_G\begin{pmatrix} b \\ a \end{pmatrix}\right](h, r) = (h, \pi_i(r))\,, \quad \left[\Lambda_G\begin{pmatrix} b \\ c \end{pmatrix}\right](h, r) = (h, r)$$

for $h \in H_i$, $i = 1, 2$, $r \in R$. Clearly, all $\Lambda_G(o)$, $o \in \{a, b, c\}$, are metrizable spaces, all $\Lambda_G\begin{pmatrix} o' \\ o \end{pmatrix}$ are open local homeomorphisms, so they are morphisms of K. Given a compatible mapping $f : G \to G'$, put $\Phi f = \Lambda_f$, where

$$\begin{aligned}
[\Lambda_f(a)]\,(h, x) &= (h, f(x)) \quad \text{for all} \quad h \in M\,, \quad x \in X\,,\\
[\Lambda_f(b)]\,(h, r) &= (h, \bar{f}(r)) \quad \text{for all} \quad h \in H\,, \quad r \in R\,,\\
[\Lambda_f(c)]\,(h, r) &= (h, \bar{f}(r)) \quad \text{for all} \quad h \in N\,, \quad r \in R\,.
\end{aligned}$$

Clearly, all $\Lambda_f(o)$ are open local homeomorphisms. Thus Φ is really an embedding of $\mathbf{G}$ into K^h.

c) We show that Φ is a full embedding. Let $\tau : \Phi G \to \Phi G'$ be a morphism of K^h, $G = (X, R)$, $G' = (X', R')$. We have to find a compatible mapping $f : G \to G'$ with $\Phi f = \tau$. As $\tau = \{\tau_a, \tau_b, \tau_c\} : \Lambda_G \to \Lambda_{G'}$ is a transformation in K, each mapping τ_o is an open continuous mapping. The mapping τ_a maps each component $M \times \{x\}$ of $\Lambda_G(a)$ into a component $M \times \{y\}$ of $\Lambda_{G'}(a)$. Put $y = f(x)$. Thus $f : X \to X'$ is a mapping such that

$$\tau_a(M \times \{x\}) \subset M \times \{f(x)\} \quad \text{for all} \quad x \in X\,.$$

Since τ_a maps the image of $\Lambda_G\begin{pmatrix} b \\ a \end{pmatrix}$ into the image of $\Lambda_{G'}\begin{pmatrix} b \\ a \end{pmatrix}$, we have, moreover,

$$\tau_a(H \times \{x\}) \subset H \times \{f(x)\} .$$

Since τ_a is open, we obtain by Basic Lemma that

$$\tau_a(h, x) = (h, f(x)) \quad \text{for all} \quad h \in H\,, \; x \in X\,,$$

and, because of the density of H in M, also for $h \in M$. Analogously, there is $g : R \to$ $\to R'$ such that

$$\tau_c(h, r) = (h, g(r)) \quad \text{for all} \quad h \in N\,, \quad r \in R\,.$$

Now it is easy to verify that $g = \bar{f}$ $\left(\text{see the definition of } \Lambda_G\begin{pmatrix} b \\ a \end{pmatrix} \text{ and } \Lambda_G\begin{pmatrix} b \\ c \end{pmatrix}!\right)$ and $\tau = \Phi f$.

d) Now we show that K^k is alg-universal whenever h can be fully embedded into k. Let us suppose that h is a full subcategory of k. Each presheaf ΦG over h, described in b), can be extended to a presheaf $\Phi^* G$ over k and each $\Phi f : \Phi G \to \Phi G'$ can be extended to a $\Phi^* f : \Phi^* G \to \Phi^* G'$ such that Φ^* is a full embedding of $\mathbf{G}$ into K^k. It suffices to put

α) $(\Phi^* G)(o) = L$ whenever $o > a$ and $o > c$,
β) $(\Phi^* G)(o) = \Lambda_G(a)$ whenever $o > a$ but not $o > c$,
γ) $(\Phi^* G)(o) = \Lambda_G(c)$ whenever $o > c$ but not $o > a$,
δ) $(\Phi^* G)(o) = \Lambda_G(b)$ whenever $o > b$ but neither $o > a$ nor $o > c$,
ε) $(\Phi^* G)(o) = \emptyset$ otherwise;

if $o, p \in k$, $p < o$ then $(\Phi^* G)\begin{pmatrix} p \\ o \end{pmatrix}$ is defined as follows. If o is as in α) while p is as in β) or γ) or δ), then

$$\left[(\Phi^* G)\begin{pmatrix} p \\ o \end{pmatrix}\right](h, z) = h\,;$$

otherwise $(\Phi^* G)\begin{pmatrix} p \\ o \end{pmatrix}$ is the identity or the inclusion (according to the cases for o and p). Φ^* just defined is obviously again a full embedding.

