Czechoslovak Mathematical Journal

Štefan Porubský

Commutative semi-primary x-semigroups

Czechoslovak Mathematical Journal, Vol. 27 (1977), No. 3, 467–472

Persistent URL: http://dml.cz/dmlcz/101483

Czechoslovak Mathematical Journal, 27 (102) 1977, Praha

COMMUTATIVE SEMI-PRIMARY x-SEMIGROUPS

ŠTEFAN PORUBSKÝ, Bratislava

(Received July 25, 1975)

In this paper we intend to prove some results on semi-primary semigroups equipped with a system of x-ideals. Our effort was motivated by the results of the almost coincidently entitled paper [3].

Let S be a commutative semigroup written multiplicatively. We say that a *system of x-ideals* (or an *x-system*) is defined in S if to every subset A of S there corresponds a subset A_x of S such that

(1) $$A \subseteq A_x$$

(2) $$A \subseteq B_x \quad \text{implies} \quad A_x \subseteq B_x$$

(3) $$AB_x \subseteq B_x \cap (AB)_x$$

where AB is the set of all products ab with a in A and b in B. If $A = A_x$ we shall say that A is an *x-ideal*. For the sake of brevity we shall call a semigroup equipped with an x-system *x-semigroup*. This concept of x-ideals was introduced by AUBERT in [1] following an idea of Lorenzen. For the details and the relationships to the known ideal theories [1] may be consulted.

Proposition 1 of [1] shows that the family of all x-ideals of S forms a complete lattice with respect to set-inclusion. The union within this lattice will be called the *x-union* and denoted by $\bigcup_x$, i.e.

$$\bigcup_{x\, i\in I} A_x^{(i)} = \Big(\bigcup_{i\in I} A_x^{(i)}\Big)_x .$$

The *x-product* of two subsets A and B of S is defined as the set $(AB)_x$ and denoted by $A \circ B$. It follows from Theorem 1 of [1] that $A \circ B = A \circ B_x = A_x \circ B_x$ for any subsets A and B of S. In the sequel A_x^n or A^n means the x-product of n factors A_x or the usual product of n factors A, respectively.

An x-ideal P_x is said to be prime if $ab \in P_x$ implies $a \in P_x$ or $b \in P_x$. An x-ideal Q_x is said to be primary if $ab \in Q_x$ and $a \notin Q_x$ imply $b^n \in Q_x$ for some positive integer n.

Lemma 1 (Aubert [1]). *The x-ideal P_x is prime if and only if $A_x \circ B_x \subseteq P_x$ and $A_x \nsubseteq P_x$ imply $B_x \subseteq P_x$.*

The (nilpotent) *radical* $\sqrt{A_x}$ of A_x is the set of all elements b in S such that $b^n \in A_x$ for some (positive) integer n. The operation of forming the radical has a number of expected properties, e.g. $\sqrt{(A_x \circ B_x)} = \sqrt{(A_x \cap B_x)} = \sqrt{A_x} \cap \sqrt{B_x}$ and the following lemma holds.

Lemma 2 (Aubert [1]). *The radical of an x-ideal A_x is the intersection of all prime x-ideals containing A_x.*

An x-system is said to be of *finite character* if for N finite, $A_x = \bigcup_{N \subseteq A} N_x$ for every subset A of S.

Lemma 3 (Aubert [1]). *An x-system is of finite character if and only if the set-theoretic union of any chain of x-ideals is an x-ideal.*

Let S be an x-semigroup. Put $\Omega_x = \bigcap_{A \subseteq S} A_x$. (It may happen that Ω_x is void.) An element t is *nilpotent* if $t^n \in \Omega_x$ for some n. An x-ideal A_x is *nil* if each element of A_x si nilpotent. The radical of Ω_x (*the x-nilradical of S*) is the set of all nilpotent elements in S and according to Lemma 2 it equals the intersection of all prime x-ideals in S. The x-nilradical of S is *half-prime*, that is it coincides with its radical. Proposition 12 of [1] immediately yields the following lemma.

Lemma 4. *For any two x-ideals A_x and B_x we have $A_x \circ B_x \subseteq \sqrt{\Omega_x}$ if and only if $A_x \cap B_x \subseteq \sqrt{\Omega_x}$.*

An x-ideal A_x is called *semi-primary* if its radical $\sqrt{A_x}$ is a prime x-ideal. Commutative x-semigroup is *semi-primary* if its, each x-ideal is semi-primary. For instance, valuation rings are semi-primary.

Theorem 1. *Let S be an x-semigroup with non-void x-radical $\sqrt{\Omega_x}$ and let $\sqrt{\Omega_x}$ be a proper ($\neq S$) prime x-ideal. Then an x-ideal A_x is nil if and only if there is a non-nil B_x with $A_x \cap B_x \subseteq \sqrt{\Omega_x}$.*

Proof. We know from Lemma 4 that $A_x \cap B_x \subseteq \sqrt{\Omega_x}$ is equivalent to $A_x \circ B_x \subseteq \subseteq \sqrt{\Omega_x}$. Since $\sqrt{\Omega_x}$ is a proper x-ideal, there is a non-nilpotent element t in S. Now, if A_x is nil we have, say, $(t)_x \circ A_x \subseteq A_x \subseteq \sqrt{\Omega_x}$. The reverse statement of the theorem follows from the fact that $\sqrt{\Omega_x}$ is prime.

