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STRUCTURE THEOREMS ON CERTAIN REGULAR AND INVERSE SEMIGROUPS

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1.

One of the basic classes of rings is that of semisimple artinian rings. Ring theory had its very beginning with the investigation of this class of rings initiated by E. ARTIN and E. NOETHER in the late twenties. Even now one can say that a considerable part of ring theory is centred around semisimple artinian rings. Accordingly, they have been characterized in a lot of different ways (s.e.g. A. KERTÉSZ [2]), starting with the classical theorems of Wedderburn-Artin and E. Noether. Some of these characterizations describe semisimple artinian rings as sums of minimal one- or two-sided or quasi-ideals. Following the exposition of O. STEINFELD [7], we give here only these ones, all the others will be left out of consideration, our aim being the search for generalizations and analogies of exactly these characteristics.

Theorem I (s. O. Steinfeld [7]). *The following conditions on a ring A are equivalent:*

(a_l) *A is semiprime, and it is the sum of a finite number of its minimal left ideals.*

(a_r) *A is semiprime, and it is the sum of a finite number of its minimal right ideals.*

(b) *A is the direct sum of a finite number of its two-sided ideals which are, on the one hand, simple subrings of A, on the other hand, sums of a finite number of their minimal left [right] ideals.*

(c) *A is the sum of such of its quasi-ideals which form a finite complete system.*

(d) *A is semiprime, and it is the sum of a finite number of its minimal quasi-ideals.*

Later F. SZÁSZ and C. FAITH (independently) succeeded in extending several results on artinian rings to rings with minimum condition on principal left [right] ideals.

Among others, analogues of the Wedderburn-Artin and the E. Noether theorems were obtained (s. Conditions (A_l), (A_r), and (B) below). Recently O. Steinfeld managed to carry over the whole Theorem I above to this case.

Theorem II (O. Steinfeld [6]; s. also O. Steinfeld [7]). *The following conditions on a ring A are equivalent*:

(A_l) *A is semiprime, and it is the sum of its minimal left idels.*

(A_r) *A is semiprime, and it is the sum of its minimal right ideals.*

(B) *A is the discrete direct sum of two-sided ideals which are simple subrings of A containing at least one minimal left [right] ideal.*

(C) *A is the sum of such of its quasi-ideals which form a complete system.*

(D) *A is semiprime, and it is the sum of its minimal quasi-ideals.*

This latter theorem has its analogue in semigroup theory, which has been known for some time already. In fact, there is an important class of semigroups which can be characterized, among others, by conditions corresponding to those in $(A_l)-(D)$.

Theorem II′ (O. Steinfeld [5] and G. Lallement - M. Petrich [3]; s. also O. Steinfeld [7]). *The following conditions on a semigroup S with* 0 *are equivalent*:

(A_l') *S is regular, and it is the union of its* 0*-minimal left ideals.*

(A_r') *S is regular, and it is the union of its* 0*-minimal right ideals.*

(B_l') *S is a union of* 0*-minimal left ideals generated by idempotents.*

(B_r') *S is a union of* 0*-minimal right ideals generated by idempotents.*

(C′) *S is a* 0*-direct union of two-sided ideals which are completely* 0*-simple subsemigroups of S.*

(D′) *S is a union of some of its quasi-ideals, and these quasi-ideals form a complete system.*

(E′) *S is regular, and it is the union of its* 0*-minimal quasi-ideals.*

(F′) *S is primitive regular.*

The analogy of Theorems I and II on the one hand, and that of Theorems II and II′ on the other hand suggest to look for a Theorem I′ which would complete these analogies. If we consider the formal difference between the corresponding conditions of Theorems I and II, respectively, then we see at once that it consists in introducing in Theorem I a finiteness restriction on the number of the components in the conditions of Theorem II. Thus one is led to ask the following question: Does the relation of conditions $(A_l')-(F')$ alter when introducing an appropriate finiteness condition in each of them? And if it does, then how?

It is this question that we are going to answer in the next section.

2.

Before describing the relation of these modified conditions, we give some definitions. For the non-defined notions see A. H. Clifford - G. B. Preston [1].

The union of some subsemigroups of a semigroup with 0 is said to be 0-*direct* if the intersection of any two of them is 0. A subset Q of a semigroup S is a *quasi-ideal* of S if $QS \cap SQ \subseteq Q$.

Consider a system of quasi-ideals $Q_{\lambda\lambda'}$ $(\lambda, \lambda' \in \Lambda)$ of a semigroup S with 0. We shall say that these quasi-ideals form a *complete system* if the following three conditions hold:

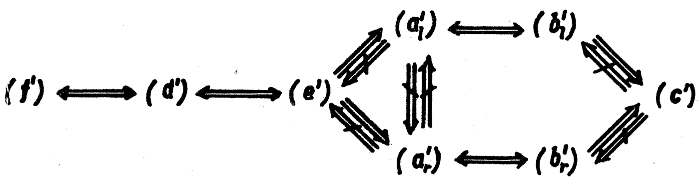
1) either $Q_{\lambda\lambda'} = 0$ or $Q_{\lambda\lambda'}$ is a 0-minimal quasi-ideal of S,
2) if $Q_{\lambda\lambda'} \neq 0$ then $Q_{\lambda\lambda'} = e_\lambda S e_{\lambda'}$ with suitable idempotent elements e_λ, $e_{\lambda'}$ in S,
3) $Q_{\lambda\lambda'} \neq 0$ implies $Q_{\lambda\lambda'} Q_{\lambda'\lambda} \neq 0$ $(\lambda, \lambda' \in \Lambda)$.*)

Now we can answer the question asked in the previous section.

