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VARIETIES OF ALGEBRAS WITH EQUATIONALLY DEFINABLE ZEROS

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INTRODUCTION

One of the most important tasks of universal algebra is to study interconnections between various special properties of varieties (equational classes) of algebras. We have in mind such properties as residual smallness, amalgamation, distributivity and permutability of congruences, regularity, the Schreier property, solvability of the word problem etc, and their various weak and strong forms. To be able to form a picture of the possible interconnections, one must have a suitable supply of examples. The system of "natural" varieties offers a large number of examples good in many respects, but one would rather prefer a more uniform system, satisfying the following requirements:

(1) there is a set M of constructive objects and a natural mapping of M onto the system of varieties;

(2) for many interesting properties of varieties, the problem of deciding for which elements of M the corresponding variety has the given property can be solved algorithmically;

(3) for many interesting properties, the solution of the above formulated problem is non-trivial.

The aim of this paper is to show that the system of finitely based EDZ-varieties (defined in Section 3) is good from this point of view. We shall be concerned namely with the following five properties of varieties: the amalgamation property; having enough subdirectly irreducible algebras; being residually small; having enough simple algebras; having few simple algebras. For each of them we shall describe all finitely based EDZ-varieties of groupoids (or in some cases, of algebras of an arbitrary type) with the property. The answers are fairly non-trivial except the case of the residual smallness.

EDZ-varieties are interesting from other reasons, too. In [11] they proved to be useful in the study of the lattice of all varieties of algebras of a given type; in Section 11 we show their usefulness from another view-point, in connection with the subdirect product of varieties.

1. PRELIMINARIES

By a type we mean a set Δ of operation symbols. Every operation symbol $F \in \Delta$ is associated with a non-negative integer n_F, called the arity of F.

By an algebra of type Δ (or briefly Δ-algebra) we mean an ordered pair $A = \langle M, \varphi \rangle$, where M is a non-empty set (called the underlying set of A) and φ is a mapping, assigning to any $F \in \Delta$ an n_F-ary operation on M. The operation $\varphi(F)$ will be denoted by F_A. If there is no confusion possible, we identify A with its underlying set. For the basic concepts of universal algebra see [5].

Let X be a fixed infinite countable set of symbols, called the variables. Given a type Δ, the set of Δ-terms is defined as follows. Δ-terms are formal expressions which can be obtained by applying a finite number times the following three rules:

(1) every variable is a Δ-term;
(2) if $F \in \Delta$ and $n_F = 0$, then F is a Δ-term;
(3) if $F \in \Delta$, $n_F \geqq 1$ and $t_1, \ldots, t_{n_F}$ are Δ-terms, then the inscription $F(t_1, \ldots, t_{n_F})$ is a Δ-term, too.

For every type Δ we define a Δ-algebra W_Δ in this way: its underlying set is the set of all Δ-terms; if $F \in \Delta$ and $t_1, \ldots, t_{n_F} \in W_\Delta$, put $F_{W_\Delta}(t_1, \ldots, t_{n_F}) = F(t_1, \ldots, t_{n_F})$. The algebra W_Δ is an absolutely free Δ-algebra over X, i.e. X is its generating set and every mapping of X into an arbitrary Δ-algebra A can be uniquely extended to a homomorphism of W_Δ into A.

Evidently there exists exactly one mapping λ of W_Δ into the set of positive integers such that $\lambda(x) = 1$ for every variable x and $\lambda(t) = 1 + \lambda(t_1) + \ldots + \lambda(t_{n_F})$ whenever $t = F(t_1, \ldots, t_{n_F})$. The number $\lambda(t)$ is called the length of the term t.

There are many syntactical notions (such as the notion of subterm), the meaning of which is clear without giving its formal definition. However, it is often of advantage to have a precise inductive definition of such a notion at hand, since proofs of many properties of terms turn out to be trivial if they are carried out by the induction on the length of the terms.

The notion of a subterm of a Δ-term t is defined inductively in this way: if t is a variable, then t is the only subterm of t; if $t = F(t_1, \ldots, t_{n_F})$, then u is a subterm of t iff either $u = t$ or u is a subterm of one of the terms $t_1, \ldots, t_{n_F}$.

By a proper subterm of a term t we mean a subterm u such that $u \neq t$.

We say that a variable x is contained in a term t (or that x occurs in t) if x is a subterm of t.

By an identity we mean an ordered pair of Δ-terms. An identity $\langle u, v \rangle$ is denoted by $u \doteq v$. We say that a Δ-algebra A satisfies an identity $u \doteq v$ if $f(u) = f(v)$ for any homomorphism $f : W_\Delta \to A$. A class of Δ-algebras is called a variety if it is the class of all Δ-algebras satisfying a given set of identities. The lattice of varieties of Δ-algebras is antiisomorphic to the lattice of fully invariant congruences of W_Δ, i.e. congruences r such that $\langle u, v \rangle \in r$ implies $\langle f(u), f(v) \rangle \in r$ for any endomorphism f of W_Δ.

Let Δ be a fixed type.

Given two Δ-terms u and v, we write $u \leqq v$ if there exists an endomorphism f of W_Δ such that $f(u)$ is a subterm of v. Evidently, $\leqq$ is a quasiordering on W_Δ. If $u \leqq v$ and $v \leqq u$, then we write $u \sim v$ and call the terms u, v similar. Evidently, $u \sim v$ iff $v = f(u)$ for an automorphism f of W_Δ. If $u \leqq v$ and u, v are not similar, we write $u < v$. If $T \subseteq W_\Delta$, $u \in T$ and there is no $v \in T$ with $v < u$, then u is called a minimal element of T.

2.1. Proposition. *There exists no infinite sequence $t_1, t_2, t_3, \ldots$ of Δ-terms such that $t_{i+1} < t_i$ for all i. Consequently, if $T \subseteq W_\Delta$, then for any $u \in T$ there exists a minimal $v \in T$ with $v \leqq u$.*

Proof is easy. ■

A set $T \subseteq W_\Delta$ is called full if $u \in T$, $v \in W_\Delta$ and $u \leqq v$ imply $v \in T$.

2.2. Proposition. *The intersection and the union of any system of full subsets of W_Δ is a full subset of W_Δ. The system of all full sets of Δ-terms is thus a complete distributive lattice with respect to $\cap$ and $\cup$; the empty set is its smallest and the set W_Δ its greatest element.*

Proof is evident. ■

2.3. Proposition. *Suppose that Δ either contains no nullary symbols or contains at least one symbol of arity $\geqq 2$. Then the intersection of any pair of non-empty full subsets of W_Δ is non-empty, so that the system of non-empty full subsets is a sublattice of the lattice of all full subsets of W_Δ.*

Proof. Let T_1 and T_2 be two non-empty full subsets of W_Δ; let $t_1 \in T_1$ and $t_2 \in T_2$. If $F \in \Delta$ and $n_F \geqq 2$, then evidently $F(t_1, t_2, t_2, \ldots, t_2) \in T_1 \cap T_2$. If Δ contains only unary symbols, then $t_2 = F_1(F_2(\ldots(F_n(x))))$ for some $F_1, \ldots, F_n \in \Delta$ and $x \in X$; evidently $F_1(F_2(\ldots(F_n(t_1)))) \in T_1 \cap T_2$. ■

For every subset M of W_Δ denote by $\Phi(M)$ the set of all $u \in W_\Delta$ such that $v \leqq u$ for some $v \in M$. Evidently, $\Phi(M)$ is just the smallest full subset of W_Δ containing M; it is called the full set generated by M and M is called its generating subset.

A set $J \subseteq W_\Delta$ is called irreducible if $u, v \in J$ and $u \leqq v$ imply $u = v$.

2.4. Proposition. *Let T be a full subset of W_Δ. Then T has at least one irreducible generating subset. Moreover, every two irreducible generating subsets of T have the same cardinality and are similar (i.e. there exists a one-to-one correspondence between their elements such that the corresponding terms are similar).*

Proof follows from 2.1. ■

For every set T of Δ-terms we denote by $\mathscr{Z}_T$ the variety of Δ-algebras satisfying any identity $u \doteq v$ such that $u, v \in \Phi(T)$. By an EDZ-variety of Δ-algebras we mean a variety K such that $K = \mathscr{Z}_T$ for a set T of Δ-terms.

An element a of a Δ-algebra A is called a zero element of A if $F_A(b_1, \ldots, b_{n_F}) = a$ whenever $F \in \Delta$, $b_1, \ldots, b_{n_F} \in A$ and $a \in \{b_1, \ldots, b_{n_F}\}$. Evidently, if Δ contains at least one at least binary symbol, then any Δ-algebra has at most one zero element. If A has exactly one zero element, then the zero element of A will be denoted by 0_A.

3.1. Proposition. *Let T be a non-empty set of Δ-terms and let A be a Δ-algebra. Then $A \in \mathscr{Z}_T$ iff A has a zero element a such that $f(t) = a$ for every $t \in T$ and every homomorphism $f : W_\Delta \to A$.*

Proof is easy. ■

The zero element a from 3.1 is uniquely determined by A and T. It will be denoted by $0_{A,T}$ or only by 0_A.

