Arun Kumar Srivastava

Near reflections

Czechoslovak Mathematical Journal, Vol. 27 (1977), No. 4, 619–625

Persistent URL: http://dml.cz/dmlcz/101500

Czechoslovak Mathematical Journal, 27 (102) 1977, Praha

NEAR REFLECTIONS

ARUN K. SRIVASTAVA*), Delhi

(Received October 15, 1975)

0. INTRODUCTION

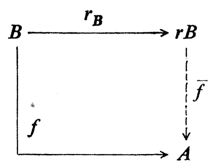
Near reflections first made their appearance in HARRIS [3] where they were called stable-reflections. The name near-reflections was suggested in SRIVASTAVA [7]. Recently SKULA [6] studied the same concept and called it 'quasi-reflection. Near-reflections have also been dealt with in SHUKLA and Srivastava [5] from a point of view different from Skula's [6]. The name near-reflection appears to be preferable at present (BARON [1] uses the name 'quasi-reflection' for a concept different from Skula's [6]).

In Skula [6] the main concern was to find analogs of certain familiar results relating (genuine) reflections with limits. This led to a consideration of limits with respect to certain special diagrams. The present note reports a few observations relating near reflections, limits and near limits including one which says exactly when a nearly reflective subcategory's inclusion functor preserves near limits. A kind of Herrlich-Strecker's version of Freyd-Isbell Theorem for nearly reflective subcategories is also included.

All undefined categorical concepts come from HERRLICH and STRECKER [4]. All subcategories are assumed to be full and replete. Dual notions are not defined but their use is made.

1. DEFINITIONS

1.1. A subcategory $\mathbf{A}$ of $\mathbf{B}$ is *nearly reflective* iff for each $B \in \mathrm{Ob}\mathbf{B}$, there exists an object $rB \in \mathbf{A}$ and a morphism $r_B : B \to rB$ such that for any morphism $f : B \to A$, $A \in \mathrm{Ob}\mathbf{A}$, there exists a (not necessarily unique) morphism $\bar{f}$ rendering commutative the following diagram



*) Financial support from the Council of Scientific and Industrial Research is gratefully acknowledged.

Moreover, $q \circ r_B = r_B$ is possible only when $q = \mathrm{Id}rB$. The morphism r_B is a *near universal arrow* from B to the inclusion functor from ***A*** to ***B*** (near universality carries the obvious meaning and need not be defined). It is clear what *near limits* must mean.

Examples ([5], [6]). The following list contains some examples of nearly reflective/nearly coreflective subcategories. ***A*** denotes a subcategory of ***B***.

	B	**A**	Nearly reflective/ nearly coreflective	Near reflection/ near coreflection
i)	T_1-spaces and continuous maps extendable to their Wallman compactifications	Compact spaces	nearly reflective	Wallman compactification
ii)*	Compact Hausdorff spaces and continuous maps	extremally disconnected spaces	nearly coreflective	extremally disconnected cover
iii)	Partially ordered sets and orderpreserving maps	complete partially ordered sets	nearly reflective	Mc Neille completion
iv)	Semigroups and homomorphisms	semigroups with identity elements	nearly reflective	adding identity
v)	Banach spaces and linear transformations of norm at most 1	injective Banach spaces	nearly reflective	injective envelope
vi)	Metric spaces with functions not increasing distance	injective metric spaces	nearly reflective	injective envelope
vii)	Distributive lattices and lattice homomorphisms	injective distributive lattices	nearly reflective	injective envelope

*) See Shukla and Srivastava [5] for a most of similar examples.

Note that in a balanced category the only epi-near coreflections which are genuine coreflections are isomorphisms. Thus, in example (ii) above, extremally disconnected spaces are 'totally non-coreflective' in the sense that no extremally disconnected cover can work as a coreflection unless it is a cover of an extremally disconnected space.

Several constructions in mathematics suggest considering a weaker version of near reflection; it is (in the notations of Definition 1.1) a near reflection except that $q \circ r_B = r_B$ is possible even if q is an automorphism. Let us call the resulting concept 'almost near reflection'. The algebraic closure construction of a field makes the subcategory of algebraically closed fields almost nearly reflective in the category of fields and their unitary homomorphisms. Enochs [2] has provided a torsion free cover for every module over an integral domain; via this, the subcategory of torsion free modules becomes almost nearly coreflective in the category of modules over a fixed integral domain. W. TAYLOR*) has some examples of almost nearly reflective subcategories from mathematical logic also.

