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AN UPPER BOUND FOR THE MINIMUM DEGREE OF A GRAPH

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Let G be a graph (in the sense of BEHZAD and CHARTRAND [1] or HARARY [5]) with the vertex set $V(G)$, the edge set $E(G)$, the connectivity $\kappa(G)$, and the minimum degree $\delta(G)$. Obviously, $0 \leqq \kappa(G) \leqq \delta(G)$. We say that $e \in E(G)$ is a κ-critical edge of G if $\kappa(G - e) = \kappa(G) - 1$. Analogously, we say that $v \in V(G)$ is a κ-critical vertex of G if $\kappa(G - v) = \kappa(G) - 1$. R. HALIN [3] proved that if each edge of G is κ-critical, then $\delta(G) = \kappa(G)$. G. CHARTRAND, A. KAUGARS, and D. R. LICK [2] proved that if each vertex of G is κ-critical and $\kappa(G) \geqq 2$, then $\delta(G) < (3\kappa(G) - 1)/2$ (they also proved that this inequality is – in a certain sense – the best possible). In the present note, these theorems will be generalized and extended.

It is clear that $|V(G)| > \kappa(G)$. If $|V(G)| = \kappa(G) + 1$, then $\delta(G) = \kappa(G)$. We shall assume that $|V(G)| \geqq \kappa(G) + 2$. We denote by Cut the set of all $R \subseteq V(G)$ such that the graph $G - R$ is disconnected. Obviously, Cut $\neq \emptyset$, and $\kappa(G) = \min\{|R|; R \in \text{Cut}\}$.

We say that a graph T is a *territory in* G if there exists $R \in$ Cut such that the following conditions hold:

(1) T is a component of $G - R$;

(2) if R_0 is a proper subset of R, then T is not a component of $G - R_0$;

(3) if T' is a proper subgraph of T and $R' \in$ Cut such that T' is a component of $G - R'$, then $|R| < |R'|$.

Let T be a territory in G. It is easy to see that there exists precisely one $R \in$ Cut such that the conditions (1)–(2) hold; denote $B(T) = R$. We denote by $C(T)$ the graph $G - B(T) - V(T)$. Moreover, we denote $b(T) = |B(T)|$. Obviously, $b(T) \geqq \kappa(G)$.

The concept of a territory in G is a generalization of the concept of an end of G (Ende von G) studied by W. MADER in [6]; a territory T in G is an end of G if and only if $b(T) = \kappa(G)$.

If x is a real number, then we denote by $[x]$ the maximum integer i such that $i \leqq x$.

We shall prove the following lemma. For $b(T) = \kappa(G)$, our lemma is closely related to Theorem 6 in [4].

Lemma. *Let T be a territory in G. If T contains a κ-critical edge of G, then*

$$|V(C(T))| \leqq \kappa(G) - \max\,(b(T) + 3 - |V(T)|,\ [(b(T) + 3)/2])\,.$$

Proof. Let T contain a κ-critical edge of G. Then there exists $S \subseteq V(G)$ such that $|S| = \kappa(G) - 1$ and that the graph $G - S - e$ is disconnected. Let H be a component of $G - S - e$. We denote by H' the graph $G - S - V(H)$. Obviously, there exist $u \in V(H)$ and $u' \in V(H')$ such that $e = uu'$. Denote

$$\text{(1)}\quad \begin{aligned} &W_{11} = V(T) \cap V(H)\,, && W_{12} = V(T) \cap S\,, && W_{13} = V(T) \cap V(H')\,,\\ &W_{21} = B(T) \cap V(H)\,, && W_{22} = B(T) \cap S\,, && W_{23} = B(T) \cap V(H')\,,\\ &W_{31} = V(C(T)) \cap V(H)\,, && W_{32} = V(C(T)) \cap S\,, && W_{33} = V(C(T)) \cap V(H')\,. \end{aligned}$$

Moreover, for $j, k = 1, 2, 3$, we denote

$$\text{(2)}\qquad f_{jk} = |W_{jk}|\,.$$

Clearly, $f_{11}, f_{13} \geqq 1$, $f_{21} + f_{22} + f_{23} = b(T)$, $f_{31} + f_{32} + f_{33} \geqq 1$, and $f_{12} + f_{22} + f_{32} = \kappa(G) - 1$. Since the graphs $G - (W_{21} \cup W_{22} \cup W_{23})$ and $G - (\{e\} \cup W_{12} \cup W_{22} \cup W_{32})$ are disconnected, it holds for any $j, j', k, k' = 1, 2, 3$ with either $|j - j'| = 2$ or $|k - k'| = 2$ that if $v \in W_{jk}$, $v' \in W_{j'k'}$, and $vv' \neq e$, then v and v' are not adjacent in G.