3.2. Proposition. *Let T be a set of Δ-terms and let u, v be two Δ-terms. The identity $u \doteq v$ is satisfied in $\mathscr{Z}_T$ iff either $u = v$ or $u, v \in \Phi(T)$.*

Proof. Define a binary relation r on W_Δ as follows: $\langle u, v\rangle \in r$ iff either $u = v$ or $u, v \in \Phi(T)$. It is easy to prove that r is a fully invariant congruence of W_Δ. ■

By 3.2 every EDZ-variety K of Δ-algebras can be expressed in the form $K = \mathscr{Z}_T$ where T is a full set of Δ-terms and, at the same time, it can be expressed in the form $K = \mathscr{Z}_J$ where J is an irreducible set of Δ-terms. Moreover, if Δ contains at least one symbol of arity $\geqq 1$, then evidently the full set T is uniquely determined and the irreducible set J is (by 2.4) almost uniquely determined by K. It will be more convenient for our purposes to express EDZ-varieties in the form $K = \mathscr{Z}_J$, since we shall be interested in finding algorithms deciding which finitely based EDZ-varieties have a given special property and we have the following

3.3. Proposition. *Let Δ be a finite type and let J be an irreducible set of Δ-terms. The variety $\mathscr{Z}_J$ is finitely based iff J is finite.*

Proof. Let $u_1 \doteq v_1, \ldots, u_n \doteq v_n$ be a finite base for the identities of $\mathscr{Z}_J$. Denote by U the set of all the u_i such that $u_i \neq v_i$ and by V the set of all the v_i such that $u_i \neq v_i$. The union $U \cup V$ contains a finite irreducible subset I such that $U \cup V \subseteq \Phi(I)$. It is easy to see that if J has at least two elements, then the sets I, J are similar and have the same cardinality, so that J is finite, too.

Now let $J = \{t_1, \ldots, t_n\}$ be finite. Let us fix pairwise distinct variables $x_1, x_2, x_3, \ldots$ not contained in t_1. It is easy to see that the set composed of the identities $t_1 \simeq t_2$, $t_2 \simeq t_3, \ldots, t_{n-1} \simeq t_n$ and $t_1 \simeq F(x_1, \ldots, x_{i-1}, t_1, x_{i+1}, \ldots, x_{n_F})$ (for any $F \in \Delta$ and $i \in \{1, \ldots, n_F\}$) is a finite base for the identities of $\mathscr{Z}_J$ if either Δ contains an at least binary symbol or t_1 contains no variables. In the remaining case it is enough to add the identity $t_1 \simeq w$ where w is obtained from t_1 by substituting x_1 for the variable contained in t_1. ■

We shall conclude this section by several remarks on the lattice of EDZ- varieties.

3.4. Proposition. *Suppose that Δ either contains no nullary symbols or contains at least one symbol of arity $\geqq 2$. Then the system of all EDZ-varieties of Δ-algebras is a complete distributive sublattice of the lattice of all varieties of Δ-algebras; the mapping, assigning to any full subset T of W_Δ the variety $\mathscr{Z}_T$, is an antiisomorphism of the lattice of full subsets of W_Δ onto the lattice of EDZ-varieties of Δ-algebras.*

Proof is easy. ■

The smallest EDZ-variety of Δ-algebras is the trivial variety $\mathcal{O}_\Delta$ of one-element algebras. The greatest EDZ-variety of Δ-algebras is the variety $\mathscr{A}_\Delta$ of all Δ-algebras. For every type Δ put $\mathscr{C}_\Delta = \mathscr{Z}_T$ where $T = W_\Delta \setminus X$. Evidently, $\mathscr{C}_\Delta$ is an EDZ-variety of Δ-algebras, $\mathscr{C}_\Delta \neq \mathcal{O}_\Delta$ and $\mathscr{C}_\Delta \subseteq K$ for every EDZ-variety K of Δ-algebras such that $K \neq \mathcal{O}_\Delta$. In the case of groupoids (i.e. $\Delta = \{F\}$ where $n_F = 2$) $\mathscr{C}_\Delta$ is the variety of groupoids with constant multiplication.

3.5. Proposition. *Suppose that Δ is the same as in* 3.4. *Then the variety $\mathscr{A}_\Delta$ is not generated by any finite system of its proper EDZ-subvarieties.*

Proof follows from 2.3. ■

Let us remark that (by [7]) there exist two proper varieties of groupoids generating the variety of all groupoids.

3.6. Proposition. *Let Δ contain either at least one at least binary symbol or at least two unary symbols. Then there exists an infinite irreducible set of Δ-terms. Consequently, there are uncountably many EDZ-varieties of Δ-algebras.*

Proof. Let $F \in \Delta$ be at least binary and let x, y be two distinct variables. For every non-negative integer n define a Δ-term u_n by $u_0 = x$ and $u_{n+1} = F(u_n, y, y, \ldots \ldots, y)$. Put $t_n = F(u_n, x, x, \ldots, x)$. The set $\{t_1, t_2, t_3, \ldots\}$ is evidently irreducible.

Now let F and G be two distinct unary symbols from Δ. The set of all terms $F(G^n(F(x)))$, where x is a fixed variable and n ranges over positive integers, is evidently irreducible.

Every two distinct, at least two-element subsets of an irreducible set of Δ-terms evidently define distinct EDZ-varieties. ■

4. FREE ALGEBRAS IN EDZ-VARIETIES

Let a type Δ be given. For every non-empty subset T of W_Δ we define a Δ-algebra W_T as follows: its underlying set is the set $(W_\Delta \setminus \Phi(T)) \cup \{0\}$; if $F \in \Delta$, $t_1, \ldots, t_{n_F} \in W_\Delta \setminus \Phi(T)$ and $F(t_1, \ldots, t_{n_F}) \notin \Phi(T)$, then we put $F_{W_T}(t_1, \ldots, t_{n_F}) = F(t_1, \ldots, t_{n_F})$; if $F \in \Delta$, $p_1, \ldots, p_{n_F} \in W_T$ and if $F_{W_T}(p_1, \ldots, p_{n_F})$ is not yet defined, then we put $F_{W_T}(p_1, \ldots, p_{n_F}) = 0$.

4.1. Proposition. *Let T be a non-empty subset of W_Δ and let no variable belong to T. Then W_T is just the $\mathscr{Z}_T$-free algebra over X.*

Proof follows from 3.2. ∎

We have thus described free algebras of infinite countable ranks in all EDZ-varieties except $\mathcal{O}_\Delta$ and $\mathscr{A}_\Delta$. In $\mathscr{A}_\Delta$ the free algebra is W_Δ and in $\mathcal{O}_\Delta$ free algebras of infinite ranks do not exist.

4.2. Proposition. *Let T be a non-empty subset of W_Δ and let $A \in \mathscr{Z}_T$. If B is an arbitrary Δ-algebra such that $0_A \in B \subseteq A$ and $F_B(b_1, \ldots, b_{n_F}) \in \{0_A, F_A(b_1, \ldots, b_{n_F})\}$ for all $F \in \Delta$ and $b_1, \ldots, b_{n_F} \in B$, then $B \in \mathscr{Z}_T$.*

Proof. Let f be an arbitrary homomorphism of W_Δ into B. Denote by g the homomorphism of W_Δ into A such that $g(x) = f(x)$ for all $x \in X$. It is easy to prove by the induction on $\lambda(t)$ that if t is a Δ-term, then $f(t) \in \{0_A, g(t)\}$. Since $A \in \mathscr{Z}_T$, we have $g(t) = 0_A$ for all $t \in T$, so that $f(t) = 0_A$ for all $t \in T$. By 3.1, $B \in \mathscr{Z}_T$. ∎

The combination of 4.1 and 4.2 yields a uniform method for constructing many examples of algebras in $\mathscr{Z}_T$, which will be needed in the following sections.

It follows from 4.2 that every EDZ-variety has the finite embeddability property. By [4], the word problem is solvable for finitely presented algebras in any finitely based EDZ-variety of algebras of a finite type.

5. THE AMALGAMATION PROPERTIES

A class K of Δ-algebras is said to have the amalgamation property (AP) if for any triple $A, B, C \in K$ and any pair of injective homomorphisms $f : A \to B$, $g : A \to C$ there exists an algebra $D \in K$ and two injective homomorphisms $f' : B \to D$, $g' : C \to D$ such that $f' \circ f = g' \circ g$.

A class K of Δ-algebras is said to have the strong amalgamation property (SAP) if for any triple $A, B, C \in K$ and any pair of injective homomorphisms $f : A \to B$, $g : A \to C$ there exists an algebra $D \in K$ and two injective homomorphisms $f' : B \to D$, $g' : C \to D$ such that $f' \circ f = g' \circ g$ and $f'(B) \cap g'(C) = f'(f(A))$.

We shall give in 6.1 a complete answer to the problem which EDZ-varieties have the amalgamation properties. Before that we incorporate two re-formulations for both AP and SAP. The essential part of Theorem 5.2 is contained in [12]. A class of algebras is called abstract if it is closed with respect to isomorphic images.

5.1. Proposition. *An abstract class K of Δ-algebras has the AP iff the following holds for any triple $A, B, C \in K$: if A is a subalgebra of both B and C and $A = B \cap C$, then there exists an algebra $D \in K$ and two injective homomorphisms $f : B \to D$, $g : C \to D$ coinciding on A.*

An abstract class K of Δ-algebras has the SAP iff the following holds for any triple $A, B, C \in K$: if A is a subalgebra of both B and C and $A = B \cap C$, then there exists an algebra $D \in K$ such that both B and C are subalgebras of D.

Proof is evident. ∎

Let an algebra H and its subset Y be given. By an H, Y-situation we shall mean a six-tuple I, J, B, C, r, s such that $I \subseteq Y$, $J \subseteq Y$, $I \cap J$ is non-empty, $Y = I \cup J$, B is the subalgebra of H generated by I, C is the subalgebra of H generated by J, r is a congruence of B, s is a congruence of C and r, s coincide on $B \cap C$. By a solution of an H, Y-situation I, J, B, C, r, s we mean a congruence of H which is an extension of both r and s. By a strong solution of an H, Y-situation I, J, B, C, r, s we mean a solution t such that if $b \in B$, $c \in C$ and $\langle b, c\rangle \in t$, then there exists an $a \in B \cap C$ with $\langle b, a\rangle \in r$ and $\langle a, c\rangle \in s$.