e) Now we show that if h cannot be fully embedded in k, then K^k is not alg-universal. In fact, we prove that the monoid $M = \{1, c, c^2 = c^3\}$ cannot be represented as the endomorphism monoid of an object of K^k. Given a presheaf Γ in K over k and a transformation $\gamma : \Gamma \to \Gamma$ such that $1_\Gamma \neq \gamma \neq \gamma^2 = \gamma^3 \neq 1_\Gamma$, we shall find a transformation $\tau : \Gamma \to \Gamma$ such that $\tau \notin \{1_\Gamma, \gamma, \gamma^2\}$.

f) We recall the following assertion proved in [9]. Let H be a Hausdorff space, $g : H \to H$ an idempotent continuous mapping which is either open or locally one-to-one. Then $g(H)$ is open-and-closed in H.

g) Thus, let $\Gamma : k \to K$ be a presheaf, $\gamma : \Gamma \to \Gamma$ a transformation such that $1_\Gamma \neq \gamma \neq \gamma^2 = \gamma^3 \neq 1_\Gamma$. For each o in k put

$$G_o = \gamma_o^2(\Gamma(o)), \quad D_o = \gamma_o^{-1}(G_o).$$

By f) G_o (and consequently also D_o) is open-and-closed in $\Gamma(o)$. Consider the following two cases.

Case I. $\left[\Gamma\begin{pmatrix} o \\ p \end{pmatrix}\right](\Gamma(o) \setminus D_o) \subset (\Gamma(p) \setminus D_p) \cup G_p$ for all o, p in k with $o < p$. Then define $\tau : \Gamma \to \Gamma$ by

$$\tau_o(x) = x \quad \text{for all} \quad x \in \Gamma(o) \setminus D_o, \quad \tau_0(x) = \gamma_0(x) \quad \text{for} \quad x \in D_o$$

for all σ in k. Obviously, each τ_o is a sum of the identical map on $\Gamma(o) \setminus D_o$ and the domain-range-restriction of γ_o to D_o and so τ_o is a morphism of K because of the regularity of K. Now, it is routine to prove that τ is a transformation and $\tau \notin \{1_\Gamma, \gamma, \gamma^2\}$.

Case II. $\left(\left[\Gamma\begin{pmatrix} o_0 \\ p_0 \end{pmatrix}\right](\Gamma(o_0) \setminus D_{o_0})\right) \cap D_{p_0} \setminus G_{p_0} \neq \emptyset$ for some o_0, p_0 in k with $o_0 < p_0$. Here, put

$$\tau_o(x) = \gamma_o(x) \quad \text{whenever} \quad o < p_0, \quad x \in \left(\Gamma\begin{pmatrix} o \\ p_0 \end{pmatrix}\right)^{-1} G(p_0),$$

$$\tau_o(x) = x \quad \text{otherwise}.$$

Then τ_o is a morphism of K. One can verify that τ is really a transformation. Indeed, if $o < p_0$ and $o < o'$ for some o', then, since h cannot be fully embedded into k, either $o' \leqq p_0$ or $p_0 < o'$, hence

$$\tau_{o'} \circ \Gamma\begin{pmatrix} o \\ o' \end{pmatrix} = \Gamma\begin{pmatrix} o \\ o' \end{pmatrix} \circ \tau_o.$$

(In the other cases, this equality holds trivially.) We show that $\tau \notin \{1_\Gamma, \gamma, \gamma^2\}$. Since τ_{p_0} is the identity, $\tau \neq \gamma$ as well as $\tau \neq \gamma^2$. Since τ_{o_0} is distinct from the identity, $\tau \neq 1_\Gamma$. This completes the proof.