Since $f_{11} \geqq 1$ and $f_{31} + f_{32} + f_{33} \geqq 1$, we have that $G - (\{u'\} \cup W_{12} \cup W_{22} \cup W_{21})$ is disconnected. Since T is a territory in G, we have that $f_{12} + f_{22} + f_{21} \geqq b(T)$. Therefore, $f_{12} \geqq b(T) - f_{21} - f_{22} = f_{23}$. Analogously we obtain that $f_{12} + f_{22} + f_{23} \geqq b(T)$, and thus $f_{12} \geqq f_{21}$. Since $|V(T)| \geqq f_{12} + 2$, we have that $|V(T)| \geqq \max\,(f_{21}, f_{23}) + 2$.

Assume that $f_{32} + f_{22} + f_{21} \geqq \kappa(G)$. Then $b(T) + \kappa(G) - 1 = f_{21} + f_{22} + f_{23} + f_{12} + f_{22} + f_{32} \geqq b(T) + \kappa(G)$, which is a contradiction. Hence, $f_{32} + f_{22} + f_{21} < \kappa(G)$. Therefore $G - (W_{32} \cup W_{22} \cup W_{21})$ is connected. Thus $f_{31} = 0$. Analogously we obtain that $f_{32} + f_{22} + f_{23} < \kappa(G)$ and $f_{33} = 0$. This implies that

$$\begin{aligned} |V(C(T))| = f_{32} &\leqq \kappa(G) - \max\,(f_{22} + f_{21}, f_{22} + f_{23}) - 1 = \\ &= \kappa(G) - b(T) + \min\,(f_{23}, f_{21}) - 1\,. \end{aligned}$$

We have

$$\min\,(f_{23}, f_{21}) \begin{cases} \leqq \max\,(f_{21}, f_{23}) \leqq f_{12} \leqq |V(T)| - 2\,. \\ \leqq [b(T)/2]\,. \end{cases}$$

Since $1 + b(T) - [b(T)/2] = [(b(T) + 3)/2]$, the statement of the lemma follows.

The lemma implies that if there exists a territory in G containing a κ-critical edge of G, then $\kappa(G) \geqq 4$. An example for $\kappa(G) = 4$ is in Fig. 1.

Let $n \geqq 0$ be an integer, and let T be a territory in G. We shall write "$P_n(T)$" instead of the statement

"either $\delta(T) \leqq n$ or T contains a κ-critical vertex of G"; analogously, we shall write "$Q_n(T)$" instead of the statement

"either $\delta(T) \leqq n$ or T contains a κ-critical edge of G". (Note that $\delta(T)$ denotes the minimum degree of T).

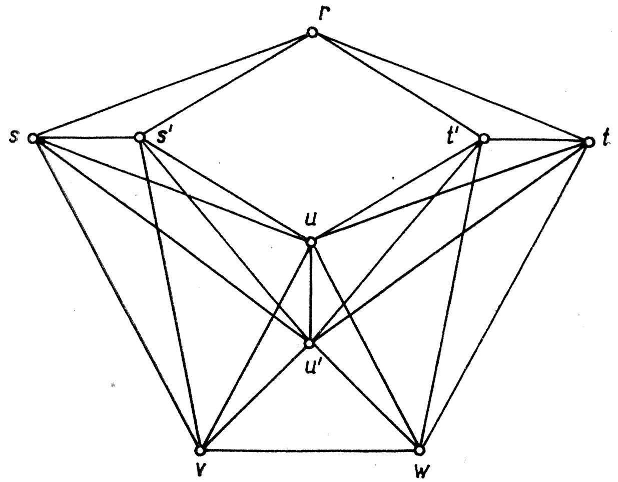

Fig. 1.