5.2. Theorem. *The following assertions are equivalent for any variety $K \neq \mathcal{O}_\Delta$:*

(1) *K has the AP;*
(2) *the class of all finitely generated K-algebras has the AP;*
(3) *if Y is a set and H is the K-free algebra over Y, then any H, Y-situation has a solution;*
(4) *if Y is a finite set and H is the K-free algebra over Y, then any H, Y-situation has a solution.*

Moreover, the following assertions are equivalent:

(1′) *K has the SAP;*
(2′) *the class of all finitely generated K-algebras has the SAP;*
(3′) *if Y is a set and H is the K-free algebra over Y, then any H, Y-situation has a strong solution;*
(4′) *if Y is a finite set and H is the K-free algebra over Y, then any H, Y-situation has a strong solution.*

Proof. (1) ⇒ (2) is evident. (2) ⇒ (4): Let I, J, B, C, r, s be an H, Y-situation. Put $A = B \cap C$. We shall show first that A is just the subalgebra of H generated by $I \cap J$. Since H is K-free over Y, there exist two endomorphisms φ, ψ of H such that φ is identical on I, ψ is identical on J, φ maps $J \setminus I$ into $I \cap J$ and ψ maps $I \setminus J$ into $I \cap J$. Evidently, φ is identical on B and ψ is identical on C, so that $\psi \circ \varphi$ is identical on A. On the other hand, $\psi \circ \varphi$ maps Y onto $I \cap J$, so that it maps H onto the subalgebra generated by $I \cap J$. Hence it follows easily that, in fact, A is just the subalgebra of H generated by $I \cap J$. Put $z = r \cap (A \times A) = s \cap (A \times A)$. We denote by π_r the canonical homomorphism of B onto B/r. Since z is the kernel of $\pi_r \circ id_A$, there exists an injective homomorphism $f : A/z \to B/r$ with $\pi_r \upharpoonright A = f \circ \pi_z$. Similarly there exists an injective homomorphism $g : A/z \to C/s$ with $\pi_s \upharpoonright A = g \circ \pi_z$. By the AP for finitely generated K-algebras there exists an algebra $D \in K$ and two injective homomorphisms $f' : B/r \to D$, $g' : C/s \to D$ with $f' \circ f = g' \circ g$. As H is K-free, there exists a homomorphism $h : H \to D$ with $h(x) = f'(\pi_r(x))$ for all $x \in I$ and $h(x) = g'(\pi_s(x))$ for all $x \in J$. This definition is correct, since $x \in I \cap J$ implies $f'(\pi_r(x)) = f'(f(\pi_z(x))) = g'(g(\pi_z(x))) = g'(\pi_s(x))$. Evidently $h \upharpoonright B = f' \circ \pi_r$ and $h \upharpoonright C = g' \circ \pi_s$. Denote by t the kernel of h. As f' and g' are injective, t extends both r and s. It is easy to verify that if $f'(B/r) \cap g'(C/s) = f'(f(A/z))$, then the solution t is strong.

(4) $\Rightarrow$ (3): Let I, J, B, C, r, s be an H, Y-situation. For every finite $M \subseteq Y$ such that $M \cap I \cap J$ is non-empty denote by H_M the subalgebra of H generated by M and put $I_M = I \cap M$, $J_M = J \cap M$, $B_M = B \cap H_M$, $C_M = C \cap H_M$, $r_M = r \cap (B_M \times B_M)$, $s_M = s \cap (C_M \times C_M)$. The H_M, M-situation $I_M, J_M, B_M, C_M, r_M, s_M$ has at least one solution by (4). Denote by t_M the intersection of all solutions of this H_M, M-situation, so that t_M is a congruence of H_M extending both r_M and s_M. If $M_1 \subseteq M_2$, then $t_{M_2} \cap (H_{M_1} \times H_{M_1})$ is a congruence of H_{M_1} extending both r_{M_1} and s_{M_1}, so that $t_{M_1} \subseteq t_{M_2} \cap (H_{M_1} \times H_{M_1}) \subseteq t_{M_2}$. The union t of the updirected system of all t_M is a congruence of H. Let us prove that it extends r. Let $a, b \in B$. There exists M with $a, b \in B_M$. If $\langle a, b \rangle \in r$, then $\langle a, b \rangle \in r_M \subseteq t_M \subseteq t$. If $\langle a, b \rangle \in t$, then there exists N with $\langle a, b \rangle \in t_N$; this implies $\langle a, b \rangle \in t_{N \cup M}$, so that $\langle a, b \rangle \in r_{N \cup M} \subseteq r$. Analogously, t extends s. It is easy to verify that if the solutions t_M are strong, then t is strong, too.

(3) $\Rightarrow$ (1): Let A, B, C be three algebras from K such that A is a subalgebra of both B and C and $A = B \cap C$. Put $Y = B \cup C$ and denote by H the K-free algebra over Y, by B' the subalgebra of H generated by B, and by C' the subalgebra generated by C. As H is K-free, there exist homomorphisms $h : B' \to B$ and $k : C' \to C$ such that $h \supseteq id_B$ and $k \supseteq id_C$. The H, Y-situation B, C, B', C', Ker(h), Ker(k) has a solution r. Put $f = \pi_r \upharpoonright B$ and $g = \pi_r \upharpoonright C$. It is easy to see that f is an injective homomorphism of B into H/r, g is an injective homomorphism of C into H/r and $f \upharpoonright A = g \upharpoonright A = \pi_r \upharpoonright A$. Moreover, $f(B) \cap g(C) = f(A)$ if the solution r is strong. ∎

For the importance of the amalgamation properties see e.g. [1], [3], [8] and [12]. It is proved in [10] that every variety of unary algebras has the SAP.

6.1. Theorem. *Let Δ be an arbitrary type and let J be an irreducible set of Δ-terms. The following assertions are equivalent:*

(1) *$\mathscr{Z}_J$ has the AP;*

(2) *$\mathscr{Z}_J$ has the SAP;*

(3) *if $t \in J$ and if x, y are two distinct variables occurring in t, then there exists a symbol $F \in \Delta$ and terms $u_1, \ldots, u_{n_F}$ such that $F(u_1, \ldots, u_{n_F})$ is a subterm of t and $x, y \in \{u_1, \ldots, u_{n_F}\}$.*

Proof. All the three conditions are evidently satisfied if J is empty. Now let J be non-empty.

(3) $\Rightarrow$ (2): Let A, B, C be three algebras from $\mathscr{Z}_J$ such that A is a subalgebra of both B and C and $A = B \cap C$. Define a Δ-algebra D as follows: its underlying set is the union $B \cup C$; if $F \in \Delta$ and $p_1, \ldots, p_{n_F} \in D$, then $F_D(p_1, \ldots, p_{n_F}) = F_B(p_1, \ldots, p_{n_F})$ if $p_1, \ldots, p_{n_F} \in B$, $F_D(p_1, \ldots, p_{n_F}) = F_C(p_1, \ldots, p_{n_F})$ if $p_1, \ldots, p_{n_F} \in C$ and $F_D(p_1, \ldots, p_{n_F}) = 0_A$ in all other cases. Evidently, B and C are subalgebras of D and $0_A = 0_D$. It remains to prove $D \in Z_J$. Suppose that this is not true, so that $f(t) \neq 0_A$ for some $t \in J$ and a homomorphism $f : W_\Delta \to D$. Since $B, C \in \mathscr{Z}_J$, there exist two variables x, y occurring in t such that $f(x) \in B \setminus C$ and $f(y) \in C \setminus B$. By (3) there exists a symbol F and terms $u_1, \ldots, u_{n_F}$ such that $F(u_1, \ldots, u_{n_F})$ is a subterm of t and $x, y \in \{u_1, \ldots, u_{n_F}\}$. Since $\{f(u_1), \ldots, f(u_{n_F})\}$ is a subset of neither B nor C, we have $f(F(u_1, \ldots, u_{n_F})) = F_D(f(u_1), \ldots, f(u_{n_F})) = 0_A$, a contradiction to $f(t) \neq 0_A$.

(2) $\Rightarrow$ (1) is obvious. (1) $\Rightarrow$ (3): Suppose that there exists a term $t \in J$ and two distinct variables x, y contained in t such that t has no subterm of the form $F(u_1, \ldots, u_{n_F})$ with $x, y \in \{u_1, \ldots, u_{n_F}\}$. Let S denote the set of subterms of t. Define a Δ-algebra B as follows: $B = (S \setminus \{x\}) \cup \{0\}$; if $F \in \Delta$, $u_1, \ldots, u_{n_F} \in S \setminus \{x\}$ and $F(u_1, \ldots, u_{n_F}) \in S$, put $F_B(u_1, \ldots, u_{n_F}) = F(u_1, \ldots, u_{n_F})$; if $F \in \Delta$, $p_1, \ldots, p_{n_F} \in B$ and $F_B(p_1, \ldots, p_{n_F})$ is not yet defined, put $F_B(p_1, \ldots, p_{n_F}) = 0$.

