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ON SINGULAR PERTURBATION OF NONLINEAR TWO-POINT BOUNDARY VALUE PROBLEMS*)

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1. Introduction. We are concerned here with the two point boundary value problem

$$\varepsilon y'' + f(x, y, y', \varepsilon) = 0\,, \quad 0 \leqq x \leqq 1\,, \tag{1.1}$$

$$a_0(\varepsilon)\, y(0) - a_1(\varepsilon)\, y'(0) = \alpha(\varepsilon)\,, \quad |a_0(\varepsilon)| + |a_1(\varepsilon)| > 0 \tag{1.2}$$

$$b_0(\varepsilon)\, y(1) + b_1(\varepsilon)\, y'(1) = \beta(\varepsilon)\,, \quad |b_0(\varepsilon)| + |b_1(\varepsilon)| > 0\,. \tag{1.3}$$

where $\varepsilon > 0$ is a small parameter and "prime" denotes differentiation with respect to x. Ever since CODDINGTON and LEVINSON [1] produced an example of such a problem for which no solution exists for ε in some interval $(0, \varepsilon_0)$, there has been interest (see, e.g., ERDELYI [6], [7]) in the question of the continued existence of a solution as $\varepsilon \to 0$.

Our basic problem is the formulation of hypotheses under which (1.1), (1.2), (1.3) has a unique solution $y(x, \varepsilon)$ for sufficiently small $\varepsilon > 0$. Granting that such a unique solution does exist for $0 < \varepsilon < \varepsilon_0$, we are interested in the behavior of $y(x, \varepsilon)$ as $\varepsilon \to 0$.

Because of well-known existence and uniqueness theorems [2], [3], [10] for two-point boundary value problems which do not involve a parameter, one would expect that it would be necessary to assume in (1.1), (1.2), (1.3) that $a_0(\varepsilon)\, a_1(\varepsilon) \geqq 0$, $b_0(\varepsilon)\, b_1(\varepsilon) \geqq 0$, or equivalently

$$a_0(\varepsilon) \geqq 0\,, \quad a_1(\varepsilon) \geqq 0\,, \quad b_0(\varepsilon) \geqq 0\,, \quad b_1(\varepsilon) \geqq 0\,. \tag{1.4}$$

However, it turns out that if $b_1(\varepsilon)$ is not small relative to $b_0(\varepsilon)$ as $\varepsilon \to 0$, serious difficulties may arise. In Section 2, an example is discussed to exhibit these difficulties. In Section 3, conditions are formulated which guarantee the existence, uniqueness, and boundedness as $\varepsilon \to 0$ of the solution to (1.1), (1.2), (1.3).

*) This research was done while the author was visiting at the University of Illinois, Champaign-Urbana.

In several papers, O'MALLEY (see, e.g., [12], [13], [14]) has obtained asymptotic expansions of the solutions $y(x, \varepsilon)$ of various singular perturbation problems. In these problems, the crucial step was to show that a certain equation had a unique solution which was bounded as $\varepsilon \to 0$. O'Malley's technique has been to pass to an equivalent Volterra integral equation and use successive approximations. The use of integral equations in singular perturbation theory has been widely used (see, e.g., Erdelyi [8]). In Section 4, we apply the results of Section 3 to the problem studied by O'Malley in [14] (see also [9]). In fact, this work was initially motivated by the idea that one could get such asymptotic expansions using the maximum principle rather than the theory of Volterra integral equations. Further applications of the main result of section 3 will appear elsewhere.

We use the maximum principle in a crucial way; its use in singular perturbation theory is not new. Indeed, the maximum principle has been used by ECKHAUS and DEJAGER [5], PARTER [15], [16], [17], DORR, PARTER, and SHAMPINE [4], O'MALLEY [12] and by others. We refer to the book by PROTTER and WEINBERGER [18] for background results concerning the maximum principle.

2. An example. Consider the elementary problem

$$\varepsilon y'' + 2y' + 4y = 0\,, \tag{2.1}$$

$$a_0\, y(0) - a_1\, y'(0) = \alpha\,, \quad |a_0| + |a_1| > 0\,, \tag{2.2}$$

$$b_0\, y(1) + b_1\, y'(1) = \beta\,, \quad |b_0| + |b_1| > 0\,, \tag{2.3}$$

where $a_0 \geqq 0$, $a_1 \geqq 0$, $b_0 \geqq 0$, $b_1 \geqq 0$.

The general solution of (2.1) is $u = A \exp(\nu_1 x) + B \exp(\nu_2 x)$ where $\nu_1 = \nu_1(\varepsilon)$ and $\nu_2 = \nu_2(\varepsilon)$ are the roots of the characteristic polynomial $\varepsilon\nu^2 + 2\nu + 4 = 0$. One easily sees that

$$\nu_1 = -2 + O(\varepsilon)\,,$$

$$\nu_2 = -\frac{2}{\varepsilon} + O(1)\,.$$

The criterion that (2.1), (2.2), (2.3) has a unique solution is that the determinant

$$\begin{vmatrix} a_0 - a_1\nu_1\,, & a_0 - a_1\nu_2 \\ (b_0 + b_1\nu_1)\exp(\nu_1), & (b_0 + b_1\nu_2)\exp(\nu_2) \end{vmatrix}$$

not vanish. If $b_1 = 0$, the expanded determinant is

$$a_0 b_0(\exp(\nu_2) - \exp(\nu_1)) - a_1 b_0(\nu_1 \exp(\nu_2) - \nu_2 \exp(\nu_1))$$

which is negative for $\varepsilon > 0$ sufficiently small; thus if $b_1 = 0$, a unique solution exists.