Proposition. *Let $n \geqq 0$ be an integer, and let T be a territory tn G. If $Q_n(T)$, then $P_n(T)$.*

Proof. Assume that $Q_n(T)$. If $\delta(G) \leqq n$, then $P_n(T)$. Let $\delta(T) > n$. Then T contains a κ-critical edge $e = uv$ of G. This means that there exists $S \subseteq V(G)$ such that $|S| = \kappa(G) - 1$ and that $G - S - e$ is disconnected. If either $G - S - u$ or $G - S - v$ is disconnected, then $P_n(T)$. Assume that both $G - S - u$ and $G - S - v$ are connected. Then it is easy to see that $V(G - S) = \{u, v\}$. Therefore $|V(G)| = \kappa(G) + 1$, which is a contradiction. Hence the proof is complete.

We say that territories T_1 and T_2 in G are separated if there exist $R \in \text{Cut}$ and distinct components H_1 and H_2 of $G - R$ such that T_1 is a subgraph of H_1 and T_2 is a subgraph of H_2.

The following theorem is the main result of this note:

Theorem. *Let $m \geqq \kappa(G)$ and $n \geqq 0$ be integers. Assume that*

either (A) $n < \kappa(G) - [(m + 5)/2]$ *and there exist separated territories T_1 and T_2 in G such that $Q_n(T_1)$, $Q_n(T_2)$ and* $\max(b(T_1), b(T_2)) \leqq m$;

or (B) $\kappa(G) - [(m + 5)/2] \leqq n < [(m - 2)/2]$ *and there exists a territory* T_0 *in* G *such that* $Q_n(T_0)$ *and* $b(T_0) \leqq m$;

or (C) $[(m - 2)/2] \leqq n$ *and there exists a territory* T *in* C *such that* $P_n(T)$ *and* $b(T) \leqq m$.

Then

$$\delta(G) \leqq m + n . \tag{3}$$

Proof. (A) If either $\delta(T_1) \leqq n$ or $\delta(T_2) \leqq n$, then (3) holds. Assume that both $\delta(T_1) > n$ and $\delta(T_2) > n$. Since $Q_n(T_1)$ and $Q_n(T_2)$, the lemma implies

$$|V(C(T_1))| \leqq |V(T_1)| + \kappa(G) - b(T_1) - 3 < |V(T_1)| ,$$

and analogously $|V(C(T_2))| < |V(T_2)|$.

Since T_1 and T_2 are separated, there exist $R \in \text{Cut}$ and the graphs H_1 and H_2 such that H_1 and H_2 are distinct components of $G - R$, T_1 si a subgraph of H_1 and T_2 is a subgraph of H_2. We denote by H'_1 the graph $G - R - V(H_1)$ and by H'_2 the graph $G - R - V(H_2)$.

Assume that there exists $u \in B(T_1) \cap V(H'_1)$. Then u is adjacent to no vertex of T_1. This implies that $B(T_1) - \{u\} \in \text{Cut}$ and that T_1 is a component of $G - - (B(T_1) - \{u\})$, which is a contradiction. Hence $B(T_1) \cap V(H'_1) = \emptyset$. This means that $V(H'_1) \subseteq V(C(T_1))$. Analogously we obtain that $V(H'_2) \subseteq V(C(T_2))$. Since $V(H_1) \subseteq V(H'_2)$ and $V(H_2) \subseteq V(H'_1)$, we have

$$|V(T_1)| \leqq |V(H_1)| \leqq |V(H'_2)| \leqq |V(C(T_2))| < |V(T_2)| \leqq$$
$$\leqq |V(H_2)| \leqq |V(H'_1)| \leqq |V(C(T_1))| < |V(T_1)| ,$$

which is a contradiction.

(B) If $\delta(T_0) \leqq n$, then (3) holds. Assume that $\delta(T_0) > n$. Since $Q_n(T_0)$, the lemma implies

$$\delta(G) \leqq b(T_0) + |V(C(T_0))| - 1 \leqq b(T_0) + \kappa(G) - [(b(T_0) + 3)/2] - 1 =$$
$$= b(T_0) + \kappa(G) - [(b(T_0) + 5)/2] \leqq m + \kappa(G) - [(m + 5)/2] \leqq m + n .$$