Remark. It is easy to show that a near reflection is unique upto equivalence (Harris [3], Skula [6]). It is not only unique upto an equivalence but, as is the case with genuine reflections, it is unique upto a unique equivalence. Almost near reflections are also unique upto equivalence but the equivalence is not so 'sharp' (recall that $\mathbb{C}$ is an algebraic closure of $\mathbb{R}$ and that the identity map and the complex conjugation are two distinct automorphisms of $\mathbb{C}$ which keep $\mathbb{R}$ 'fixed').

2. CLOSURE PROPERTIES

Not all closure properties of reflective subcategories carry over to nearly reflective subcategories. However, the following holds and can easily be verified.

2.1. Theorem. (Skula [6]). *Nearly reflective subcategories are closed under products.*

2.2. Theorem. *Nearly reflective subcategories are closed under the formation of coretractions.*

Nearly reflective subcategories in general are not closed under equalizers as shown by the following example.

The subcategory $\mathbf{C}$ of complete partially ordered sets is nearly reflective in the category $\mathbf{P}$ of partially ordered sets and order-preserving maps**) (Skula [6]). Let S be the family of all closed subsets of $\mathbb{R}$ (with the usual topology). It, under inclusion, is a complete lattice (the g.l.b. being the intersection and the l.u.b. being the closure of the union). Let A and C be two intervals in the negative part of $\mathbb{R}$ with 0 as the end point of C and $A \subseteq C$. Define two functions $f : S \to S$ and $g : S \to S$ by setting

$$f(X) = X \cap A\,, \quad g(X) = X \cap C$$

for each $X \in S$. Clearly, f and g are order-preserving.

The equalizer of f and g is

$$E = \{X \in S \mid X \cap A = X \cap C\}\,.$$

Each member of the family $\{[1/n, 2] \mid n \geqq 1\}$ belongs to E. Consider the subset $\{A\} \cup \{[1/n, 2] \mid n \geqq 1\}$ of E. Then its l.u.b. is $A \cup [0, 2]$. Since $A \cup [0, 2] \cap A =$

---

*) Private communication.

**) This example and the subsequent observation (which inspired Theorem 2.3) were made known to the author by B. BANASCHEWSKI.

$= A$ and $A \cup [0, 2] \cap C = A \cup \{0\}$, it follows that $A \cup [0, 2] \notin E$. Thus, E is not complete showing that $\mathbf{C}$ is not closed under equalizers.

Next, we observe that nearly reflective subcategories when closed under equalizers are necessarily reflective.

2.3. Theorem. *For a nearly reflective subcategory* $\mathbf{A}$ *of a category* $\mathbf{B}$ *with equalizers equivalent are*:

(a) $\mathbf{A}$ is closed under equalizers

(b) $\mathbf{A}$ is reflective.

Proof. If (a) holds and $r_B : B \to rB$ is a near reflection of an object $B \in \mathrm{Ob}\mathbf{B}$, then we show that r_B is a reflection. For any $f : B \to A$, $A \in \mathrm{Ob}\mathbf{A}$, let $m, n : rB \to A$ be two morphisms such that $m \circ r_B = f = n \circ r_B$. Let $e : E \to rB$ be the equalizer of (m, n). Then $E \in \mathrm{Ob}\mathbf{A}$ and there must exist a unique morphism $p : B \to E$ with $e \circ p = r_B$. Also, since r_B is a near reflection, there is a $q : rB \to E$ such that $q \circ r_B = p$. Hence $e \circ q = \mathrm{Id}rB$. Thus e is an epimorphism showing that $m = n$ and (b) holds. The converse needs no proof. Q.E.D.

2.4. Corollary. *Nearly reflective subcategories are not closed under pullbacks.*

Proof. We consider the subcategory consisting of complete partially ordered sets of the category of partially ordered sets and order-preserving maps. If it were closed under pullbacks then since it has an initial object, it must have equalizers which by Theorem 2.3 cannot happen. Q.E.D.