Suppose now that $b_1 > 0$ and, for convenience, that $a_0 = 0$. The expanded determinant is

$$a_1 b_0(\nu_2 \exp(\nu_1) - \nu_1 \exp(\nu_2)) + a_1 b_1 \nu_1 \nu_2(\exp(\nu_1) - \exp(\nu_2)).$$

Now the first term above is non-positive, and the second term is non-negative; in fact the determinant will be zero if

$$\frac{b_0}{b_1} = \frac{\nu_1 \nu_2(\exp(\nu_1) - \exp(\nu_2))}{\nu_1 \exp(\nu_2) - \nu_2 \exp(\nu_1)}.$$

Thus, given $\varepsilon_0 > 0$, we can choose b_0 and b_1 so that the determinant is zero for some $\varepsilon \in (0, \varepsilon_0)$. So even if for certain b_0 and b_1, a unique solution exists for all ε in some interval $(0, \varepsilon_0)$, the value of ε_0 will depend critically on the values of b_0 and b_1. It is clear that if b_0 and b_1 are allowed to be functions of ε, a unique solution may fail to exist for every ε in some interval $(0, \varepsilon_0)$.

3. Boundary value problems. Our fundamental assumptions concerning the boundary value problem (1.1), (1.2), (1.3) are as follows: we require the existence of $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$,

A1:
$$f(x, y, z, \varepsilon),\ f_2(x, y, z, \varepsilon) \equiv \frac{\partial}{\partial y} f(x, y, z, \varepsilon),\ f_3(x, y, z, \varepsilon) \equiv \frac{\partial}{\partial z} f(x, y, z, \varepsilon)$$

are all continuous for fixed ε in the region

$$R = \{(x, y, z) : 0 \leqq x \leqq 1,\ y^2 + z^2 < \infty\}.$$

A2:
$$f_2(x, y, z, \varepsilon) < B(\varepsilon) < \infty, \quad 0 < a(\varepsilon) \leqq f_3(x, y, z, \varepsilon) \leqq M(\varepsilon) < \infty$$

hold throughout R.

A3:
$$a_0(\varepsilon) \geqq 0,\ a_1(\varepsilon) \geqq 0,\ b_1(\varepsilon) \geqq 0, \quad \text{and there exists} \quad p(\varepsilon) \geqq 0$$

$$\text{such that} \quad p(\varepsilon) \geqq \frac{B(\varepsilon)}{a(\varepsilon)}, \quad b_0(\varepsilon) \geqq p(\varepsilon)\, b_1(\varepsilon) \quad \text{and}$$

$$2\varepsilon\, p^2(\varepsilon) \leqq p(\varepsilon)\, a(\varepsilon) - B(\varepsilon).$$

Comments. The upper bound $B(\varepsilon)$ on $f_2(x, y, z, \varepsilon)$, the lower bound $a(\varepsilon)$ on $f_3(x, y, z, \varepsilon)$ of A2 and all the requirements of A3 will be explicitly used below. The smoothness hypotheses of A1 and the upper bound $M(\varepsilon)$ on $f_3(x, y, z, \varepsilon)$ of A2 appear so that the maximum principle [18] and existence-uniqueness theorem of KELLER [10] and BEBERNES and GAINES [2] may be invoked.

The key to our work is a simple change of dependent variable. Let

$$w = \exp(-p(\varepsilon)\, x)\,. \tag{3.1}$$

The following lemma results from a simple calculation.

Lemma 3.1. *Under assumptions* A1–A3, *the boundary value problem* (1.1), (1.2), (1.3) *has the solution* y *if and only if* $u = y/w$ *is a solution of*

$$u'' + H(x, u, u', \varepsilon) = 0\,, \tag{3.2}$$

$$[a_0(\varepsilon) + p(\varepsilon)\, a_1(\varepsilon)]\, u(0) - a_1(\varepsilon)\, u'(0) = \alpha(\varepsilon)\,, \tag{3.3}$$

$$[b_0(\varepsilon) - p(\varepsilon)\, b_1(\varepsilon)]\, u(1) + b_1(\varepsilon)\, u'(1) = \beta(\varepsilon) \exp(p(\varepsilon))\,, \tag{3.4}$$

where

$$H(x, u, u', \varepsilon) = \frac{1}{\varepsilon}\left[\varepsilon\, p^2(\varepsilon)\, u - 2\varepsilon p(\varepsilon)\, u' + \frac{f(x, wu, (wu)', \varepsilon)}{w}\right]. \tag{3.5}$$

Remark. We shall see below that if $0 < \varepsilon < \varepsilon_0$, the maximum principle may be applied to u. Such changes of dependent variable, designed for the purpose of applying the maximum principle, have been used by Dorr, Parter, and Shampine [4] and by Parter [15], [16], [17]. These authors have used in (3.1) a function of the form $p(\varepsilon) = q/\varepsilon$ where $q > 0$. This choice of $p(\varepsilon)$ has the disadvantage that $1/w$ is unbounded as $\varepsilon \to 0$. We shall see that $p(\varepsilon)$ can often be chosen independent of ε so that this disadvantage does not occur.

Theorem 3.1. *Under assumptions* A1–A3, *the problem* (1.1), (1.2), (1.3) *has a unique solution for* $0 < \varepsilon < \varepsilon_0$.