Theorem I'. *Consider the following conditions on a semigroup S with* 0:

(a'_l) *S is regular, and it is the union of a finite number of its 0-minimal left ideals.*
(a'_r) *S is regular, and it is the union of a finite number of its 0-minimal right ideals.*
(b'_l) *S is a union of finitely many 0-minimal left ideals of the form Se_λ $(e_\lambda^2 = e_\lambda$, $\lambda \in \Lambda)$.*
(b'_r) *S is a union of finitely many 0-minimal right ideals of the form $e_\varrho S$ $(\varepsilon_\varrho^2 = e_\varrho$, $\varrho \in P)$.*
(c') *S is a 0-direct union of finitely many two-sided ideals which are completely 0-simple subsemigroups of S.*
(d') *S is the union of a finite number of its quasi-ideals, and these quasi-ideals form a complete system.*
(e') *S is regular, and it is the union of a finite number of its 0-minimal quasi-ideals.*
(f') *S is primitive regular with finitely many idempotent elements.*

The relation of these conditions is as shown by the following diagram:



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*) If we replace condition 2) by the following condition "2') there exists a set of non-zero idempotent elements $\{e_\lambda \mid \lambda \in \Lambda\}$ in S such that $Q_{\lambda\lambda'} = e_\lambda S e_{\lambda'} (\lambda, \lambda' \in \Lambda)$" then we consider those complete systems which do not contain superfluous indices (i.e. indices $\lambda \in \Lambda$ such that $Q_{\lambda\lambda'} = 0$ for all $\lambda' \in \Lambda$). Therefore the structure theorems II' and I' are valid with these modified definitions, too.

Proof. By Theorem II′ we have to take into consideration only the finiteness restrictions of our conditions, since the rest is everywhere equivalent. Now (f′) ⇒ ⇒ (d′) ⇒ (e′) follows immediately from the definition of complete systems. The validity of (e′) ⇒ (f′) is a consequence of Theorem 4 of O. Steinfeld [4] which states that a 0-minimal quasi-ideal of a semigroup with 0 is either a zero semigroup or a group with 0, thus contains at most one non-zero idempotent. (f′) ⇒ (b'_l) ⇒ (a'_l) and (f′) ⇒ (b'_r) ⇒ (a'_r) are evident. Since each 0-minimal one-sided ideal of a regular semigroup is generated by an idempotent, we also have (a'_l) ⇒ (b'_l) and (a'_r) ⇒ (b'_r). Now we shall prove (a'_l) ⇒ (c′); (a'_r) ⇒ (c′) can be obtained dually. Suppose that S is a 0-direct union of its two-sided ideals, A_i $(i \in I)$ and the union of a finite number of 0-minimal left ideals, then every A_i is a union of some of these 0-minimal left ideals. Since by (a'_l) the latter ones are of a finite number, so are the A_i. Thus we have proved all the desired implications.

Finally, consider a left zero semigroup with a zero element 0 adjoined: $ab = a$ if $b \neq 0$. Clearly, this semigroup S is regular; it has two left ideals and two ideals: 0 and S; 0 and an arbitrary one of the non-zero elements constitute a 0-minimal right ideal and a 0-minimal quasi-ideal; each non-zero element of S is primitive idempotent; S is completely 0-simple. If S is infinite, it satisfies (a'_l), (b'_l), and (c′), but none of (a'_r), (b'_r), (d′), (e′), and (f′). On the other hand, an infinite right zero semigroup with a zero element adjoined satisfies (a'_r), (b'_r), (c′), but none of (a'_l), (b'_l), (d′), (e′), and (f′). On account of these examples we can establish the non-implications indicated in the diagram.

Remark. (a'_l) and (a'_r) together are equivalent with (d′), (e′) and (f′) already. In fact, if both (a'_l) and (a'_r) hold, then $S = \bigcup_{i=1}^{n} L_i = \bigcup_{j=1}^{m} R_j$, where the L_i and the R_j are 0-minimal left and right ideals, respectively, so we also have $S = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} (L_i \cap R_j)$, where each $L_i \cap R_j$ is either 0 or a 0-minimal quasi-ideal of S by Theorem 8 of O. Steinfeld [5].

3.

In this section we introduce the notion of a special complete system by means of which we can complete some known results so as to obtain, in full analogy with Theorem II′, a characterization of semigroups being unions of completely 0-simple inverse semigroups. A result corresponding to Theorem I′ will also be formulated.