4. We consider the categories

$\boldsymbol{H}_{l1-1}$ of all Hausdorff spaces and all locally one-to-one continuous mappings and

$H_{o,lh}^{\mathrm{metr}}$ of all metrizable topological spaces and all open local homeomorphisms

in the following theorem.

Theorem. *Let K be a regular category such that*

$$\boldsymbol{H}_{o,lh}^{\mathrm{metr}} \subset K \subset \boldsymbol{H}_{l1-1}\,.$$

Let k a poset. Then the following conditions are equivalent.

(i) *K^k is alg-universal.*
(ii) *h can be fully embedded into k.*

The proof is the same as that of the previous theorem.

VI. Presheaves in categories with closed mappings

In this part, we shall characterize the classes of all posets k such that the presheaves categories $\boldsymbol{Comp}^k$ and $\boldsymbol{T}_{1,cl}^k$ are alg-universal, where

$\boldsymbol{Comp}$ is the category of all compact Hausdorff spaces and all continuous mappings.

$\boldsymbol{T}_{1,cl}$ is the category of all T_1-spaces and all closed continuous mappings.

The corresponding class of posets for $\boldsymbol{T}_{1,cl}$ appears to be the same as that for $\boldsymbol{T}$ but the proof requires different constructions. Both the following theorems will be proved under the assumption of non-existence of measurable cardinals.

1. In fact, we shall assume that each metrizable space is realcompact. This makes it possible to use following

Lemma [11]. *Let M, M' be metric spaces, M connected, M' realcompact. Let $g : Y \to Y'$ be a continuous mapping where $M \subset Y \subset \beta M$, $M' \subset Y' \subset \beta M'$. Then g is a constant or g is the unique extension of a continuous mapping $g' : M \to M'$.*

2. Let us recall that the poset l_1 is described in VI.1.

Theorem. *Let k be a poset. The following conditions are equivalent.*

(i) *$\boldsymbol{Comp}^k$ is alg-universal.*
(ii) *l_1 can be fully embedded into k.*

Proof. a) First, we prove that $\boldsymbol{Comp}^{l_1}$ is alg-universal. Denote by K the category of all metrizable spaces and all continuous mappings. Let $\Phi : \boldsymbol{G} \to K^{l_1}$ be the full embedding described in the proof in IV.2. Let us use the notation of this proof, i.e. $\Phi G = \Lambda_G$. Put $\tilde{\Phi} G = \beta \circ \Lambda_G$, $\tilde{\Phi} f = \beta \circ \Lambda_f$ (more in detail $(\tilde{\Phi} G)(o) = \beta \Lambda_G(o)$, $(\tilde{\Phi} G)\binom{p}{o} = \beta \Lambda_G \binom{o}{p}$, $o, p \in \{a, b, c\}$), where β denotes the Čech-Stone compactification functor. $\tilde{\Phi}$ is obviously an embedding of $\boldsymbol{G}$ into $\boldsymbol{Comp}^{l_1}$. To prove that $\tilde{\Phi}$ is a full embedding it suffices to show that for each continuous transformation $\tau : \tilde{\Phi}G \to \tilde{\Phi}G'$ there is a transformation $\sigma : \Phi G \to \Phi G'$ such that $\tau = \beta\sigma$ (i.e. $\tau_o = \beta\sigma_o$, $o \in \{a, b, c\}$). Let us observe that $\Lambda_G(a)$ and $\Lambda_G(c)$ are functionally separated C-embedded subspaces of $\Lambda_G(b)$; remember that $\Lambda_G\begin{pmatrix} a \\ b \end{pmatrix}$ and $\Lambda_G\begin{pmatrix} b \\ c \end{pmatrix}$ are embeddings. This implies that $(\tilde{\Phi}G)(a)$ and $(\tilde{\Phi}G)(c)$ are disjoint subspaces of $(\tilde{\Phi}G)(b)$ and $(\tilde{\Phi}G)\begin{pmatrix} a \\ b \end{pmatrix}$, $(\tilde{\Phi}G)\begin{pmatrix} c \\ b \end{pmatrix}$ are also embeddings; analogously for G'. Thus, τ_b is non-constant and τ_a, τ_c are domain-range-restrictions of τ_b. Now, apply the above lemma to $g = \tau_b$.

b) Let k be a poset such that l_1 can be fully embedded into it. To prove that $\boldsymbol{Comp}^k$ is alg-universal, we proceed as in IV.2 c); replace only Φ by $\tilde{\Phi}$.