Let us prove $B \in \mathscr{Z}_J$. Suppose, on the contrary, that $f(w) \neq 0$ for some $w \in J$ and a homomorphism $f : W_\Delta \to B$. Of course, $f(z) \neq 0$ for every variable z contained in w. Hence there exists an endomorphism g of W_Δ such that $g(z) = f(z)$ for every variable z contained in w. Let us prove by induction with respect to $\lambda(u)$ that if u is a subterm of w, then either $f(u) = g(u)$ or $f(u) = 0$. If u is a variable, this follows from the definition of g. Let $u = F(u_1, \ldots, u_{n_F})$. We have $f(u) = F_B(f(u_1), \ldots, f(u_{n_F}))$. If one of the elements $f(u_1), \ldots, f(u_{n_F})$ equals 0, then, of course, $f(u) = 0$. In the opposite case $f(u_1) = g(u_1), \ldots, f(u_{n_F}) = g(u_{n_F})$ by the induction assumption. From the construction of B it follows that if $g(u) \in S$, then $f(u) = F_B(f(u_1), \ldots, f(u_{n_F})) = F_B(g(u_1), \ldots, g(u_{n_F})) = F(g(u_1), \ldots, g(u_{n_F})) = g(u)$, while if $g(u) \notin S$, then $f(u) = 0$. The induction is thus completed. Since $f(w) \neq 0$, we get $f(w) = g(w)$. Consequently $g(w)$ is a subterm of t. Since J is irreducible, we get $w = t$. For $g(t)$ to be a subterm of t we have only one possibility: $g(z) = z$ for every variable z contained in t. Especially $x = g(x) = f(x) \in B$, a contradiction.

Similarly as in the case of B, we may define Δ-algebras C and A with the underlying sets $C = (S \setminus \{y\}) \cup \{0\}$ and $A = (S \setminus \{x, y\}) \cup \{0\}$ and prove $C \in \mathscr{Z}_J$ analogously. Evidently, A is a subalgebra of both B, C and $A = B \cap C$. Suppose that there exists an algebra $D \in \mathscr{Z}_J$ and two monomorphisms $f : B \to D$, $g : C \to D$ coinciding on A. Put $h = f \cup g$, so that h is a mapping of $B \cup C$ into D. There exists a homomorphism $h' : W_\Delta \to D$ such that $h'(z) = h(z)$ for every variable z contained in t. Let us prove by induction with respect to $\lambda(u)$ that if u is a subterm of t, then $h'(u) = h(u)$. If u is a variable, this follows from the definition of h'. Let $u = F(u_1, \ldots, u_{n_F})$. By the induction assumption $h'(u_1) = h(u_1), \ldots, h'(u_{n_F}) = h(u_{n_F})$. At least one of the variables x, y does not belong to $\{u_1, \ldots, u_{n_F}\}$, so that either $\{u_1, \ldots, u_{n_F}\} \subseteq B$ or $\{u_1, \ldots, u_{n_F}\} \subseteq C$. In the first case $h'(u) = F_D(h(u_1), \ldots, h(u_{n_F})) = F_D(f(u_1), \ldots$ $\ldots, f(u_{n_F})) = f(F_B(u_1, \ldots, u_{n_F})) = f(u) = h(u)$; in the other case $h'(u) = h(u)$ similarly. The induction is thus completed. Especially, $h'(t) = h(t)$. However, $D \in \mathscr{Z}_J$ implies $h'(t) = 0_D = f(0) \neq f(t) = h(t)$, a contradiction. This shows that $\mathscr{Z}_J$ has not the AP. ∎

Let us mention a corollary of 6.1: there exists an infinite increasing sequence of varieties of groupoids such that the varieties with odd indexes have the SAP and the varieties with even indexes have not the AP.

7. EDZ-VARIETIES WITH ENOUGH SUBDIRECTLY IRREDUCIBLE ALGEBRAS

Let K be a class of Δ-algebras. We say that K has enough subdirectly irreducible algebras if every algebra from K can be embedded into a subdirectly irreducible algebra from K and K does not consist only of one-element algebras.

7.1. Theorem. *Let Δ be a type without nullary symbols; let J be an irreducible set of Δ-terms. The following assertions are equivalent:*

(1) *$\mathscr{Z}_J$ has enough subdirectly irreducible algebras;*

(2) *no variable belongs to J and there exists a pair F, i with the following properties: $F \in \Delta$; $n_F \geqq 2$; $i \in \{1, \ldots, n_F\}$; if $F(x_1, \ldots, x_{i-1}, u, x_{i+1}, \ldots, x_{n_F}) \in J$ for a term u and pairwise distinct variables $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n_F}$, then at least one of the variables $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n_F}$ occurs in u.*

Proof. We shall assume that no variable belongs to J and that Δ contains an at least binary symbol, since in the other cases clearly neither (1) nor (2) is fulfilled.

(1) $\Rightarrow$ (2): Denote by I the set of all pairs $\langle F, i\rangle$ such that $F \in \Delta$, $n_F \geqq 2$ and $i \in \{1, \ldots, n_F\}$. Suppose that for any $\langle F, i\rangle \in I$ there exists a term $u_{F,i}$ such that $F(x_1, \ldots, x_{i-1}, u_{F,i}, x_{i+1}, \ldots, x_{n_F}) \in J$ for some pairwise distinct variables $x_1, \ldots$ $\ldots, x_{i-1}, x_{i+1}, \ldots, x_{n_F}$ not occurring in $u_{F,i}$. For every $\langle F, i\rangle \in I$ define a Δ-algebra $A_{F,i}$ as follows: its underlying set is the disjoint union $U_{F,i} \cup \{0\}$ where $U_{F,i}$ denotes the set of subterms of $u_{F,i}$; if $G \in \Delta$, $v_1, \ldots, v_{n_G} \in U_{F,i}$ and $G(v_1, \ldots, v_{n_G}) \in U_{F,i}$, put $G_{A_{F,i}}(v_1, \ldots, v_{n_G}) = G(v_1, \ldots, v_{n_G})$; if $G \in \Delta$, $p_1, \ldots, p_{n_G} \in A_{F,i}$ and $G_{A_{F,i}}(p_1, \ldots$ $\ldots, p_{n_G})$ is not yet defined, put $G_{A_{F,i}}(p_1, \ldots, p_{n_G}) = 0$. By 4.2, $A_{F,i} \in \mathscr{L}_J$. Denote by A the direct product of the family $A_{F,i}(\langle F, i\rangle \in I)$; we have $A \in \mathscr{L}_J$. For every $\langle F, i\rangle \in I$ and $v \in U_{F,i}$ define an element $h_{F,i,v}$ of A by $h_{F,i,v}(\langle F, i\rangle) = v$ and $h_{F,i,v}(\langle G, j\rangle) = 0$ if $\langle G, j\rangle \neq \langle F, i\rangle$. Define a binary relation r on A by $\langle p, q\rangle \in r$ iff either $p = q \in A$ or $p = h_{F,i,u_{F,i}}$ and $q = h_{G,j,u_{G,j}}$ for some $\langle F, i\rangle, \langle G, j\rangle \in I$. It is easy to see that r is a congruence of A. Put $B = (A/r) \times (A/r)$, so that $B \in \mathscr{L}_J$. Let $C \in \mathscr{L}_J$ be an extension of B. We shall show that C is not subdirectly irreducible. Denote by a the element of A/r satisfying $a = \pi_r(h_{F,i,u_{F,i}})$ for all $\langle F, i\rangle \in I$. Define two binary relations s_1 and s_2 on C in this way: $\langle p, q\rangle \in s_1$ iff either $p = q \in C$ or $p, q \in \{0_B, \langle a, 0_{A/r}\rangle\}$; $\langle p, q\rangle \in s_2$ iff either $p = q \in C$ or $p, q \in \{0_B, \langle 0_{A/r}, a\rangle\}$. Evidently, s_1 and s_2 are non-trivial equivalences on C and $s_1 \cap s_2 = id_C$. Hence it is sufficient to prove that s_1 and s_2 are congruences of C. To this end it is enough to show that if $F \in \Delta$, $p_1, \ldots, p_{n_F} \in C$ and either $\langle a, 0_{A/r}\rangle$ or $\langle 0_{A/r}, a\rangle$ belongs to $\{p_1, \ldots, p_{n_F}\}$, then $F_C(p_1, \ldots, p_{n_F}) = 0_C$. We shall assume $p_i = \langle a, 0_{A/r}\rangle$ for some $i \in \{1, \ldots, n_F\}$, since the case $p_i = \langle 0_{A/r}, a\rangle$ is similar.

Let $n_F = 1$. We may write

$$F_C(\langle a, 0_{A/r}\rangle) = F_B(\langle a, 0_{A/r}\rangle) = \langle F_{A/r}(a), 0_{A/r}\rangle =$$

$$= \langle F_{A/r}(\pi_r(h_{G,j,u_{G,j}})), 0_{A/r}\rangle = \langle \pi_r(F_A(h_{G,j,u_{G,j}})), 0_{A/r}\rangle = \langle \pi_r(0_A), 0_{A/r}\rangle = 0_B\,.$$

Let $n_F \geqq 2$. Then $\langle F, i\rangle \in I$. There exist pairwise distinct variables $x_1, \ldots, x_{i-1}$, $x_{i+1}, \ldots, x_{n_F}$ not occurring in $u_{F,i}$ such that $F(x_1, \ldots, x_{i-1}, u_{F,i}, x_{i+1}, \ldots, x_{n_F}) \in J$. There exists a homomorphism $f : W_\Delta \to C$ such that $f(x_k) = p_k$ for $k \in \{1, \ldots, i-1, i+1, \ldots, n_F\}$, $f(x) = \langle \pi_r(h_{F,i,x}), 0_{A/r}\rangle$ for all variables x occurring in $u_{F,i}$ and $f(z) = 0_B$ for all other variables. It is easy to prove by induction with respect to $\lambda(v)$ that if v is a subterm of $u_{F,i}$, then $f(v) = \langle \pi_r(h_{F,i,v}), 0_{A/r}\rangle$. Especially, $f(u_{F,i}) = \langle a, 0_{A/r}\rangle = p_i$. Since $C \in \mathscr{L}_J$, we get $0_C = f(F(x_1, \ldots, x_{i-1}, u_{F,i}, x_{i+1}, \ldots, x_{n_F}) = F_C(p_1, \ldots, p_{n_F})$.