Corollary. *If S is a semi-primary x-semigroup with at least one nilpotent element and $A_x \cap B_x$ is nil then at least one of the x-ideals A_x and B_x is nil.*

Theorem 2. *Let S be an x-semigroup and $\mathcal{S}$ such a system of x-ideals in S that each x-ideal not in $\mathcal{S}$ is semi-primary. Then for any two prime x-ideals A_x and B_x we have either $A_x \cap B_x \in \mathcal{S}$ or A_x and B_x are ordered under the set-inclusion.*

Proof. If $A_x \cap B_x$ is not in $\mathscr{S}$ then $A_x \cap B_x$ is semi-primary. But $A_x \cap B_x = \sqrt{A_x} \cap \sqrt{B_x} = \sqrt{(A_x \cap B_x)}$ which implies that $A_x \cap B_x$ is prime. On the hand, $A_x \cap B_x \supseteq A_x \circ B_x$ and therefore $A_x \cap B_x \supseteq A_x$ or $A_x \cap B_x \supseteq B_x$.

Corollary 1. *In a semi-primary x-semigroup, prime x-ideals form a chain.*

Corollary 2. *If in an x-semigroup the x-ideals different from Ω_x are semi-primary then for any two prime x-ideals A_x and B_x either $A_x \cap B_x = \Omega_x$ or $A_x \supseteq B_x$ or $B_x \supseteq A_x$.*

Corollary 3. *A semi-primary x-semigroup S with $S \circ S = S$ is quasi-local (i.e. with a unique maximal x-ideal).*

Theorem 3. *Let S be an x-semigroup. Then the following statements are equivalent:*

(1) *S is a semi-primary x-semigroup.*
(2) *Every principal x-ideal of S is semi-primary.*
(3) *Prime x-ideals of S form a chain.*
(4) *The radicals of all the x-ideals in S form a chain.*

The proof of the equivalence of the first three statements runs along the same lines as the proof of Theorem 1 in [3]. The equivalence between (3) and (4) is an easy exercise.

Theorem 4. *Let S be a semi-primary x-semigroup. Then for any two finitely generated x-ideals A_x and B_x we have either $A_x^n \subseteq B_x$ or $B_x^n \subseteq A_x$ for some n.*

Proof. Since S is semi-primary, $\sqrt{A_x} \subseteq \sqrt{B_x}$ may be assumed. Let $A_x = \bigcup_{x\, i=1}^{k} (a_i)_x$. Then $a_i^m \in B_x$ for some m and every $i = 1, \ldots, k$. According to Theorem 1 of [1] the x-multiplication is distributive with respect to the x-union and therefore $A_x^{km} \subseteq B_x$.

Corollary. *In a semi-primary x-semigroup the principal x-ideals generated by idempotents form a chain.*

To show that Theorem 4 cannot be extended to all x-ideals we borrow the following example from [4]. Let $R = F[x_1] \cup F[x_1, x_2] \cup F[x_1, x_2, x_3] \cup \ldots$, where F is a field and $\{x_1, x_2, x_3, \ldots\}$ is a countable set of indeterminates such that $x_i^2 = 0$, $x_i x_j = x_j x_i$ and $x_i a = a x_i$ for all $a \in F$ and i. Every ideal in R is primary and thus R is semi-primary. Let A_1 and A_2 be ideals generated by $\{x_1, x_3, x_5, \ldots\}$ and $\{x_2, x_4, x_6, \ldots\}$ respectively. If $A_i^n \subseteq A_2$ for some n, then $f^n = 0$ for every $f \in A_1$ which is not true. Similarly $A_2^n \nsubseteq A_1$ for all n.

Theorem 5. *A sufficient condition for an x-semigroup to be semi-primary is that for any two x-ideals A_x and B_x there is an integer n (depending on A_x and B_x) with $A_x^n \subseteq B_x$ or $B_x^n \subseteq A_x$.*

Proof. If A_x and B_x are any two prime x-ideals in S then we have, say, $A_x^n \subseteq B_x$ for some n. But then $A_x \subseteq B_x$ and Theorem 3 completes the proof.

Corollary. *Let S be an x-semigroup of finite character satisfying the ascending chain condition for x-ideals. Then S is semi-primary if and only if to any two x-ideals A_x and B_x there is an integer n with $A_x^n \subseteq B_x$ or $B_x^n \subseteq A_x$.*

To prove this corollary recall that in an x-semigroup of finite character with ACC for x-ideals each x-ideal is finitely generated.