Proof. From (3.5) we calculate

$$\begin{aligned} H_2(x, u, u', \varepsilon) &= \\ &= \frac{1}{\varepsilon}[\varepsilon\, p^2(\varepsilon) - p(\varepsilon) f_3(x, wu, w'u + wu', \varepsilon) + f_2(x, wu, w'u + wu', \varepsilon)] \end{aligned} \tag{3.6}$$

so that by A2 and A3

$$\begin{aligned} H_2(x, u, u', \varepsilon) &\leqq \frac{1}{\varepsilon}[\varepsilon p^2(\varepsilon) - p(\varepsilon)\, a(\varepsilon) + B(\varepsilon)] \\ &\leqq \frac{B(\varepsilon) - p(\varepsilon)\, a(\varepsilon)}{2\varepsilon} \\ &\leqq 0 \end{aligned} \tag{3.7}$$

if $0 < \varepsilon < \varepsilon_0$. It follows from A1 and A2 that $H(x, u, u', \varepsilon)$, $H_2(x, u, u', \varepsilon)$, $H_3(x, u, u', \varepsilon)$ are all continuous in R for fixed $\varepsilon \in (0, \varepsilon_0)$ and that $H_3(x, u, u', \varepsilon)$ is bounded in R for fixed $\varepsilon \in (0, \varepsilon_0)$. Further, because of A3, the boundary conditions (3.3) and (3.4) (which are not trivial (see (1.2), (1.3)) satisfy the conditions

$$a_0(\varepsilon) + p(\varepsilon)\, a_1(\varepsilon) \geqq 0\,, \quad a_1(\varepsilon) \geqq 0\,,$$

$$b_0(\varepsilon) - p(\varepsilon)\, b_1(\varepsilon) \geqq 0\,, \quad b_1(\varepsilon) \geqq 0\,.$$

Thus, using Bebernes and Gaines [2], it follows from Lemma 3.1 that (1.1), (1.2), (1.3) has a unique solution.

Remark. If in A2, $B(\varepsilon) \leqq 0$, then the choice $p(\varepsilon) \equiv 0$ suffices; i.e., no change of variable is necessary in order to conclude existence and uniqueness.

Theorem 3.2. *Assume* A1 – A3 *are valid and let* $c(\varepsilon)$ *be the smallest number satisfying all of the inequalities*

(i) $|\alpha(\varepsilon)| \leqq c(\varepsilon)\,[a_0(\varepsilon) + p(\varepsilon)\, a_1(\varepsilon)]$,

(ii) $|\beta(\varepsilon)| \exp(p(\varepsilon)) \leqq c(\varepsilon)\,[b_0(\varepsilon) - p(\varepsilon)\, b_1(\varepsilon)]$,

(iii) $|f(x, 0, 0, \varepsilon)| \exp(p(\varepsilon)\, x) \leqq c(\varepsilon)\,[p(\varepsilon)\, a(\varepsilon) - B(\varepsilon)]$.

Then for $0 < \varepsilon < \varepsilon_0$, *the unique solution* $y(x, \varepsilon)$ *of* (1.1), (1.2), (1.3) *satisfies*

$$|y(x, \varepsilon)| \leqq c(\varepsilon) \exp(-p(\varepsilon)\, x)\,.$$

In particular, if $c(\varepsilon) = O(1)$ *as* $\varepsilon \to 0$, *then* $y(x, \varepsilon)$ *is uniformly bounded as* $\varepsilon \to 0$.

Proof. We first calculate bounds on the corresponding solution $u = y/w$ of (3.2), (3.3), (3.4). The proof of Theorem 3.1 shows that the maximum principle [18, p. 48] may be applied to u, and we will use it to obtain upper and lower bounds of u. Thus, we seek two functions $z_1(x)$ and $z_2(x)$ such that

$$z_1'' + H(x, z_1, z_1', \varepsilon) \leqq 0\,,$$

$$[a_0(\varepsilon) + p(\varepsilon)\, a_1(\varepsilon)]\, z_1(0) - a_1(\varepsilon)\, z_1'(0) \geqq \alpha(\varepsilon)\,,$$

$$[b_0(\varepsilon) - p(\varepsilon)\, b_1(\varepsilon)]\, z_1(1) + b_1(\varepsilon)\, z_1'(1) \geqq \beta(\varepsilon) \exp(p(\varepsilon))$$

and

$$z_2'' + H(x, z_2, z_2', \varepsilon) \geqq 0\,,$$

$$[a_0(\varepsilon) + p(\varepsilon)\, a_1(\varepsilon)]\, z_2(0) - a_1(\varepsilon)\, z_2'(0) \leqq \alpha(\varepsilon)\,,$$

$$[b_0(\varepsilon) - p(\varepsilon)\, b_1(\varepsilon)]\, z_2(1) + b_1(\varepsilon)\, z_2'(1) \leqq \beta(\varepsilon) \exp(p(\varepsilon))\,.$$

For then, by the maximum principle, $z_2(x) \leqq u(x) \leqq z_1(x)$, for $0 \leqq x \leqq 1$.

In both cases we choose constants; that is, we let $z_1(x) = c_1(\varepsilon)$, $z_2(x) = c_2(\varepsilon)$. Thus we require, using the mean value theorem, that

$$H(x, c_1(\varepsilon), 0, \varepsilon) = H(x, 0, 0, \varepsilon) + H_2(x, \hat{c}_1(\varepsilon), 0, \varepsilon)\, c_1(\varepsilon) \leqq 0\,, \tag{3.8}$$

$$[a_0(\varepsilon) + p(\varepsilon)\, a_1(\varepsilon)]\, c_1(\varepsilon) \geqq \alpha(\varepsilon)\,, \tag{3.9}$$

$$[b_0(\varepsilon) - p(\varepsilon)\, b_1(\varepsilon)]\, c_1(\varepsilon) \geqq \beta(\varepsilon) \exp(p(\varepsilon))\,,$$

where $\hat{c}_1(\varepsilon)$ is between 0 and $c_1(\varepsilon)$.