(2) $\Rightarrow$ (1): Let $A \in \mathscr{L}_J$. Define a Δ-algebra B as follows: its underlying set is the set $(A \times \{1, 2, 3, \ldots\}) \cup \{\alpha\}$ where α is an element not belonging to $A \times \{1, 2, 3, \ldots\}$; $G_B(\langle a_1, 1\rangle, \ldots, \langle a_{n_G}, 1\rangle) = \langle G_A(a_1, \ldots, a_{n_G}), 1\rangle$ for all $G \in \Delta$ and $a_1, \ldots, a_{n_G} \in A$; $F_B(p_1, \ldots, p_{n_F}) = \alpha$ if $p_i = \langle a, m\rangle \in B \setminus \{\langle 0_A, 1\rangle, \alpha\}$ and $p_j = \langle a, m + j\rangle$ for all $j \in \{1, \ldots, i-1, i+1, \ldots, n_F\}$; $G_B(q_1, \ldots, q_{n_G}) = \langle 0_A, 1\rangle$ in all other cases. Evidently, B is an algebra with zero element $0_B = \langle 0_A, 1\rangle$ and the mapping $a \mapsto \langle a, 1\rangle$

is an embedding of A into B. Define a binary relation r on B: $\langle p, q\rangle \in r$ iff either $p = q \in B$ or $p, q \in \{0_B, \alpha\}$. It is easy to see that r is a non-trivial congruence of B and that r is contained in any non-trivial congruence of B. In order to prove that $\mathscr{X}_J$ has enough subdirectly irreducible algebras it remains to show that $B \in \mathscr{X}_J$. Suppose, on the contrary, that $f(t) \neq 0_B$ for some $t \in J$ and a homomorphism $f: W_\Delta \to B$. Since $A \in \mathscr{X}_J$, there exists a variable x contained in t such that $f(x) \notin A \times \{1\}$. Evidently $f(x) = \langle a, m\rangle$ for some $a \in A$ and some $m \geqq 2$ and there exist terms $u_1, \ldots, u_{n_F}$ such that $x \in \{u_1, \ldots, u_{n_F}\}$ and such that $F(u_1, \ldots, u_{n_F})$ is a subterm of t. Evidently $f(F(u_1, \ldots, u_{n_F})) = \alpha$ and $t = F(u_1, \ldots, u_{n_F})$.

Suppose $u_i = x$. Then $f(u_j) = \langle a, m + j\rangle$ for all $j \neq i$. Since $\langle a, m + j\rangle$ does not belong to the range of any fundamental operation of B, u_j is a variable. All the variables are pairwise distinct, since $\langle a, m\rangle, \langle a, m + 1\rangle, \ldots, \langle a, m + n_F\rangle$ are pairwise distinct. We get a contradiction to (2).

Hence $u_j = x$ for some $j \neq i$, so that $f(u_i) = \langle a, m - j\rangle \neq \langle 0_A, 1\rangle$ and $f(u_k) = \langle a, m - j + k\rangle$ for all $k \in \{1, \ldots, i - 1, i + 1, \ldots, n_F\}$. This implies that $u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_{n_F}$ are pairwise distinct variables. By (2) some of them, say u_k, must be contained in u_i. There exist terms $v_1, \ldots, v_{n_F}$ such that $u_k \in \{v_1, \ldots, v_{n_F}\}$ and $F(v_1, \ldots, v_{n_F})$ is a subterm of u_i. Evidently $f(F(v_1, \ldots, v_{n_F})) = \alpha$. Since $G_B(q_1, \ldots, q_{n_G}) = 0_B$ whenever $\alpha \in \{q_1, \ldots, q_{n_G}\}$, we get $f(t) = 0_B$. ∎

8. RESIDUALLY SMALL EDZ-VARIETIES

A class K of Δ-algebras is called residually small if it has a representative subset of subdirectly irreducible algebras, i.e. if there exists a cardinal number $\varkappa$ such that $\operatorname{Card}(A) < \varkappa$ for any subdirectly irreducible algebra $A \in K$. For various equivalent definitions and the importance of residually small varieties see [2] and [14]. In the case of EDZ-varieties we have, unfortunately,

8.1. Theorem. *Let Δ be an arbitrary type and let J be an irreducible set of Δ-terms. The following assertions are equivalent:*

(1) *$\mathscr{X}_J$ is residually small;*

(2) *if no variable belongs to J, then for any $F \in \Delta$ with $n_F \geqq 2$ there exist pairwise distinct variables $x_1, \ldots, x_{n_F}$ such that $F(x_1, \ldots, x_{n_F}) \in J$.*

Proof. (2) ⇒ (1) is easy. (1) ⇒ (2): Suppose that no variable belongs to J and that there exists a symbol $F \in \Delta$ such that $n_F \geqq 2$ and $F(x_1, \ldots, x_{n_F}) \notin J$ whenever $x_1, \ldots, x_{n_F}$ are pairwise distinct variables. Let M be an arbitrary set. Define a Δ-algebra A as follows: $A = (M \times \{1, 2, 3, \ldots\}) \cup \{0, \alpha\}$; $F_A(\langle a, i\rangle, \langle a, i + 1\rangle, \ldots, \langle a, i + n_F - 1\rangle) = \alpha$; in all other cases $G_A(p_1, \ldots, p_{n_F}) = 0$. It is easy to prove that A is subdirectly irreducible and $A \in \mathscr{X}_J$. Since M was arbitrary, $\mathscr{X}_J$ is not residually small. ∎

An algebra is called simple if it has not more than two congruences. We say that a class K of Δ-algebras has enough simple algebras if every algebra from K can be embedded into a simple algebra from K and K does not consist only of one-element algebras.

The aim of this section is to characterize finitely based EDZ-varieties with enough simple algebras. However, in the case of EDZ-varieties of algebras of an arbitrary type the situation seems to be complicated and so we find the characterization only for types consisting of a single operation symbol. The symbol will be at least binary, since evidently no variety of unary algebras has enough simple algebras.

Throughout this section F is a fixed at least binary operation symbol and n denotes its arity.

Let A be an algebra of the type $\{F\}$. An element $a \in A$ is called irreducible if it cannot be expressed in the form $a = F_A(a_1, \ldots, a_n)$ where $a_1, \ldots, a_n \in A$.

Let A be an $\{F\}$-algebra with a zero element 0_A. Let $a \in A$ and let $e_1, \ldots, e_n$ be pairwise distinct elements not belonging to A. Then we define an $\{F\}$-algebra $E = E(A; a; e_1, \ldots, e_n)$ as follows: its underlying set is the union $A \cup \{e_1, \ldots, e_n\}$; A is a subalgebra of E; $F_E(e_1, \ldots, e_n) = a$; if $p_1, \ldots, p_n \in E$ and $F_E(p_1, \ldots, p_n)$ is not yet defined, put $F_E(p_1, \ldots, p_n) = 0_A$. Evidently, 0_A is the zero element of E.

9.1. Lemma. *Let J be a non-empty irreducible set of $\{F\}$-terms. The variety $\mathscr{X}_J$ has enough algebras without irreducible elements iff $E(A; a; e_1, \ldots, e_n) \in \mathscr{X}_J$ for any $A \in \mathscr{X}_J$, any $a \in A$ and any pairwise distinct elements $e_1, \ldots, e_n$ not belonging to A.*

Proof. Suppose that $\mathscr{X}_J$ has enough algebras without irreducible elements but $E(A; a; e_1, \ldots, e_n) \notin \mathscr{X}_J$. There exists an extension $B \in \mathscr{X}_J$ of A and elements $b_1, \ldots, b_n \in B$ with $a = F_B(b_1, \ldots, b_n)$. On the other hand, $f(t) \neq 0_A$ for some $t \in J$ and a homomorphism $f : W_{\{F\}} \to E(A; a; e_1, \ldots, e_n)$. Since $W_{\{F\}}$ is absolutely free, there exists a unique homomorphism $g : W_{\{F\}} \to B$ with the following properties: if x is a variable and $f(x) \in A$, then $g(x) = f(x)$; if x is a variable and $f(x) = e_i$ (where $i \in \{1, \ldots, n\}$), then $g(x) = b_i$. Let us prove by induction with respect to $\lambda(u)$ that if u is a subterm of t and $u = F(u_1, \ldots, u_n)$ for some terms $u_1, \ldots, u_n$, then $g(u) = f(u) \in A$. It follows from $f(u) \neq 0_A$ that either $f(u_1) = e_1, \ldots, f(u_n) = e_n$ or $f(u_1), \ldots, f(u_n) \in A$. In the first case all the terms $u_1, \ldots, u_n$ are variables, since no e_i is in the range of the fundamental operation of $E(A; a; e_1, \ldots, e_n)$; we get $g(u) = F_B(g(u_1), \ldots, g(u_n)) = F_B(b_1, \ldots, b_n) = a = f(u)$. In the second case $g(u_i) = f(u_i) \in A$ for all i, since this follows from the definition of g if u_i is a variable, while if u_i is not a variable, we may use the induction assumption; hence $g(u) = F_B(g(u_1), \ldots, g(u_n)) = F_A(f(u_1), \ldots, f(u_n)) = f(u)$. The induction is thus finished. Especially $g(t) = f(t) \neq 0_A$, a contradiction to $B \in \mathscr{X}_J$.