(C) If $\delta(T) \leqq n$, then (3) holds. Assume that $\delta(T) > n$. Since $P_n(T)$, we have that T contains a κ-critical vertex u of G. There exists $S \in \text{Cut}$ such that $|S| = \kappa(G)$ and $u \in S$. Let H be a component of $G - S$. We denote by H' the graph $G - S - - V(H)$. For $j, k = 1, 2, 3$ we define W_{jk} and f_{jk} by equalities (1) and (2). Clearly, $f_{12} \geqq 1, f_{21} + f_{22} + f_{23} = b(T), f_{31} + f_{32} + f_{33} \geqq 1, f_{11} + f_{21} + f_{31} \geqq 1, f_{12} + + f_{22} + f_{32} = \kappa(G)$, and $f_{13} + f_{23} + f_{33} \geqq 1$. Since the graphs $G - (W_{21} \cup \cup W_{22} \cup W_{23})$ and $G - (W_{12} \cup W_{22} \cup W_{32})$ are disconnected, we have that for any $j, j', k, k' = 1, 2, 3$ with either $|j - j'| = 2$ or $|k - k'| = 2$ it holds that if $v \in W_{jk}$ and $v' \in W_{j'k'}$, then v and v' are not adjacent in G.

Assume that $f_{21} = 0$. First, let $f_{31} \geqq 1$. Then $W_{22} \cup W_{32} \in \text{Cut}$. Since $f_{12} \geqq 1$, we have $f_{22} + f_{32} < \kappa(G)$, which is a contradiction. Next, let $f_{31} = 0$. Then $f_{11} \geqq 1$. This implies that W_{11} is a component of the graph $G - (W_{12} \cup W_{22} \cup W_{32})$, which contradicts the fact that T is a territory. This means that $f_{21} \geqq 1$. Analogously we obtain $f_{23} \geqq 1$.

If $f_{12} + f_{22} + f_{21} > b(T)$ and $f_{32} + f_{22} + f_{23} \geqq \kappa(G)$, then $b(T) + \kappa(G) = f_{12} + f_{22} + f_{32} + f_{21} + f_{22} + f_{23} > b(T) + \kappa(G)$ which is a contradiction. Hence either $f_{12} + f_{22} + f_{21} \leqq b(T)$ or $f_{32} + f_{22} + f_{23} < \kappa(G)$. Analogously we obtain that either $f_{12} + f_{22} + f_{23} \leqq b(T)$ or $f_{32} + f_{22} + f_{21} < \kappa(G)$.

We distinguish the following cases:

(1) $f_{12} + f_{22} + f_{21} \leqq b(T)$ and $f_{12} + f_{22} + f_{23} \leqq b(T)$. Since T is a territory, $f_{11} = 0 = f_{13}$. Hence, $|V(T)| = f_{12}$. Since $f_{21} + f_{22} + f_{23} = b(T)$, we have $|V(T)| \leqq \min(f_{23}, f_{21}) \leqq [b(T)/2]$. This implies that $\delta(T) \leqq [(b(T) - 2)/2] \leqq [(m - 2)/2] \leqq n$, which is a contradiction.

(2) $f_{12} + f_{22} + f_{21} \leqq b(T)$ and $f_{12} + f_{22} + f_{23} > b(T)$. Then $f_{21} < f_{23}$. Hence, $f_{21} \leqq [(b(T) - 1)/2]$. We have $f_{11} = 0$ and $f_{32} + f_{22} + f_{21} < \kappa(G)$. Thus $f_{31} = 0$. This implies that for each $u \in W_{21}$, $\deg_G u \leqq \kappa(G) + [(b(T) - 3)/2] \leqq m + n$.

(3) $f_{12} + f_{22} + f_{21} > b(T)$ and $f_{12} + f_{22} + f_{23} \leqq b(T)$. Then analogously $\deg_G u' \leqq \kappa(G) + [(b(T) - 3)/2] \leqq m + n$ for each $u' \in W_{23}$.

(4) $f_{12} + f_{22} + f_{21} > b(T)$ and $f_{12} + f_{22} + f_{23} > b(T)$. Then $f_{32} + f_{22} + f_{23} < \kappa(G)$ and $f_{32} + f_{22} + f_{21} < \kappa(G)$. Hence, $f_{31} = 0 = f_{33}$ and $|V(C(T))| = f_{32} \leqq \kappa(G) - [(b(T) + 3)/2]$. This means that $\delta(C(T)) \leqq \kappa(G) - [(b(T) + 5)/2]$. Since $n \geqq [(m - 2)/2]$, we have $\delta(C(T)) \leqq n$. Therefore, $\delta(G) \leqq m + n$.

Thus the proof is complete.