Using (3.7) and assuming $c_1(\varepsilon) > 0$, we may satisfy (3.8) by requiring that

$$H(x, 0, 0, \varepsilon) + \left(\frac{B(\varepsilon) - p(\varepsilon)\, a(\varepsilon)}{2\varepsilon}\right) c_1(\varepsilon) = \tag{3.11}$$

$$= \frac{1}{2\varepsilon}\, [f(x, 0, 0, \varepsilon) \exp(p(\varepsilon)\, x) - (p(\varepsilon)\, a(\varepsilon) - B(\varepsilon))\, c_1(\varepsilon)] \leqq 0\,.$$

By (i), (ii), (iii), we may take $c_1(\varepsilon) = c(\varepsilon)$. Similarly, we may choose $c_2(\varepsilon) = -c(\varepsilon)$.

Thus

$$|y| = |u|\, w \leqq c(\varepsilon) \exp(-p(\varepsilon)\, x)\,,$$

and the theorem is proved.

Remark. Even if in A2, $B(\varepsilon) \leqq 0$, the choice $p(\varepsilon) \equiv 0$, which suffices for Theorem 3.1, may not be satisfactory for Theorem 3.2. For if $b_0(\varepsilon) \equiv 0$, (ii) will be violated unless $\beta(\varepsilon) \equiv 0$ and if $B(\varepsilon) \equiv 0$, choosing $p(\varepsilon) \equiv 0$ will violate (iii) unless $f(x, 0, 0, \varepsilon) \equiv \equiv 0$. Thus, choosing $p(\varepsilon)$ for Theorem 3.2 often requires more care than choosing for Theorem 3.1. It is clear that a choice of the form $p(\varepsilon) = q/\varepsilon$ $(q > 0)$ will often be damaging. For then, unless $\beta(\varepsilon) \equiv 0$ and $f(x, 0, 0, \varepsilon) \equiv 0$, conditions (ii) and (iii) might require that $b_0(\varepsilon)$ or $a(\varepsilon)$ grow exponentially. Thus, for problems which are non-homogeneous, $p(\varepsilon) = q/\varepsilon$ $(q > 0)$ is often, though not always (see section 4), an inappropriate choice. We now pass to situations in which $p(\varepsilon)$ can be chosen independent of ε.

Corollary 3.1. *In addition to assumption* A1, *suppose* A2 *holds with* $0 < B = B(\varepsilon) < \infty$, $a = a(\varepsilon) > 0$ *independent of* ε, *and suppose that* $a_0(\varepsilon) \geqq 0$, $a_1(\varepsilon) \geqq 0$, $b_0(\varepsilon) > 0$, $b_1(\varepsilon) \geqq 0$ *and* $b_1(\varepsilon) = o(b_0(\varepsilon))$ *as* $\varepsilon \to 0$. *Then the problem* (1.1), (1.2), (1.3) *has a unique solution for* $\varepsilon > 0$ *sufficiently small.*

Proof. Let $p(\varepsilon) = 2B/a$; then the assumptions A1 – A3 are all satisfied for $0 < \varepsilon < \varepsilon_1$ where ε_1 is chosen so that $\varepsilon_1 \leqq \varepsilon_0$, $\varepsilon_1 \leqq a^2/8B$, and $b_0(\varepsilon) \geqq (2B/a)\, b_1(\varepsilon)$ for $0 < \varepsilon < \varepsilon_1$. Thus Theorem 3.1 applies.

Corollary 3.2. *In addition to the hypotheses of Corollary* 3.1, *suppose that* $a_0 = a_0(\varepsilon)$, $b_0 = b_0(\varepsilon)$, $a_1 = a_1(\varepsilon)$ *are all independent of* ε. *Also suppose that* $|\alpha(\varepsilon)| \leqq \alpha < \infty$, $|\beta(\varepsilon)| \leqq \beta < \infty$ *and* $|f(x, 0, 0, \varepsilon)| \leqq N$, *for* $0 < \varepsilon < \varepsilon_0$. *Then the unique solution* $y(x, \varepsilon)$ *of* (1.1), (1.2), (1.3) *is uniformly bounded as* $\varepsilon \to 0$.

Proof. Letting $p(\varepsilon) = 2B/a$, this corollary follows from Theorem 3.2.

In the same way, we obtain

Corollary 3.3. *In addition to the hypotheses of Corollary* 3.1, *suppose that* $\alpha(\varepsilon) \equiv 0$, $\beta(\varepsilon) \equiv 0$ *and* $f(x, 0, 0) \equiv f(x, 0, 0, \varepsilon)$ *is independent of* ε. *Then the unique solution of* (1.1), (1.2), (1.3) *satisfies*

$$|y(x, \varepsilon)| \leqq \frac{1}{B} \exp\left(\frac{2B}{a}\right) \max_{0 \leqq x \leqq 1} |f(x, 0, 0)|, \quad 0 \leqq x \leqq 1,$$

for all sufficiently small $\varepsilon > 0$.

This last result is essentially a generalization of a lemma of Kreiss and Parter [11, Lemma 2.3], who require $f(x, y, z, \varepsilon)$ to be linear in y and z and the coefficients in the boundary conditions to be $a_0(\varepsilon) \equiv b_0(\varepsilon) \equiv 1$ and $a_1(\varepsilon) \equiv b_1(\varepsilon) \equiv 0$.