Theorem 6. *Let S be a semi-primary x-semigroup. If A_x is finitely generated then A_x^n is contained in a principal x-ideal for some n.*

Proof. Let $A_x = \bigcup_{i=1}^{k}{}_x (a_i)_x$. We know that the radicals $\sqrt{(a_i)_x}$ for $i = 1, \ldots, k$ form a chain. Let $\sqrt{(a_k)_x}$ contain all the remaining ones. Then $a_i^m \in (a_k)_x$ for some m and all $i = 1, \ldots, k$ which yields $A_x^{km} \subseteq (a_k)_x$ and the proof is complete.

The (unique) maximal ideal of the ring R from the example above shows that the theorem does not hold for arbitrary x-ideals.

Corollary 1. *If P_x is a finitely generated prime x-ideal in a semi-primary x-semigroup then P_x is the radical of a principal ideal.*

We know from the previous proof that $P_x^n \subseteq (a)_x$ for some $a \in P_x$ and thus $\sqrt{(a)_x} = P_x$.

Corollary 2. *In a semi-primary x-semigroup no finitely generated prime x-ideal P_x is the set-theoretic union of prime x-ideals properly contained in P_x.*

P_x is the radical of a principal x-ideal which is generated by one of its elements and therefore this element cannot belong to a prime x-ideal properly contained in P_x.

The next theorem originates in [2]. It is only another version of Lemma 3.4 of [2] in terms of x-ideals and its proof can also be rewritten from [2] without difficulties.

Theorem 7. *Let S be a quasi-local x-semigroup of finite character with $S \circ S = S$. Then S is a semi-primary x-semigroup satisfying the ACC for prime x-ideals if and only if for any prime x-ideal P_x different from Ω_x there exists a prime x-ideal $N(P_x)$ properly contained in P_x such that for each prime x-ideal P'_x properly in P_x we have $P'_x \subseteq N(P_x)$.*

Corollary. *Let S be a local (i.e. quasi-local with ACC for x-ideals) x-semigroup of finite character with $S \circ S = S$. Then S is semi-primary if and only if for every prime x-ideal $P_x \neq \Omega_x$ the x-union of all prime x-ideals properly contained in P_x is a prime x-ideal properly contained in P_x.*

An x-semigroup S will be called (*von Neumann*) *quasi-regular* if $a \in (a)_x^2$ for each $a \in S$. Plainly, an x-semigroup is quasi-regular if and only if $A_x^2 = A_x$ for every x-ideal A_x. An x-semigroup S is said to be *regular* if it is quasi-regular and its x-system enjoys the property that $(a)_x = \{a\} \cup aS$ for every $a \in S$ (this is the so-called Lorenzen's x-system, cf. [1]). A semigroup S equipped with a Lorenzen's x-system is regular if and only if every principal x-ideal in S is generated by an idempotent. Quasi-regular x-semigroup S need not be regular. Perhaps the simplest example of this kind is obtained in the case when S is equipped with an x-system consisting from only one x-ideal S.

Theorem 8. *Let S be a quasi-regular x-semigroup. Then the following statements are equivalent:*

(1) *Every x-ideal in S is prime.*

(2) *S is a semi-primary x-semigroup.*

(3) *Principal x-ideals of S form a chain.*

(4) *All x-ideals of S form a chain.*

If in addition S is regular then the statements above are equivalent with

(5) *Principal x-ideals generated by idempotents form a chain.*

Proof. (2) implies (3). For any two principal x-ideals $(a)_x$ and $(b)_x$ we have, say, $(a)_x^n \subseteq (b)_x$ according to Theorem 4. But S is quasi-regular and thus $(a)_x^n = (a)_x$.
(3) implies (4). Let $A_x \nsubseteq B_x$. Then there exists $a \in A_x - B_x$. Now for any b in B_x we have $b \in (b)_x \subseteq (a)_x$, that is $B_x \subseteq (a)_x$ and also $B_x \subseteq A_x$.

The rest of the proof can follow the lines of proof of Theorem 2 in [3].

Corollary 1. *Let S be a semi-primary x-semigroup. Then every x-ideal in S is prime if and only if S is quasi-regular.*

The proof of this and the next corollary is based on the same ideas as the proofs of Corollary 1 and 2 of [3].

Corollary 2. *Let S be a semigroup equipped with a Lorenzen's x-system. Then every x-ideal in S is prime if and only if S is regular and principal x-ideals generated by idempotents form a chain.*

References

[1] *Aubert, K. E.:* Theory of x-ideals. Acta Math. *107*, 1—52 (1962).
[2] *Gilmer, R., Ohm, J.:* Primary ideals and valuation ideals. Trans. Amer. Math. Soc. *117*, 237—250 (1965).
[3] *Lal, H.:* Commutative semi-primary semigroups. Czechoslovak Math. J. *25 (100)*, 1—3 (1975).
[4] *Satyanarayana, M.:* Generalized primary rings. Math. Ann. *179*, 109—114 (1969).

Author's address: 886 25 Bratislava, ul. Obrancov mieru 41, ČSSR (Matematický ústav SAV)