The proof of the converse implication is the standard method using the technique of updirected unions. ■

9.2. Lemma. *Let J be a finite non-empty irreducible set of $\{F\}$-terms. The following assertions are equivalent*:

(1) *$\mathscr{L}_J$ has enough simple algebras*;

(2) *$\mathscr{L}_J$ has enough subdirectly irreducible algebras and enough algebras without irreducible elements.*

Proof. (1) $\Rightarrow$ (2) is evident: every simple algebra with at least three elements is subdirectly irreducible and has no irreducible elements.

(2) $\Rightarrow$ (1): By 7.1 there exists a number $i \in \{1, \ldots, n\}$ such that if $F(x_1, \ldots, x_{i-1}, u, x_{i+1}, \ldots, x_n) \in J$ for a certain term u and pairwise distinct variables $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$, then at least one of the variables $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ occurs in u. Since J is finite, there exists a positive integer m such that if $t \in J$, then the number of subterms of t is smaller than m.

Let $A \in \mathscr{L}_J$ and let a, b be two elements from $A \setminus \{0_A\}$. By 9.1 there exist pairwise distinct elements $e_{1,1}, \ldots, e_{1,n}, \ldots, e_{m,1}, \ldots, e_{m,n}$ such that the algebras

$$B_1 = E(A; b; e_{1,1}, \ldots, e_{1,n}),$$
$$B_2 = E(B_1; e_{1,i}; e_{2,1}, \ldots, e_{2,n}),$$
$$\ldots$$
$$B_m = E(B_{m-1}; e_{m-1,i}; e_{m,1}, \ldots, e_{m,n})$$

belong to $\mathscr{L}_J$. Define an $\{F\}$-algebra B as follows: its underlying set is the set $B_m \cup \cup \{\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n\}$ where $\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n$ are pairwise distinct elements not belonging to B_m; B_m is a subalgebra of B; $F_B(\alpha_1, \ldots, \alpha_{i-1}, a, \alpha_{i+1}, \ldots \ldots, \alpha_n) = e_{m,i}$; if $p_1, \ldots, p_n \in B$ and $F_B(p_1, \ldots, p_n)$ is not yet defined, put $F_B(p_1, \ldots \ldots, p_n) = 0_A$.

Let us prove $B \in \mathscr{L}_J$. Suppose, on the contrary, that $f(t) \neq 0_A$ for some $t \in J$ and a homomorphism $f : W_{\{F\}} \to B$. Since $B_m \in \mathscr{L}_J$, there exists a variable x occurring in t such that $f(x) \notin B_m$. From this and from $t \neq x$ it follows that there exist terms $u_1, \ldots, u_n$ such that $x \in \{u_1, \ldots, u_n\}$ and $F(u_1, \ldots, u_n)$ is a subterm of t. By $f(t) \neq 0_A$ and by the definition of B we get $f(u_i) = a$ and $f(u_j) = \alpha_j$ for all $j \neq i$. Since the elements α_j are not in the range of the operation F_B, the terms u_j $(j \neq i)$ are variables. The elements α_j are pairwise distinct; consequently, the variables u_j are pairwise distinct, too. This shows that there exists a term u such that $f(u) = a$ and $F(x_1, \ldots \ldots, x_{i-1}, u, x_{i+1}, \ldots, x_n)$ is a subterm of t for a sequence of pairwise distinct variables $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ satisfying $f(x_j) = \alpha_j$ $(j \neq i)$. Let us fix a minimal term u with these properties. (Minimal in the sense that no its proper subterm has these properties.) No variable from $\{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\}$ occurs in u, since in the opposite case we could repeat the construction of the terms $u_1, \ldots, u_n$ and get a contradiction with the minimality of u. Put $v_0 = F(x_1, \ldots, x_{i-1}, u, x_{i+1}, \ldots, x_n)$. By the property of the number i, v_0 is a proper subterm of t. Evidently $f(v_0) = e_{m,i}$.

Denote by M the set of all $j \in \{0, 1, \ldots, m-1\}$ such that $f(v_j) = e_{m-j,i}$ for a proper subterm v_j of t. We have just proved $0 \in M$. Since the number of subterms of t is smaller than m, there exists a number $j \in M$ such that $j + 1 \notin M$. Let us fix such a number j and the corresponding proper subterm v_j with $f(v_j) = e_{m-j,i}$. There exist terms $w_1, \ldots, w_n$ such that $v_j \in \{w_1, \ldots, w_n\}$ and $F(w_1, \ldots, w_n)$ is a subterm of t. We have $f(w_1) = e_{m-j,1}, \ldots, f(w_n) = e_{m-j,n}$, because $f(v_j) = e_{m-j,i}$ and $v_j \in \{w_1, \ldots, w_n\}$ imply that this is the only possibility for $F_B(f(w_1), \ldots, f(w_n)) \neq 0_A$. Consequently, the terms $w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_n$ are pairwise distinct variables and $f(F(w_1, \ldots, w_n)) = e_{m-(j+1),i}$. Since $j + 1 \notin M$, we get $t = F(w_1, \ldots, w_n)$. By the property of i, there exists a number $k \in \{1, \ldots, i-1, i+1, \ldots, n\}$ such that the variable w_k occurs in w_i. We have $w_k \neq w_i$ (as $e_{m-j,k} \neq e_{m-j,i}$) and so there exist terms $z_1, \ldots, z_n$ such that $w_k \in \{z_1, \ldots, z_n\}$ and $F(z_1, \ldots, z_n)$ is a subterm of w_i. Again, this is possible only if $f(z_1) = e_{m-j,1}, \ldots, f(z_n) = e_{m-j,n}$, so that $f(F(z_1, \ldots, z_n)) = e_{m-(j+1),i}$. However, $F(z_1, \ldots, z_n)$ is a proper subterm of t. This contradiction to $j + 1 \notin M$ proves $B \in \mathscr{Z}_J$.

Now let C be an arbitrary extension of B. If r is a congruence of C and if $\langle a, c \rangle \in r$ for some $c \in B$ different from a, then $\langle b, 0_A \rangle \in r$. Indeed, we have

$$\langle F_C(\alpha_1, \ldots, \alpha_{i-1}, a, \alpha_{i+1}, \ldots, \alpha_n),\ F_C(\alpha_1, \ldots, \alpha_{i-1}, c, \alpha_{i+1}, \ldots, \alpha_n) \rangle \in r\,,$$

$$\text{i.e.}\quad \langle e_{m,i}, 0_A \rangle \in r\,; \quad \text{hence}$$

$$\langle F_C(e_{m,1}, \ldots, e_{m,n}),\ F_C(e_{m,1}, \ldots, e_{m,i-1}, 0_A, e_{m,i+1}, \ldots, e_{m,n}) \rangle \in r\,,$$

$$\text{i.e.}\quad \langle e_{m-1,i}, 0_A \rangle \in r\,, \quad \text{etc.; finally} \quad \langle e_{1,i}, 0_A \rangle \in r\,, \quad \text{so that}$$

$$\langle F_C(e_{1,1}, \ldots, e_{1,n}),\ F_C(e_{1,1}, \ldots, e_{1,i-1}, 0_A, e_{1,i+1}, \ldots, e_{1,n}) \rangle \in r\,,$$

$$\text{i.e.}\quad \langle b, 0_A \rangle \in r\,.$$

We have proved that for any $A \in \mathscr{Z}_J$ and any $a, b \in A \setminus \{0_A\}$ there exists an extension $B \in \mathscr{Z}_J$ of A such that if r is a congruence of an arbitrary extension C of B and $\langle a, c \rangle \in r$ for some $c \in B \setminus \{a\}$, then $\langle b, 0_A \rangle \in r$. The standard argument on up-directed unions implies that $\mathscr{Z}_J$ has enough simple algebras. ■

By an admissible $n + 1$-tuple we mean an $n + 1$-tuple $t, Y_1, \ldots, Y_n$ with the following two properties:

(A1) t is an $\{F\}$-term and $Y_1, \ldots, Y_n$ are non-empty, pairwise disjoint sets of variables occurring in t;

(A2) if $F(u_1, \ldots, u_n)$ is a subterm of t and $\{u_1, \ldots, u_n\} \cap (Y_1 \cup \ldots \cup Y_n)$ is non-empty, then $u_1 \in Y_1, \ldots, u_n \in Y_n$.

Let $t, Y_1, \ldots, Y_n$ be an admissible $n + 1$-tuple and let z be a variable not occurring in t. Then for every subterm u of t we define a term $\Psi_{Y_1,\ldots,Y_n,z}(u)$ by the induction on $\lambda(u)$ in this way: if u is a variable, then $\Psi_{Y_1,\ldots,Y_n,z}(u) = u$; if $u = F(x_1, \ldots, x_n)$ and $x_1 \in Y_1, \ldots, x_n \in Y_n$, then $\Psi_{Y_1,\ldots,Y_n,z}(u) = z$; if $u = F(u_1, \ldots, u_n)$ and

$\{u_1, \ldots, u_n\} \cap (Y_1 \cup \ldots \cup Y_n)$ is empty, then $\Psi_{Y_1,\ldots,Y_n,z}(u) = F(\Psi_{Y_1,\ldots,Y_n,z}(u_1), \ldots$ $\ldots, \Psi_{Y_1,\ldots,Y_n,z}(u_n))$. Hence $\Psi_{Y_1,\ldots,Y_n,z}(u)$ can be obtained from u by substituting z for any subterm $F(x_1, \ldots, x_n)$ such that $x_1 \in Y_1, \ldots, x_n \in Y_n$.