4. An application to chemical flow reactors. The boundary value problem (see [9])

$$\varepsilon y'' - y' - a y^N = 0, \quad 0 \leqq x \leqq 1, \tag{4.1}$$

$$y(0) - \varepsilon y'(0) = 1, \tag{4.2}$$

$$y'(1) = 0, \tag{4.3}$$

where $a > 0$, $N \geqq 0$ are constants and ε is a small positive parameter, arises in the study of chemical flow reactors. O'Malley [14] attacked the problem of obtaining an asymptotic solution of the more general problem

$$\varepsilon y'' - b(x)\, y' - g(x, y) = 0, \quad 0 \leqq x \leqq 1, \tag{4.4}$$

$$y(0) - \varepsilon y'(0) = \alpha, \tag{4.5}$$

$$y'(1) = \beta, \tag{4.6}$$

where $b(x)$ is strictly positive, and both $b(x)$ and $g(x, y)$ are infinitely differentiable. Although O'Malley does not explicitly state other hypotheses, it is clear that these assumptions are not sufficient to guarantee the conclusion of his theorem; for he explicitly requires the existence of a global solution on $[0, 1]$ of the reduced problem

$$b(x)\, y' + g(x, y) = 0, \quad 0 \leqq x \leqq 1$$

$$y(0) = \alpha.$$

However, $b(x) \equiv 1$, $g(x, y) \equiv y^2$, $\alpha = -2$ furnishes a counterexample. We also note that non-integral values of n, which are of interest in the applications (see [9]), certainly violate the smoothess assumptions on $g(x, y)$.

We shall obtain a form of O'Malley's result here, (with suitable hypotheses on $g(x, y)$) as an application of our results in the previous section. However, we prefer to change variables by replacing x by $1 - x$ so that the problem (4.4), (4.5), (4.6) assumes the form

$$\varepsilon y'' + b(x)\, y' - g(x, y) = 0\,, \quad 0 \leqq x \leqq 1\,, \tag{4.7}$$

$$-y'(0) = \beta\,, \tag{4.8}$$

$$y(1) + \varepsilon\, y'(1) = \alpha\,. \tag{4.9}$$

Of course, the functions $b(x)$ and $g(x, y)$ are not quite the same functions as before. For the formal work to follow, we assume for the time being that $g(x, y)$ is analytic and that $b(x)$ is strictly positive and has a continuous derivative.

From a careful study of the procedure in [14] or generalizing from the linear case [12], one may decide to assume a solution of the form

$$y = \sum_{k=0}^{\infty} P_k(x, \varepsilon)\, \varepsilon^k\,, \tag{4.10}$$

where each $P_k(x, \varepsilon)$ is a polynomial in

$$E_1(x, \varepsilon) \equiv \exp\left(-\varepsilon^{-1} \int_0^x b(s)\, \mathrm{d}s\right) \quad \text{of degree} \quad k - 1 \quad \text{for} \quad k \geqq 2$$

and of degree k for $k = 0, 1$ and whose coefficients are functions of x only. Thus we write

$$P_0(x, \varepsilon) = a_{00}(x)\,, \quad P_1(x, \varepsilon) = a_{01}(x) + a_{11}(x)\, E_1(x, \varepsilon)\,,$$

and, for $k \geqq 2$,

$$P_k(x, \varepsilon) = \sum_{j=0}^{k-1} a_{jk}(x)\, E_j(x, \varepsilon)\,,$$

where

$$E_j(x, \varepsilon) = E_1^j(x, \varepsilon)\,.$$

Before substituting (4.10) formally into (4.7), we observe that if we expand $g(x, y)$ as a power series in $y - P_0(x, \varepsilon)$, substitute for y from (4.10), expand the powers involved, and rearrange in terms of powers of ε, we get

$$g(x, y) = \sum_{k=0}^{\infty} g^{(k)}(x, \varepsilon)\, \varepsilon^k\,, \tag{4.11}$$

where

$$g^{(0)}(x, \varepsilon) = g(x, a_{00}(x)), \quad g^{(1)}(x, \varepsilon) = g_2(x, a_{00}(x))\, P_1(x, \varepsilon)$$

and in general,

$$g^{(k)}(x, \varepsilon) = g_2(x, a_{00}(x))\, P_k(x, \varepsilon) + \sum_{j=0}^{k} G_{jk}(x)\, E_j(x, \varepsilon),$$

where $G_{jk}(x)$ is independent of ε and depends only on the functions $a_{n,m}(x)$ for $m \leqq \leqq k - 1$.

Substituting (4.10) and (4.11) formally into (4.7) and equating the coefficient of each product $E_j(x, \varepsilon)\, \varepsilon^k$ to 0, we get the following equations for determining the a_{jk}'s:

$$b(x)\, a'_{00} - g(x, a_{00}) = 0, \tag{4.12}$$

$$\begin{aligned} b(x)\, a'_{01} - g_2(x, a_{00})\, a_{01} &= -a''_{00}, \\ b(x)\, a'_{11} + (b'(x) + g_2(x, a_{00}))\, a_{11} &= 0 \end{aligned} \tag{4.13}$$

and for each $k \geqq 2$,

$$\begin{aligned} b(x)\, a'_{0k} - g_2(x, a_{00})\, a_{0k} &= G_{0k}(x) - a''_{0,k-1}, \\ b(x)\, a'_{1k} + (b'(x) + g_2(x, a_{00})\, a_{1k} &= a''_{1,k-1} - G_{1k}(x), \\ b^2(x)\, (j^2 - j)\, a_{j,k+1} &= g_2(x, a_{00})\, a_{jk} + G_{jk}(x) + (2j - 1)\, b(x)\, a'_{jk} + \\ &\quad + j\, b'(x)\, a_{jk} - a''_{j,k-1}, \quad \text{for } j = 2, 3, \ldots, k, \end{aligned} \tag{4.14}$$

where, for convenience, we set $a_{jk}(x) \equiv 0$ for $j \geqq k \geqq 2$, and $a_{21}(x) \equiv 0$.