c) Let k be a poset such that l_1 cannot be fully embedded into it. We have to show that $\boldsymbol{Comp}^k$ is not alg-universal. We show that, in this case, a presheaf Γ (in $\boldsymbol{Comp}$ over k) possesses a non-identical idempotent endotransformation τ provided that card $\Gamma(o) > 1$ for some o in k. It will follow that no non-trivial monoid without non-identical idempotents can be represented as the monoid of all endomorphisms of an object of $\boldsymbol{Comp}^k$. Thus, let Γ and o be as above. Denote by k' the component of k containing o. Put

$$k'' = \{p \in k \mid p \leqq o,\ \Gamma(p) \neq \emptyset\}\,.$$

As k does not contain l_1, k'' is linearly ordered and so

$$\left[\Gamma\begin{pmatrix} p \\ o \end{pmatrix}\right](\Gamma(p))\,, \quad p \in k''$$

is a collection of closed subspaces of $\Gamma(o)$ with the finite intersection property. Hence $A = \bigcap_{p \in k''} \left[\Gamma\begin{pmatrix} p \\ o \end{pmatrix}\right](\Gamma(p))$ is non-empty. Choose $a \in A$. Since all spaces $\Gamma(p)$ are compact, one can find a collection $\{a_p \mid p \in k''\}$ such that $a_p \in \Gamma(p)$ for all $p \in k''$, $a_o = a$ and

$$(*) \qquad \left[\Gamma\begin{pmatrix} p \\ q \end{pmatrix}\right](a_p) = a_q$$

for each $p, q \in k''$, $p \leqq q$. Further, let us define a_q by $(*)$ also for every $q \in k'$ for which there is $p \in k''$ with $p < q$. Define $\tau : \Gamma \to \Gamma$ by

$$\tau_q(z) = a_q \quad \text{whenever } a_q \text{ is defined and} \quad z \in \Gamma(q)\,,$$

$$\tau_q = 1_{\Gamma(q)} \quad \text{otherwise}\,.$$

Obviously, τ has the required properties.

3. Let us recall that the posets l_1 and l_2 are described in Sec. IV.

Theorem. *Let K be a full subcategory of $\mathbf{T}_{1,cl}$, containing all locally compact σ-compact spaces. Let k be a poset. Then the following conditions are equivalent.*

(i) *K^k is alg-universal.*

(ii) *Either l_1 or l_2 can be fully embedded into k.*

Proof. a) The aim of the proof is to construct a full embedding of $\mathbf{G}$ into K^{l_2}. Indeed, if k is a poset such that neither l_1 nor l_2 can be fully embedded into k, then K^k is not alg-universal, we can prove it as in IV.2.d). If we have a full embedding $\Phi : \mathbf{G} \to K^{l_1}$ and a full embedding $\Psi : \mathbf{G} \to K^{l_2}$ and k is a poset containing either l_1 or l_2, we can construct a full embedding of $\mathbf{G}$ into K^k as in IV.2.c). $\mathbf{G}$ can be fully embedded into K^{l_1} by VI.2, because $K \supset \mathbf{Comp}$. Thus, we have only to construct a full embedding $\Psi : \mathbf{G} \to K^{l_2}$.

b) First, we construct a full embedding $\Theta : \mathbf{G} \to \mathscr{L}\mathbf{G}$. For any object $G = (X, R)$ of $\mathbf{G}$ put $\Theta G = (\tilde{X}, \tilde{R}, d_G)$, where

$\tilde{X} = (X \cup R) \times N$, where N is the set of all positive integers, X and R are supposed to be disjoint,

$\tilde{R} = R_1 \cup R_2 \cup R_3$, where

$$R_1 = \{((\pi_1(r), 1)\,(r, 1)) \mid r \in R\}\,,$$
$$R_2 = \{((r, 1), (\pi_2(r), 1)) \mid r \in R\}\,,$$
$$R_3 = \{((z, n), (z, n + 1)) \mid z \in X \cup R,\ n \in N\}\,,$$

$$d_G : \tilde{R} \to Z \quad \text{is defined by} \quad d(r) = -1 \quad \text{whenever} \quad r \in R_1\,,$$
$$d(r) = 1 \quad \text{whenever} \quad r \in R_2\,,$$

$$d((z, n), (z, n + 1)) = n + 1 \quad \text{for} \quad z \in X\,,$$
$$d((z, n), (z, n + 1)) = -(n + 1) \quad \text{for} \quad z \in R\,.$$