9.3. Lemma. *Let J be a non-empty irreducible set of $\{F\}$-terms such that no variable belongs to J. The following assertions are equivalent:*

(1) *$\mathscr{X}_J$ has enough algebras without irreducible elements;*

(2) *if $t \in J$ and if $t, Y_1, \ldots, Y_n$ is an admissible $n+1$-tuple, then $v \leqq \Psi_{Y_1,\ldots,Y_n,z}(t)$ for some $v \in J$ and a variable z not occurring in t.*

Proof. (1) $\Rightarrow$ (2): Let $t, Y_1, \ldots, Y_n$ be an admissible $n+1$-tuple and let z be a variable not occurring in t. Denote by A the subalgebra of W_J generated by the set $Y \cup \{z\}$ where Y is the set of variables occurring in t. By (1) there exists an extension $B \in \mathscr{X}_J$ of A and elements $b_1, \ldots, b_n \in B$ with $z = F_B(b_1, \ldots, b_n)$. There exists a homomorphism $f: W_{\{F\}} \to B$ such that if $x \in Y_i$ (where $i \in \{1, \ldots, n\}$), then $f(x) = b_i$ and if $x \in (Y \cup \{z\}) \setminus (Y_1 \cup \ldots \cup Y_n)$, then $f(x) = x$. Since $B \in \mathscr{X}_J$, we have $f(t) = 0$. Let us prove the following assertion by induction with respect to $\lambda(u)$: if u is a subterm of t and u is not a variable, then $f(\Psi_{Y_1,\ldots,Y_n,z}(u)) = f(u)$. Let $u = F(u_1, \ldots, u_n)$. If $u_1 \in Y_1, \ldots, u_n \in Y_n$, then $f(\Psi_{Y_1,\ldots,Y_n,z}(u)) = f(z) = z = F_B(b_1, \ldots, b_n) = f(u)$. In all other cases $\{u_1, \ldots, u_n\} \cap (Y_1 \cup \ldots \cup Y_n)$ is empty and $\Psi_{Y_1,\ldots,Y_n,z}(u) = F(\Psi_{Y_1,\ldots,Y_n,z}(u_1), \ldots, \Psi_{Y_1,\ldots,Y_n,z}(u_n))$. We have $f(\Psi_{Y_1,\ldots,Y_n,z}(u_i)) = f(u_i)$ for all $i \in \{1, \ldots, n\}$, since this follows from the definition of f if u_i is a variable and if u_i is not a variable, we may use the induction assumption. Hence $f(\Psi_{Y_1,\ldots,Y_n,z}(u)) = F_B(f(u_1), \ldots, f(u_n)) = f(u)$. It can be proved analogously that if u is a subterm of t, $\Psi_{Y_1,\ldots,Y_n,z}(u) \in A$ and u is not a variable, then $f(\Psi_{Y_1,\ldots,Y_n,z}(u)) = \Psi_{Y_1,\ldots,Y_n,z}(u)$.

Since $f(\Psi_{Y_1,\ldots,Y_n,z}(t)) = f(t) = 0 \neq \Psi_{Y_1,\ldots,Y_n,z}(t)$, we get $\Psi_{Y_1,\ldots,Y_n,z}(t) \notin A$, so that $\Psi_{Y_1,\ldots,Y_n,z}(t) \notin W_J$. This means that $\Psi_{Y_1,\ldots,Y_n,z}(t)$ belongs to the full set generated by J. As J is irreducible, there exists a term $v \in J$ with $v \leqq \Psi_{Y_1,\ldots,Y_n,z}(t)$.

(2) $\Rightarrow$ (1): Let $A \in \mathscr{X}_J$, $a \in A$ and let $e_1, \ldots, e_n$ be pairwise distinct elements not belonging to A. By 9.1 it is enough to prove that the algebra $E = E(A; a; e_1, \ldots, e_n)$ belongs to $\mathscr{X}_J$. Suppose, on the contrary, that $f(t) \neq 0_A$ for some $t \in J$ and a homomorphism $f: W_{\{F\}} \to E$. For every $i \in \{1, \ldots, n\}$ denote by Y_i the set of all variables x contained in t such that $f(x) = e_i$. Evidently, $Y_1, \ldots, Y_n$ are pairwise disjoint sets of variables occurring in t. Since $A \in \mathscr{X}_J$, at least one of these sets is non-empty. Now, to prove that $t, Y_1, \ldots, Y_n$ is an admissible $n+1$-tuple, it is enough to prove (A2). Let $F(u_1, \ldots, u_n)$ be a subterm of t and let $u_i \in Y_1 \cup \ldots \cup Y_n$ for some i. We have $f(u_i) \notin A$. Moreover, $f(t) \neq 0_A$ implies $F_E(f(u_1), \ldots, f(u_n)) \neq 0_A$. This is possible only if $f(u_1) = e_1, \ldots, f(u_n) = e_n$. Since the elements $e_1, \ldots, e_n$ are not in the range of F_E, the terms $u_1, \ldots, u_n$ are variables and so $u_1 \in Y_1, \ldots, u_n \in Y_n$.

Let z be a variable not occurring in t. By (2) there exists a term $v \in J$ such that $v \leqq \Psi_{Y_1,\ldots,Y_n,z}(t)$. There exists a homomorphism $g : W_{\{F\}} \to A$ such that $g(z) = a$ and $g(x) = f(x)$ for all variables $x \notin Y_1 \cup \ldots \cup Y_n \cup \{z\}$. Let us prove by induction with respect to $\lambda(u)$ that if u is a subterm of t and u is not a variable, then $g(\Psi_{Y_1,\ldots,Y_n,z}(u)) = f(u)$. Let $u = F(u_1, \ldots, u_n)$. If $u_1 \in Y_1, \ldots, u_n \in Y_n$, then

$$g(\Psi_{Y_1,\ldots,Y_n,z}(u)) = g(z) = a = F_E(e_1, \ldots, e_n) = F_E(f(u_1), \ldots, f(u_n)) = f(u) .$$

In all other cases $\{u_1, \ldots, u_n\} \cap (Y_1 \cup \ldots \cup Y_n)$ is empty. If u_i is a variable, then $g(\Psi_{Y_1,\ldots,Y_n,z}(u_i)) = g(u_i) = f(u_i)$; if u_i is not a variable, the same holds by the induction assumption. We get

$$g(\Psi_{Y_1,\ldots,Y_n,z}(u)) = F_A(g(\Psi_{Y_1,\ldots,Y_n,z}(u_1), \ldots, g(\Psi_{Y_1,\ldots,Y_n,z}(u_n))) = \\ = F_A(f(u_1), \ldots, f(u_n)) = f(u) .$$

Especially $g(\Psi_{Y_1,\ldots,Y_n,z}(t)) = f(t)$. However, $g(v) = 0_A$ implies $g(\Psi_{Y_1,\ldots,Y_n,z}(t)) = = 0_A$ so that $f(t) = 0_A$. ∎

9.4. Theorem. *Let F be an operation symbol of an arity $n \geqq 2$; let J be a finite irreducible set of $\{F\}$-terms. The variety $\mathscr{L}_J$ has enough simple algebras iff it satisfies the following three conditions:*

(1) *no variable belongs to J;*

(2) *there exists a number $i \in \{1, \ldots, n\}$ such that if $F(x_1, \ldots, x_{i-1}, u, x_{i+1}, \ldots, x_n) \in \in J$ for a term u and pairwise distinct variables $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$, then at least one of the variables $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ occurs in u;*

(3) *if $t \in J$ and if $t, Y_1, \ldots, Y_n$ is an admissible $n + 1$-tuple, then $v \leqq \Psi_{Y_1,\ldots,Y_n,z}(t)$ for a term $v \in J$ and a variable z not occurring in t.*

Proof. If J is non-empty, it follows from 7.1, 9.2 and 9.3. If J is empty, then (1), (2) and (3) are satisfied and, as is well-known (see e.g. [6]) and easy, there are enough simple $\{F\}$-algebras. ∎

Let us mention a corollary of Theorems 7.1 and 9.4: there exists an infinite increasing sequence of varieties of groupoids such that the varieties with odd indexes have enough simple groupoids while the varieties with even indexes have not enough subdirectly irreducible groupoids.

The following two problems are left open. Are 9.2 and 9.4 true for infinite irreducible sets J? Is it possible to extend Theorem 9.4 in any way from the case of the type $\{F\}$ to the case of an arbitrary type?

10. EDZ-VARIETIES WITH FEW SIMPLE ALGEBRAS

A class K of Δ-algebras is said to have few simple algebras if it has a representative subset of simple algebras, i.e. if there exists a cardinal $\varkappa$ such that $\text{Card}(A) < \varkappa$ for any simple algebra $A \in K$.

10.1. Theorem. *Let F be an operation symbol of an arity $n \geqq 2$ and let J be a fintie irreducible set of $\{F\}$-terms. The following assertions are equivalent:*

(1) *$\mathscr{Z}_J$ has few simple algebras;*

(2) *every simple algebra from $\mathscr{Z}_J$ has at most two elements;*

(3) *every algebra $A \in \mathscr{Z}_J$ without irreducible elements has only one element;*

(4) *there exists a term $t \in J$ such that every variable has at most one occurrence in t.*

Proof. (3) ⇒ (2) ⇒ (1) are easy. (4) ⇒ (3): Suppose that an algebra $A \in \mathscr{Z}_J$ has no irreducible elements. It is easy to show by induction with respect to $\lambda(u)$ that if u is an $\{F\}$-term such that every variable has at most one occurrence in u, then for any $a \in A$ there exists a homomorphism $f : W_{\{F\}} \to A$ with $a = f(u)$. Especially, for any $a \in A$ there exists a homomorphism $f : W_{\{F\}} \to A$ with $a = f(t)$, so that $a = 0_A$ for any $a \in A$.