Thus, we determine $P_0(x, \varepsilon) = a_{00}(x)$ from (4.12) and $P_1(x, \varepsilon) = a_{01}(x) + a_{11}(x)\, E_1(x, \varepsilon)$ from (4.13). Then using (4.14) with $k = 2$, we determine $P_2(x, \varepsilon) = a_{02}(x) + a_{12}(x)\, E_1(x, \varepsilon)$ and in addition $a_{23}(x)$. Afterwards, setting $k = 3, 4, \ldots$ step-by-step we determine the other a_{jk}'s. Note that on the kth step ($k \geqq 2$), we determine, in addition to $a_{0k}(x)$ and $a_{1k}(x)$, also $a_{2,k+1}(x), \ldots, a_{k,k+1}(x)$. It is interesting to observe that $a_{jk}(x)$ for $j = 0, 1$ is always determined from a first order differential equation, linear except in the case $k = 0$, and that $a_{jk}(x)$, $j \geqq 2$, is determined from a very simple algebraic equation.

Of course, the differential equations do not determine $a_{jk}(x)$, $j = 0, 1$, uniquely. To get an initial (or terminal) condition which will select a unique solution in each case, we first note that for small $\varepsilon > 0$, $E_j(1, \varepsilon) \sim 0$ for $j \geqq 1$. Thus

$$y(1) \sim \sum_{k=0}^{\infty} a_{0k}(1)\, \varepsilon^k, \quad y'(1) \sim \sum_{k=0}^{\infty} a'_{0k}(1)\, \varepsilon^k,$$

so we see that $y(1) + \varepsilon\, y'(1) = \alpha$ leads to

$$a_{00}(1) + \sum_{k=1}^{\infty} (a_{0k}(1) + a'_{0,k-1}(1))\, \varepsilon^k \sim \alpha.$$

So we set

$$a_{00}(1) = \alpha\,, \quad a_{0k}(1) = -a'_{0,k-1}(1)\,, \quad k = 1, 2, \ldots.$$

These terminal conditions, together with the differential equations for $a_{0k}(x)$ in (4.12), (4.13), (4.14), then recursively determine the a_{0k}'s uniquely.

The condition (4.8) gives

$$a'_{00}(0) - b(0)\,a_{11}(0) + (a'_{01}(0) + a'_{11}(0) - b(0)\,a_{12}(0))\,\varepsilon + \\ + \sum_{k=2}^{\infty} \Big(\sum_{j=0}^{k} a'_{jk}(0) - j\,b(0)\,a_{j,k+1}(0)\Big)\,\varepsilon^k = -\beta\,,$$

where for $j \geqq k \geqq 2$, we set $a_{jk}(x) \equiv 0$. So we require

$$b(0)\,a_{11}(0) = \beta + a'_{00}(0)\,, \quad b(0)\,a_{12}(0) = a'_{01}(0) + a'_{11}(0)\,,$$

and for $k \geqq 2$,

$$b(0)\,a_{1,k+1}(0) = a'_{0k}(0) + a'_{1k}(0) + \sum_{j=2}^{k} (a'_{jk}(0) - j\,b(0)\,a_{j,k+1}(0))\,.$$

Thus, since each $a_{jk}(x), j \neq 1$, is already uniquely determined, each $a_{1k}(x)$ may now be recursively determined uniquely using these initial conditions and the differential equations for $a_{1k}(x)$ in (4.13) and (4.14).

The remaining difficulty is the question of global existence on $[0, 1]$ of the solution of the reduced problem consisting of (4.12) and the terminal condition $a_{00}(1) = \alpha$. If we assume (and we shall below) that $g_2(x, y)$ $(= (\partial g/\partial y)(x, y))$ is bounded below, say $g_2(x, y) \geqq -\beta$ for $0 \leqq x \leqq 1$, $-\infty < y < \infty$, the easily obtained estimates

$$a'_{00} \geqq \frac{g(x, 0) - Ba_{00}}{b(x)}\,, \quad \text{for} \quad a_{00} > 0\,,$$

$$a'_{00} \leqq \frac{g(x, 0) - Ba_{00}}{b(x)}\,, \quad \text{for} \quad a_0 < 0$$

together with the standard theory of continuation of solutions [1] guarantee that the local solution of (4.12) with the terminal condition $a_{00}(1) = \alpha$ can be extended to a global solution on $[0, 1]$.

Theorem 4.1. *Suppose that $b(x) \geqq b > 0$ for $0 \leqq x \leqq 1$, $b'(x)$ is continuous on $[0, 1]$ and $g(x, y)$ together with its partial derivatives with respect to y up to and including order $n + 1$ are all continuous on $[0, 1] \times (-\infty, \infty)$. (Here $n \geqq 0$ is an integer). Suppose further that $g_2(x, y) \geqq -B$ $(B > 0)$ on $[0, 1] \times (-\infty, \infty)$. Then for $\varepsilon > 0$ sufficiently small, the boundary value problem* (4.7), (4.8), (4.9) *has a unique solution $y(x, \varepsilon)$ which is given by*

$$y(x, \varepsilon) = \sum_{k=0}^{n} P_k(x, \varepsilon)\,\varepsilon^k + \varepsilon^{n+1}\,R_n(x, \varepsilon) \tag{4.15}$$

where $P_k(x, \varepsilon)$, $k = 0, 1, \dots, n$ are the functions defined above and $R_n(x, \varepsilon)$ is uniformly bounded as $\varepsilon \to 0$.