3. NEAR REFLECTIONS, COLIMITS AND NEAR (CO) LIMITS

Let $\mathbf{A}$ be a nearly reflective subcategory of a category $\mathbf{B}$ and $E : \mathbf{D} \to \mathbf{A}$ be a diagram in B. If

$$\left(L, ED \xrightarrow{l_D} L\right)_{D \in \mathrm{Ob}\mathbf{D}}$$

is a colimit of $I \circ D$ in $\mathbf{B}$ where $I : \mathbf{A} \to \mathbf{B}$ is the inclusion functor then unlike the case of genuine reflectivity, it need not be true that

$$\left(rL, ED \xrightarrow{r_L \circ l_D} rL\right)_{D \in \mathrm{Ob}\mathbf{D}}$$

is a colimit of E in $\mathbf{A}$; it is, however, a near colimit of E in $\mathbf{B}$. For, if

$$\left(B, ED \xrightarrow{a_D} A\right)_{D \in \mathrm{Ob}\mathbf{D}}$$

with $A \in \mathbf{A}$ is any compatible family for E, then since L is a colimit, there must be a unique morphism $f : L \to A$ such that $f \circ l_D = a_D$ for each $D \in \mathrm{Ob}\mathbf{D}$. There must be, therefore, a morphism $\bar{f} : rL \to A$ with $\bar{f} \circ r_L = f$. Now if $e : rL \to rL$ is a morphism with $e \circ r_L \circ l_D = r_L \circ l_D$ for all $D \in \mathrm{Ob}\mathbf{D}$, then by the universal property of L, it must be the case that $e \circ r_L = r_L$ whence, since r_L is a near reflection, e can only be $\mathrm{Id}rL$. We have thus proved

3.1. Theorem. *If $\mathbf{A}$ is a nearly reflective subcategory of $\mathbf{B}$, then the near reflection of a colimit in $\mathbf{B}$ of a diagram in $\mathbf{A}$ is a near colimit of that diagram in $\mathbf{A}$.*

Concerning preservation of near limits by the inclusion functor $I : \mathbf{A} \to \mathbf{B}$ of a nearly reflective subcategory $\mathbf{A}$ of $\mathbf{B}$, we have the following result. First we explain certain notations we shall make use of. By $[P, Q]$ we shall denote the class of all natural transformations between any two functors P and Q having the same domain and codomain. For any two categories $\mathbf{X}$ and $\mathbf{Y}$ and any $X \in \mathrm{Ob}\mathbf{X}$, $K(X) : \mathbf{Y} \to \mathbf{X}$ shall denote the 'X-valued' constant functor. Thus, if $F : \mathbf{D} \to \mathbf{Y}$ is a diagram and

$$\left(X, X \xrightarrow{l_D} FD\right)$$

a limit in $\mathbf{Y}$ of F, then its limit diagram corresponds to a natural transformation $l : K(X) \to F$.

3.2. Theorem. *The inclusion functor $I : \mathbf{A} \to \mathbf{B}$ of a nearly reflective subcategory $\mathbf{A}$ of $\mathbf{B}$ preserves near limits iff*

(a) *for all diagrams $E : \mathbf{D} \to \mathbf{A}$ having a limit in $\mathbf{A}$ and all $B \in \mathrm{Ob}\mathbf{B}$, the function $n' : [K(rB), E] \to [K(B), I \circ E]$ defined by $n'(v)_D = n(v_D)$ for any $v \in [K(rB), E]$ is a surjection where $n : \mathbf{A}(rB, A) \to \mathbf{B}(B, A)$, $A \in \mathrm{Ob}\mathbf{A}$, is a surjection defined by*

$$n\left(rB \xrightarrow{g} A\right) = B \xrightarrow{g \circ r_B} A$$

and

(b) *if $E : \mathbf{D} \to \mathbf{A}$ is a diagram in $\mathbf{A}$ having a near limit*

$$\left(A, A \xrightarrow{v_D} ED\right)_{D \in \mathrm{Ob}\mathbf{D}},$$

then $v_D \circ e = v_D$ for all $D \in \mathrm{Ob}\mathbf{D}$, where $e : A \to A$, must imply that $e = \mathrm{Id}A$.

Proof. Suppose I preserves near limits. To prove (a), let

$$\left(A, A \xrightarrow{v_D} ED\right)_{D \in \mathrm{Ob}\mathbf{D}}$$

be a near limit in $\boldsymbol{A}$ of D. Then

$$\left(A, A \xrightarrow{v_D} ED\right)_{D \in \mathrm{Ob}\boldsymbol{D}}$$

is a near limit of $I \circ D$ in $\boldsymbol{B}$ also. If $u \in [K(B), I \circ E]$ then it represents a compatible family

$$\left(B, B \xrightarrow{u_D} ED\right)_{D \in \mathrm{Ob}\boldsymbol{D}}$$

and so there exists a morphism $f : B \to A$ with $v_D \circ f = u_D$ for all $D \in \mathrm{Ob}\boldsymbol{D}$. Clearly the natural transformation $\bar{u} \in [K(rB), E]$ whose 'value' at any $D \in \boldsymbol{D}$ is $v_D \circ \bar{f}$ is the inverse image of u under n'. This proves (a), (b) is easier to prove.