Given a compatible mapping $f : G \to G'$, put

$$(\Theta f)(x, n) = (f(x), n) \quad x \in X\,, \quad n \in N\,,$$
$$(\Theta f)(r, n) = (\bar{f}(r), n) \quad r \in R\,, \quad n \in N\,.$$

Θ is obviously a full embedding.

c) Now we construct the embedding $\Psi : G \to K^{l_2}$ as follows. Given any object $G = (X, R)$ of $\mathbf{G}$, consider the space $\mathscr{M}(\Theta G) = \mathscr{M}(\tilde{X}, \tilde{R}, d_G)$. If $r \in \tilde{R}$, we denote

$\mathscr{K}_r = e_r(K_{d_G}(r))$ as in II.4. For $i = 0, 1, 2, \ldots$ put

$$A_G^i = \bigcup_{|d_G(r)|=i} \mathscr{K}_r, \quad \Sigma_G(o_i) = \bigcup_{k>i} \overline{A_G^k},$$

where the closure is taken in $\beta\mathscr{M}(\Theta G)$. Now, we define $\Psi G = \Sigma_G : l_2 \to K$, where $\Sigma_G(o_i)$ are as above, $\Sigma_C\binom{o_m}{o_n}$ are inclusions. They are obviously closed. The spaces $\Sigma_G(o_i)$ are σ-compact. Further, $\bigcup_{k=i+1}^{s+1} \overline{A_G^k}$ is a compact neighbourhood of each point $y \in \overline{A_G^s}$ in $\Sigma_G(o_i)$ so that all spaces $\Sigma_G(o_i)$ are locally compact. Given a compatible mapping $f : G \to G'$, let $\Sigma_f(o_0) : \Sigma_G(o_0) \to \Sigma_{G'}(o_0)$ be the unique extension of $\mathscr{M}(\Theta f)$, let $\Sigma_f(o_i)$ $(i > 0)$ be the domain-range-restrictions of $\Sigma_f(o_0)$. One can verify that Ψ is really an embedding.

d) We have to prove that Ψ is full. Let $\tau : \Sigma_G \to \Sigma_{G'}$ be a morphism of K^{l_2}. It suffices to prove that τ_{o_0} is an extension of $\mathscr{M}(g)$ for some $g : \Theta G \to \Theta G'$ in $\mathscr{S}\mathbf{G}$. First, τ_{o_0} cannot be constant because $\bigcap_n \Sigma_{G'}(o_n) = \emptyset$. Thus, by VI.1, τ_{o_0} is an extension of *a* continuous mapping $\delta : \mathscr{M}(\Theta G) \to \mathscr{M}(\Theta G')$. By Basic Lemma, to complete the proof we have to show that τ_{o_0} is non-constant on any $\mathscr{K}_r \subset \mathscr{M}(\Theta G)$. Thus, let us suppose that τ_{o_0} maps $\mathscr{K}_r$ onto $\{q\}$ for some $r \in \tilde{R}$, $q \in \mathscr{M}(\Theta G')$. Denote by T the set of all $d_{G'}(t)$ such that $q \in \mathscr{K}_t \subset \mathscr{M}(\Theta G')$. It follows from the construction of Θ that card $T \leqq 3$; if card $T = 3$, then either $T = \{-1, 1, 2\}$ or $T = \{-1, 1, -2\}$; if card $T = 2$, then $T = \{n, n + 1\}$ for some $n \geqq 1$ or $n + 1 \leqq -1$; if card $T = 1$, then $T = \{n\}$, $n \neq 0$. In all these cases, by the first lemma in II.5 and by the construction of $\mathscr{M}(\Theta G)$, $\delta(\mathscr{K}_s) = \{q\}$ for all $s \in \tilde{R}$ with either $d_G(s) > |d_G(r)|$ or $d_G(s) < -|d_G(r)|$. But then necessarily $q \in \mathscr{M}(\Theta G') \cap \bigcap_{m=1}^{\infty} \Sigma_G(o_m)$, which is a contradiction.

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