Proof. The existence of a unique solution of (4.7), (4.8), (4.9) follows from Corollary 3.1. By the remarks immediately preceding the theorem, $P_{00}(x, \varepsilon) = a_{00}(x)$ exists globally on $[0, 1]$. Letting

$$y_n = \sum_{k=0}^{n} P_k(x, \varepsilon)\, \varepsilon^k , \tag{4.16}$$

using Taylor's theorem to expand $g(x, y_n)$ in powers of $y_n - a_{00}(x) = \sum_{k=1}^{n} P_k(x, \varepsilon)\, \varepsilon^k$, expanding each of the powers involved and rearranging in terms of powers of ε, we get

$$g(x, y_n) = \sum_{k=0}^{n} g^{(k)}(x, \varepsilon)\, \varepsilon^k + O(\varepsilon^{n+1}) , \tag{4.17}$$

where the functions $g^{(k)}(x, \varepsilon)$ are the same as those appearing in (4.11).

We re-write (4.7) as

$$\varepsilon y'' + b(x)\, y' - g(x, y_n) - [g(x, y) - g(x, y_n)] = 0 ,$$

substitute (4.15) and (4.17) into this equation, use the definitions of $P_k(x, \varepsilon)$, $k = 0, 1, \dots, n$, and assuming $n \geqq 2$, obtain the following differential equation for $R_n(x, \varepsilon)$:

$$\begin{aligned} \varepsilon R_n'' + b(x)\, R_n' - \frac{1}{\varepsilon^{n+1}} [g(x, y_n + \varepsilon^{n+1} R_n) - g(x, y_n)] + \\ + h_1(x, \varepsilon) + h_2(x, \varepsilon) = 0 \end{aligned} \tag{4.18}$$

where

$$\begin{aligned} h_1(x, \varepsilon) &= -\varepsilon^{-1} \sum_{j=2}^{n} (j^2 - j)\, b^2(x)\, a_{j,n+1}\, E_j(x, \varepsilon) \\ &= O\left(\varepsilon^{-1} \exp\left(-\frac{2}{\varepsilon} \int_0^x b(s)\, ds\right)\right) \end{aligned}$$

and

$$h_2(x, \varepsilon) = O(1) , \quad \text{as} \quad \varepsilon \to 0 .$$

If $n = 0$ or 1, the details are slightly different and lead to the same conclusion with the added simplification that $h_1(x, \varepsilon) \equiv 0$. In any case, we may take (4.18) as the differential equation for $R_n(x, \varepsilon)$.

Using the boundary conditions (4.8) and (4.9) for $y(x, \varepsilon)$ together with the initial and terminal conditions imposed on the $a_{j,k}$'s for $j = 0, 1$, it is easy to see that $R_n(x, \varepsilon)$ satisfies the boundary conditions

$$-\varepsilon\, R_n'(0, \varepsilon) = \sum_{j=0}^{n-1} a_{jn}'(0) \tag{4.19}$$

$$R_n(1, \varepsilon) + \varepsilon\, R_n'(1, \varepsilon) = -a_{0n}'(1) + \delta(\varepsilon) \tag{4.20}$$

where

$$\delta(\varepsilon) = O\left(\varepsilon^{-n} \exp\left(-\frac{1}{\varepsilon}\int_0^1 b(s)\, ds\right)\right) \to 0 \quad \text{as} \quad \varepsilon \to 0 .$$

Unfortunately, none of the results of section 3 apply immediately to conclude that $R_n(x, \varepsilon)$ is uniformly bounded as $\varepsilon \to 0$. However, it is possible to split the boundary value problem for $R_n(x, \varepsilon)$ into two parts and apply our results separately to each part.

Toward this end, consider the boundary value problem

$$\varepsilon u'' + b(x)\, u' - \frac{1}{\varepsilon^{n+1}} [g(x, y_n + \varepsilon^{n+1}u) - g(x, y_n)] + h_1(x, \varepsilon) = 0 , \tag{4.21}$$

$$-\varepsilon\, u'(0) = \sum_{j=0}^{n-1} a'_{jn}(0) , \tag{4.22}$$

$$u(1) + \varepsilon\, u'(1) = 0 . \tag{4.23}$$

By Corollary 3.1, this problem has a unique solution for $\varepsilon > 0$ sufficiently small; call this solution $S_n(x, \varepsilon)$. Put $Q_n(x, \varepsilon) = R_n(x, \varepsilon) - S_n(x, \varepsilon)$. Then $Q_n(x, \varepsilon)$ solves the boundary value problem

$$\varepsilon u'' + b(x)\, u' - \tag{4.24}$$
$$- \frac{1}{\varepsilon^{n+1}} [g(x, y_n + \varepsilon^{n+1}S_n + \varepsilon^{n+1}u) - g(x, y_n + \varepsilon^{n+1}S_n)] + h_2(x, \varepsilon) = 0$$

$$-\varepsilon\, u'(0) = 0 \tag{4.25}$$

$$u(1) + \varepsilon\, u'(1) = -a'_{0n}(1) + \delta(\varepsilon) \tag{4.26}$$

That $Q_n(x, \varepsilon)$ is uniformly bounded as $\varepsilon \to 0$ follows from Theorem 3.2 by taking $p(\varepsilon) = 2B/b$. That $S_n(x, \varepsilon)$ is uniformly bounded as $\varepsilon \to 0$ follows from Theorem 3.2 by choosing $p(\varepsilon) = q/\varepsilon$ where q satisfies $0 < q < 2b$, $0 < q < \frac{1}{2}$. Thus, $R_n(x, \varepsilon) = S_n(x, \varepsilon) + Q_n(x, \varepsilon)$ is uniformly bounded as $\varepsilon \to 0$, and the theorem is proved.