Conversely, suppose I satisfies (a) and (b). Let

$$\left(A, A \xrightarrow{v_D} ED\right)_{D \in \mathrm{Ob}\boldsymbol{D}}$$

be a near limit in $\boldsymbol{A}$ of a diagram $E : \boldsymbol{D} \to \boldsymbol{A}$. To prove that it also is a near limit in $\boldsymbol{B}$, let for any $B \in \mathrm{Ob}\boldsymbol{B}$, $u \in [K(B), I \circ E]$. Then there exists $\bar{u} : [K(rB), E]$ with $n'(\bar{u}) = u$. There must be, therefore, a morphism $g : rB \to A$ such that $v_D \circ g = \bar{u}_D$ for all $D \in \mathrm{Ob}\boldsymbol{D}$. Clearly, $g \circ r_B : B \to A$ is such that $v_D \circ g \circ r_B = u_D$ for all $D \in \mathrm{Ob}\boldsymbol{D}$. It remains to be verified that for any $e : A \to A$, $v_D \circ e = v_D$ for all $D \in \mathrm{Ob}\boldsymbol{D}$ means that $e = \mathrm{Id}A$, but this is just the hypothesis (b). Q.E.D.

4. FREYD-ISBELL THEOREM FOR NEARLY REFLECTIVE SUBCATEGORIES

This section is an outcome of a desire to characterize nearly reflective subcategories on the lines on which reflective subcategories are characterized in the Freyd-Isbell Theorem. (We shall adopt Herrlich-Strecker [4] version of this theorem).

In a category call a morphism e *semiepimorphism* iff $a \circ e = e$ implies that a is the identity map. Call a monomorphism m a *strong extremal monomorphism* if $m = g \circ h$ and h a semiepimorphism implies that h is an isomorphism. Categories whose each morphism is factorable as a semiepimorphism followed by a strong extremal monomorphism will be called categories with *semiepi-strong extremal mono-factorization**) property. If $A \to B$ is a semiepimorphism, B will be called a *semi-quotient* object of A. A category may be called *strongly colocally small* if its each object has only a set of 'nonisomorphic' semi-quotient objects.

Semiepimorphisms do not compose (in $\boldsymbol{S}$ these are precisely onto functions). Strong extremal monomorphisms lie between coretractions and extremal mono-

*) This terminology is not good because closely resembling name with a different meaning has been used by Herrlich and Strecker [4]. It is to be accepted only on an ad-hoc basis.

morphisms. Nearly reflective subcategories are closed under strong extremal subobjects. The main observation of this section can now be presented; the proof, since not hard, is omitted.

4.1. Theorem. *Let* **B** *be a category which*

(i) *is strongly colocally small*

(ii) *has products*

(iii) *has semiepi-strong extremal mono-factorization property.*

If **A** *is a subcategory of* **B** *then equivalent are*

(a) **A** *is nearly reflective in* **B**

(b) **A** *is closed under products and strong extremal subjects.*

Following Herrlich and Strecker [4] with appropriate modification it is easy to settle the problems of generation and intersectibility of nearly reflective subcategories.

References

[1] *Baron, S.:* Reflectors as composition of epireflectors. Trans. AMS *136*, 499—508 (1969).

[2] *Enochs, E.:* Torsion free covering modules. Proc. AMS *14*, 884—889 (1963).

[3] *Harris, D.:* Wallman compactification as a functor. Gen. Top. & Appl. *1*, 273—281 (1971).

[4] *Herrlich, H. & Strecker, G. E.:* Category Theory. An Introduction. Allyn and Bacon, Boston (1974).

[5] *Shukla, W. & Srivastava, Arun K.:* Local reflectivity + stable reflectivity = reflectivity. Gen. Top. and Appl. *5*, 61—68 (1975).

[6] *Skula, L.:* Quasi-reflections and limits. Czech. Math. J., *24*, 522—527 (1974).

[7] *Srivastava, Arun K.:* Non-Genuine Adjunctions. Thesis. Indian Institute of Technology, Delhi (1974).

Author's address: Indian Institute of Technology, Department of Mathematics, New Delhi 110029, India.