It remains to prove (1) ⇒ (4). Suppose that for every $t \in J$ there exists a variable occurring at least twice in t. Since J is finite, there exists a positive integer m such that $\lambda(t) \leqq m$ for all $t \in J$. Denote by J_m the set of all $\{F\}$-terms t such that $\lambda(t) \leqq m$ and a certain variable occurs at least twice in t. Hence $J \subseteq J_m$. Denote by I an irreducible subset of J_m such that for every $t \in J_m$ there exists $t' \in I$ with $t' \leqq t$. It is easy to verify that I has the properties (1), (2) and (3) of 9.4. Consequently, $\mathscr{Z}_I$ has enough simple algebras. Since $\mathscr{Z}_I \subseteq \mathscr{Z}_J$, we get a contradiction with (1). ∎

The variety $\mathscr{Z}_{\{t\}}$, where $t = (xx)\,y$, is (by 7.1 and 10.1) an example of a variety of groupoids which has enough subdirectly irreducible groupoids but in which every simple groupoid is trivial.

11. THE SUBDIRECT PRODUCT OF VARIETIES

Let K be a variety of Δ-algebras and let $K_1, \ldots, K_n$ be its subvarieties. We say that K is the subdirect product of $K_1, \ldots, K_n$ if any algebra from K is isomorphic to a subdirect product of some algebras $A_1 \in K_1, \ldots, A_n \in K_n$. In this case evidently K is just the join $K_1 \vee \ldots \vee K_n$ of the varieties $K_1, \ldots, K_n$ in the lattice of varieties of Δ-algebras.

For example, it is proved in [9] that if K is a variety such that the congruence lattice of any algebra from K is distributive, and if $K_1, \ldots, K_n$ are subvarieties of K such that $K = K_1 \vee \ldots \vee K_n$, then K is the subdirect product of $K_1, \ldots, K_n$. Let us remark that a quite elementary proof of this theorem can be given. On the other hand, it is easy to see that if K is a variety and the join of any pair K_1, K_2 of subvarieties of K is the subdirect product of K_1, K_2, then the lattice of subvarieties of K is distributive.

The following theorem is proved in [13] in the special case of $\mathscr{Z}_T$ being the smallest non-trivial EDZ-variety.

11.1. Theorem. *Let T be a full set of Δ-terms. Let $\mathbf{L}$ be a variety of Δ-algebras. Suppose that there exists a variable x and a term $u \in T$ such that the identity $x \doteq u$ is satisfied in $\mathbf{L}$. Then the join $\mathbf{L} \vee \mathscr{Z}_T$ is the subdirect product of $\mathbf{L}$ and $\mathscr{Z}_T$.*

Proof. We may suppose that u contains at least one variable, as in the opposite case $\mathbf{L} = \mathcal{O}_\Delta$ and everything is evident. Denote by t the term obtained from u by substituting the variable x for any variable. Evidently, $x \doteq t$ is satisfied in $\mathbf{L}$, $t \in T$ and t contains exactly one variable x.

Let $A \in \mathbf{L} \vee \mathscr{Z}_T$. Define two equivalences r and s on the set A as follows: $\langle a, b \rangle \in r$ iff $t(a) = t(b)$; $\langle a, b \rangle \in s$ iff either $a = b$ or $a = t(a)$ and $b = t(b)$.

We shall prove that r is a congruence of A. Let $F \in \Delta$ be a symbol of an arity $n \geqq 1$ and let $\langle a_1, b_1 \rangle \in r, \ldots, \langle a_n, b_n \rangle \in r$. The identity $t(F(x_1, \ldots, x_n)) \doteq F(t(x_1), \ldots, t(x_n))$ is satisfied both in $\mathbf{L}$ and $\mathscr{Z}_T$, so that it is satisfied in A. We get

$$t(F_A(a_1, \ldots, a_n)) = F_A(t(a_1), \ldots, t(a_n)) = F_A(t(b_1), \ldots, t(b_n)) = \\ = t(F_A(b_1, \ldots, b_n)),$$

so that $\langle F_A(a_1, \ldots, a_n), F_A(b_1, \ldots, b_n) \rangle \in r$.

We shall prove $A/r \in \mathbf{L}$. If an identity $v \doteq w$ holds in $\mathbf{L}$, then $t(v) \doteq t(w)$ holds in both $\mathbf{L}$ and $\mathscr{Z}_T$, so that it holds in A. This implies that $v \doteq w$ holds in A/r.

We shall prove that s is a congruence of A. Put $I = \{a \in A;\ t(a) = a\}$. Let $F \in \Delta$ be a symbol of an arity $n \geqq 1$ and let $a_1, \ldots, a_n \in A$. It is sufficient to prove that if there exists an $i \in \{1, \ldots, n\}$ with $a_i \in I$, then $F_A(a_1, \ldots, a_n) \in I$. The identity $t(F(x_1, \ldots, x_n)) \doteq F(x_1, \ldots, x_{i-1}, t(x_i), x_{i+1}, \ldots, x_n)$ holds in both $\mathbf{L}$ and $\mathscr{Z}_T$, so that it holds in A. We conclude

$$t(F_A(a_1, \ldots, a_n)) = F_A(a_1, \ldots, t(a_i), \ldots, a_n) = F_A(a_1, \ldots, a_n).$$

We shall prove $A/s \in \mathscr{Z}_T$. If an identity $v \doteq w$ holds in $\mathscr{Z}_T$ and $v \neq w$, then $v, w \in T$, so that both $v \doteq t(v)$ and $w \doteq t(w)$ hold in both $\mathbf{L}$ and $\mathscr{Z}_T$. Consequently, they hold in A. This implies that $v \doteq w$ holds in A/s.

Evidently $r \cap s = id_A$, so that A is isomorphic to a subdirect product of the algebras $A/r \in \mathbf{L}$ and $A/s \in \mathscr{Z}_T$. ∎

Given an EDZ-variety $\mathscr{Z}_T$, we may ask which varieties $\mathbf{L}$ have the property that $\mathbf{L} \vee \mathscr{Z}_T$ is the subdirect product of $\mathbf{L}$ and $\mathscr{Z}_T$. One example of such varieties is given in 11.1. Another example: every variety comparable with $\mathscr{Z}_T$ evidently has this property, too. We expect that there are not many other examples, since we have the following negative result and several similar negative propositions could be found.

11.2. Proposition. *Let T be a full set of Δ-terms and let $\mathbf{L}$ be a variety of Δ-algebras. Suppose that there exist three Δ-terms a, b, c with the following properties:*

(1) $a \notin T$; $b \in T$; $c \in T$;
(2) *the identity* $a \simeq c$ *holds in* L *but the identity* $a \simeq b$ *does not;*
(3) *if* u, v *are distinct elements of the set*

$$U = \{a\} \cup \{t \in T;\ b \simeq t \text{ holds in } L\},$$

then u *is not a subterm of* v.

Then the join $L \vee \mathscr{L}_T$ *is not the subdirect product of* L *and* $\mathscr{L}_T$.

Proof. Denote by α (resp. β) the set of all identities holding in L (resp. in $\mathscr{L}_T$). As is well-known, α (resp. β) is just the smallest congruence of W_Δ such that the corresponding quotient belongs to L (resp. to $\mathscr{L}_T$). Denote by r the congruence of W_Δ generated by $(\alpha \cap \beta) \cup \{\langle a, b\rangle\}$. Evidently $W_\Delta/r \in L \vee \mathscr{L}_T$.

Suppose that $L \vee \mathscr{L}_T$ is the subdirect product of L and $\mathscr{L}_T$. Then $r = (r \vee \alpha) \cap \cap (r \vee \beta)$. As $\langle a, c\rangle$ evidently belongs to the right hand side, we get $\langle a, c\rangle \in r$. As is well-known, this implies that there exist pairs $\langle u_i, v_i\rangle \in (\alpha \cap \beta) \cup \{\langle a, b\rangle, \langle b, a\rangle\}$ $(i = 1, \dots, n)$ and non-constant unary derived operations $h_1, \dots, h_n$ of W_Δ such that

$$a = h_1(u_1),\ h_1(v_1) = h_2(u_2),\ h_2(v_2) = h_3(u_3),\ \dots$$
$$\dots,\ h_{n-1}(v_{n-1}) = h_n(u_n),\ h_n(v_n) = c\,.$$

Let us prove $h_i(v_i) \in U$ by induction with respect to $i = 1, \dots, n$. Let $i \in \{1, \dots, n\}$ and suppose $h_j(v_j) \in U$ for all $j \in \{1, \dots, i-1\}$.

We have $h_i(u_i) \in U$. Indeed, if $i = 1$, then $h_i(u_i) = a \in U$ and if $i \geqq 2$, then $h_i(u_i) = h_{i-1}(v_{i-1}) \in U$ by the induction assumption.

As $\langle u_i, v_i\rangle \in (\alpha \cap \beta) \cup \{\langle a, b\rangle, \langle b, a\rangle\}$, we have either $\langle u_i, v_i\rangle \in \alpha \cap \beta$ or $u_i, v_i \in U$. In the first case $\langle h_i(u_i), h_i(v_i)\rangle \in \alpha \cap \beta$, so that $h_i(v_i) \in U$. Suppose $u_i, v_i \in \in U$. As u_i is a subterm of $h_i(u_i)$, we have $h_i(u_i) = u_i$ by the property of U. This implies that h_i is the identical mapping, since any non-constant unary derived operation of W_Δ with a fixed point is identical. Especially, $h_i(v_i) = v_i \in U$.

We have proved $h_i(v_i) \in U$ for any $i \in \{1, \dots, n\}$. Especially, $c = h_n(v_n) \in U$. This is evidently a contradiction. ∎

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