It has no doubt been noticed that Theorem 4.1 unfortunately does not apply to the motivating problem (4.1), (4.2), (4.3) if in (4.1) N is an even positive integer, for then the condition $g_2(x, y) \geqq -B$ is violated. Further, if N is non-integral, the requirement that $g(x, y)$ be well-behaved for $y < 0$ may be violated and if $0 < N < n + 1$, then $g(x, y) = ay^N$ surely fails to satisfy the smoothness requirements at $y = 0$. Even so, we show how Theorem 4.1 may be applied. We assume in the following that $N > 0$ and $N \neq 1$ (The cases $N = 0,1$ are already covered by Theorem 4.1). Replacing x by $1 - x$ in (4.1), (4.2), (4.3), we consider instead the problem

$$\varepsilon y'' + y' - ay^N = 0 , \quad 0 \leqq x \leqq 1 , \tag{4.27}$$

$$-y'(0) = 0 , \tag{4.28}$$

$$y(1) + \varepsilon\, y'(1) = 1 . \tag{4.29}$$

The reduced problem (in the notation of Theorem 4.1) is

$$(4.30) \qquad a'_{00} = a a_{00}^{N}, \quad 0 \leqq x \leqq 1,$$

$$(4.31) \qquad a_{00}(1) = 1.$$

It is easily verified that if $N > 1$, the unique solution of the reduced problem exists as an increasing function on $[0, 1]$ and satisfies

$$a_{00}(x) \geqq a_{00}(0) = [1 + a(N - 1)]^{-1/(N-1)}.$$

If $0 < N < 1$, then we may conclude that the unique solution of the reduced problem exists as an increasing function on $[0, 1]$ and satisfies $a_{00}(x) > 0$ on $[0, 1]$ if and only if $a < (1 - N)^{-1}$, in which case the inequality

$$a_{00}(x) \geqq a_{00}(0) = [1 - a(1 - N)]^{1/(1-N)}$$

holds for $0 \leqq x \leqq 1$.

Thus, if $N > 1$ or $0 < N < 1$ and $a < (1 - N)^{-1}$, we may choose δ so that $0 < \delta < a_{00}(0)$, say $\delta = \frac{1}{2}a_{00}(0)$, and then define

$$\hat{g}(y) = \begin{Bmatrix} a y^N, & \text{if} \quad y \geqq \delta \\ 0, & \text{if} \quad y \leqq 0 \end{Bmatrix},$$

and extend $\hat{g}(y)$ to $-\infty < y < \infty$ in such a way that $\hat{g}(y)$ is infinitely differentiable and satisfies $\hat{g}'(y) \geqq 0$ on $(-\infty, \infty)$. Then Theorem 4.1 applies to the modified problem

$$(4.32) \qquad \varepsilon y'' + y' - \hat{g}(y) = 0,$$

$$(4.33) \qquad -y'(0) = 0,$$

$$(4.34) \qquad y(1) + \varepsilon\, y'(1) = 1$$

for any integral value of $n \geqq 0$ so that in particular the unique solution $y(x, \varepsilon) \to$ $\to a_{00}(x)$ uniformly on $[0, 1]$ as $\varepsilon \to 0$. Thus for $\varepsilon > 0$ sufficiently small, $y(x, \varepsilon) \geqq \delta$ on $[0, 1]$, so that $y(x, \varepsilon)$ is also the unique solution of (4.27), (4.28), (4.29) and the asymptotic formula of Theorem 4.1 applies.

Numerical results reported in [9] would lead one to suspect that the qualitative properties of $a_{00}(x)$ ($a_{00}(x)$ is increasing on $[0, 1]$ with $0 < a_{00}(x) \leqq 1$) are true also for $y(x, \varepsilon)$. This is indeed the case, as is easy to demonstrate with the maximum principle. For $\varepsilon > 0$ sufficiently small, we saw above that $y(0, \varepsilon) > 0$; since $y'(0, \varepsilon) = 0$,

$$\varepsilon\, y''(0) = a[y(0)]^N > 0.$$

Thus $y'(x, \varepsilon)$ and hence $y(x, \varepsilon)$ is increasing in some neighborhood of $x = 0$. I $y'(x, \varepsilon) > 0$ does not hold on $(0, 1]$, then there exists $x_0 \in (0, 1]$ such that $y'(x_0, \varepsilon) = 0$, but $y'(x, \varepsilon) > 0$ for $0 < x < x_0$. But then $u = y(x, \varepsilon)$ satisfies the equation

$$\varepsilon u'' + u' - a\, y^{N-1}(x, \varepsilon) \quad u = 0$$

with $a\, y^{N-1}(x, \varepsilon) > 0$ and $y(x, \varepsilon)$ has a positive endpoint maximum on $[0, x_0]$ at x_0. By the maximum principle [18, p. 7], $y'(x_0, \varepsilon) > 0$, a contradiction. Thus $y'(x, \varepsilon) > 0$ on $(0, 1]$ and $y(x, \varepsilon) \geqq y(0, \varepsilon) > 0$. Moreover, the function $z_1(x) \equiv 1$ satisfies

$$\varepsilon z_1'' + z_1' - \hat{g}(z_1) = -a < 0\,,$$

$$-z_1'(0) = 0\,,$$

$$z_1(1) + \varepsilon\, z_1'(1) = 1\,,$$

so by the maximum principle [18, p. 48], $z_1(x) \equiv 1$ is an upper bound for $y(x, \varepsilon)$ on $[0, 1]$.

We summarize our results in

Corollary 4.1. *Let $N \geqq 0$ and if $0 < N < 1$, assume that $a < (1 - N)^{-1}$. Then for $\varepsilon > 0$ sufficiently small, the boundary value problem* (4.27), (4.28), (4.29) *has a unique solution $y(x, \varepsilon)$ which satisfies $0 < y(x, \varepsilon) \leqq 1$ for $0 \leqq x \leqq 1$ and $y'(x, \varepsilon) > 0$ for $0 < x \leqq 1$. Moreover, $y(x, \varepsilon)$ is given by an asymptotic expansion of the form* (4.15).

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