Definition. A *special complete system* is a complete system where 1) is replaced by the stronger condition

1′) each $Q_{\lambda\lambda}$ $(\lambda \in \Lambda)$ is a group with 0 which is a quasi-ideal, while for $\lambda \neq \lambda'$ $(\lambda, \lambda' \in \Lambda)$ $Q_{\lambda\lambda'}$ is either 0 or a zero semigroup which is a 0-minimal quasi-ideal.

(By O. Steinfeld [4] a quasi-ideal which is a group with 0 is necessarily a 0-minimal quasi-ideal.)

A *primitive inverse semigroup* is an inverse semigroup all idempotents of which are primitive.

Theorem II″. *The following conditions on a semigroup S with 0 are equivalent:*

(A_l'') *S is an inverse semigroup and the union of its 0-minimal left ideals.*

(A_r'') *S is an inverse semigroup and the union of its 0-minimal right ideals.*

(B_l'') *S is a union of 0-minimal left ideals each of which can be generated by one and only one idempotent.*

(B_r'') *S is a union of 0-minimal right ideals each of which can be generated by one and only one idempotent.*

(C″) *S is a 0-direct union of two-sided ideals which are completely 0-simple inverse subsemigroups of S.*

(D″) *S is a union of some of its quasi-ideals, and these quasi-ideals form a special complete system.*

(E″) *S is an inverse semigroup and the union of its 0-minimal quasi-ideals.*

(F″) *S is a primitive inverse semigroup.*

Proof. The equivalence of conditions (B_l''), (B_r''), (C″), and (F″) has been proved by P. S. Venkatesan [8], that of conditions (A_l''), (A_r''), (C″) and (E″) by O. Steinfeld [5].

Now we shall exhibit (F″) ⇒ (D″). Let S be a primitive inverse semigroup. By the equivalence of conditions (F′) and (D′) in Theorem II′, S is the union of some of its quasi-ideals $Q_{\lambda\lambda'} = e_\lambda S e_{\lambda'}$ which form a complete system. For any $\lambda \in \Lambda$ $Q_{\lambda\lambda}$ contains the idempotent e_λ, thus by Theorem 4 of O. Steinfeld [4] $Q_{\lambda\lambda}$ is a group with 0. Now consider the 0-minimal quasi-ideal $Q_{\lambda\lambda'} = e_\lambda S e_{\lambda'}$. Since e_λ is a primitive idempotent, by Lemma 6.38 of Clifford-Preston [1] it follows that $e_\lambda S$ is a 0-minimal right ideal of S. If we assume that $e_\lambda S e_{\lambda'}$ is a group with 0, then it has an identity element e, $e \in e_\lambda S e_\lambda' \subseteq e_\lambda S$,so we have $eS = e_\lambda S$. Since S is an inverse semigroup, this implies $e = e_\lambda$. Dually one can show that $e = e_{\lambda'}$, that is, $e_\lambda = e_{\lambda'}$ whence $\lambda = \lambda'$. This means that condition 1′) holds for our complete system of quasi-ideals.

Finally, we shall prove (D″) ⇒ (E″). For this purpose suppose that S is a union of some of its quasi-ideals, and these quasi-ideals form a special complete system. In view of Theorem II′, all we have to show is that S is an inverse semigroup. Consider an arbitrary non-zero $a \in S$, and let $x \in S$ be an inverse of a; $a \in e_\lambda S e_{\lambda'}$, $x \in e_\mu S e_{\mu'}$ with suitable idempotents $e_\lambda, e_{\lambda'}, e_\mu, e_{\mu'}$. Since $ax \in e_\lambda S e_{\mu'}$ is a non-zero idempotent of S, by 1′) we must have $\lambda = \mu'$, and ax has to be the identity of the group $Q_{\lambda\lambda} \setminus 0$. Dually we obtain that $\lambda' = \mu$ and that xa is the identity of $Q_{\lambda'\lambda'} \setminus 0$. Thus if both x and y are inverses of a, then we have $ax = ay$ and $xa = ya$, whence $x = xax = xay = yay = y$.

The following result is an immediate consequence of Theorems I′ and II″.

Theorem I″. *The following conditions on a semigroup S with 0 are equivalent:*

(a″) *S is an inverse semigroup and the union of a finite number of its 0-minimal left [right] ideals.*

(b″) *S is a union of finitely many 0-minimal left [right] ideals each of which can be generated by one and only one idempotent.*

(d″) *S is the union of a finite number of its quasi-ideals, and these quasi-ideals form a special complete system.*

(e″) *S is an inverse semigroup and the union of a finite number of its 0-minimal quasi-ideals.*

(f″) *S is a primitive inverse semigroup with finitely many idempotent elements.*

The condition

(c″) *S is a 0-direct union of finitely many two-sided ideals which are completely 0-simple inverse subsemigroups of S*

*is weaker than the above conditions.**)

The problems discussed in this paper have arisen while reading a preprint of the book [7] of O. Steinfeld, where they have been partly exposed. The author is also indebted to Professors O. Steinfeld and R. Šulka for their valuable criticisms.

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*) There exists a semigroup satisfying (c″) having infinitely many idempotent elements.