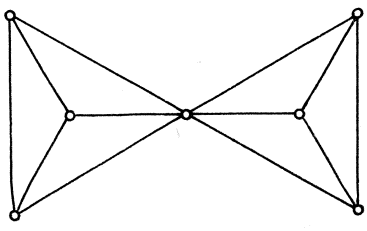

Fig. 2.

Remark. Let k, m, and n be integers such that $0 \leqq k \leqq m$ and $n \geqq 0$. Assume that G is formed by two copies G' and G'' of the complete graph K_{m+n+1} which have precisely k vertices in common. Hence $|V(G)| = 2(m + n + 1) - k$ and $\kappa(G) = k$. Consider two copies F' and F'' of K_{n+1} such that F' is a subgraph of $G - V(G'')$ and F'' is a subgraph of $G - V(G')$. It is clear that F' and F'' are separated territories in G. Since $\delta(F') = n = \delta(F'')$, we have $Q_n(F')$ and $Q_n(F'')$. Obviously, $b(F') = m =$

$= b(F'')$. (An example for $k = 1$ and $m + n = 3$ is given in Fig. 2.) Since $\delta(G) = m + n$, the inequality (3) in the theorem is – in a certain sense – the best possible.

The following corollary is an extension of a theorem of R. Halin [3] (for $|V(G)| > \kappa(G) + 1$):

Corollary 1. *Let there exist separated territories T_1 and T_2 in G such that $Q_0(T_1)$, $Q_0(T_2)$, and $b(T_1) = \kappa(G) = b(T_2)$. Then $\delta(G) = \kappa(G)$.*

Proof immediately follows from the proposition and the theorem, if we put $m = \kappa(G)$ and $n = 0$.

The next corollary is an extension of a result of G. Chartrand, A. Kaugars, and D. R. Lick [2] (for $|V(G)| > \kappa(G) + 1$; notice also the connection of this corollary with Theorem 1 in [6].

Corollary 2. *Let $\kappa(G) \geqq 2$ and let there exist a territory T in G such that $P_{[\kappa((G)-2)/2]}(T)$ and $b(T) = \kappa(G)$. Then $\delta(G) < (3\kappa(G) - 1)/2$.*

Proof. The inequality follows from the theorem, if we put $m = \kappa(G)$ and $n = [(\kappa(G) - 2)/2]$. We get $\delta(G) \leqq \kappa(G) + [(\kappa(G) - 2)/2] = [(3\,\kappa(G) - 2)/2] < (3\,\kappa(G) - 1)/2$, which completes the proof.

Note that each one-vertex subgraph of G whose vertex has degree $\kappa(G)$ in G is an example of a territory T in G with the properties that $b(T) = \kappa(G)$ and $Q_0(T)$ (and thus $P_0(T)$).

Obviously, if $2 \leqq \kappa(G) \leqq 3$, then $[(\kappa(G) - 2)/2] = 0$. Thus we get

Corollary 3. *Let $0 \leqq \kappa(G) \leqq 3$. Then $\delta(G) = \kappa(G)$ if and only if there exists a territory T in G such that $b(T) = \kappa(G)$ and $P_0(T)$.*

If $4 \leqq \kappa(G) \leqq 5$, then $\kappa(G) - [(\kappa(G) + 5)/2] = 0$. Thus we get

Corollary 4. *Let $4 \leqq \kappa(G) \leqq 5$. Then $\delta(G) = \kappa(G)$ if and only if there exists a territory T in G such that $b(T) = \kappa(G)$ and $Q_0(T)$.*

Remark 2. Let G_0 be the graph in Fig. 1. Assume that G can be obtained from G_0 by adding new vertices r', s'', t'' and u'', and new edges $r'r$, $r's$, $r's'$, $r's''$, $r't$, $r't'$, $r't''$, $s''s$, $s''s'$, $s''u$, $s''u'$, $s''u''$, $s''v$, $t''t$, $t''t'$, $t''u$, $t''u'$, $t''u''$, $t''w$, $u''u$, $u''u'$, $u''v$ and $u''w$. Then $\kappa(G) = 6$, $\delta(G) = 7$ and there exists a territory T in G such that $b(T) = \kappa(G)$ and $Q_0(T)$.

Note that in [5] the present author gave a sufficient condition for a 2-connected graph to contain a pair of distinct nonadjacent vertices of degree 2.

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