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FINITE DIMENSIONAL COVERS OF METRIC-FINE SPACES

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This paper continues the study of metric-fine (M-fine) uniform spaces begun in $[\mathrm{Fr}]_{1-3}$, $[\mathrm{Ha}]_{1-3}$, and $[\mathrm{R}]_{1-3}$. Here we consider the finite dimensional uniform covers and uniform dimension of locally sub-M-fine, sub-M-fine, and M-fine spaces. The principal results are (i) sub-M-fine spaces are determined by their finite dimensional uniform covers, (ii) the uniform dimension functions Δd and δd coincide on sub-M-fine spaces, and (iii) none of the operators m_0, m_1, and m (corresponding to the locally sub-M-fine, sub-M-fine, and M-fine classes) increase the large uniform dimension of a uniform space.

uX will denote a separated uniform space, where u is a family of covers of X that satisfies the usual axioms for a uniformity. $C(uX)$ denotes the family of uniformly continuous real-valued functions. eu is the uniformity generated by the countable u-uniform covers and cu is the uniformity generated by $C(uX)$. A uniform space αX is *fine* if α is the largest uniformity on X which has the same uniform topology. A uniform space uX is *M-fine* (respectively *sub-M-fine*) if each uniformly continuous mapping to a metric space ϱM (respectively complete metric space) is uniformly continuous with respect to the fine uniformity α on M.

Given two uniformities u and v on X, define u/v to be the quasi-uniformity with the basis of covers of the form $\{V_s \cap U^s_t\}$, where $\{V_s\}$ belongs to v and $\{U^s_t\}$ belongs to u for each s. It is implicit in $[\mathrm{I}]_1$ that u/v is a uniformity if v has a basis of point finite uniform covers. If $u = v$, we write $u/v = u^{(1)}$ and define uX to be *locally fine* if $u^{(1)} = u$. In general, λu denotes the smallest locally fine uniformity containing u. The operators corresponding to the M-fine and sub-M-fine spaces are denoted respectively by m and m_1, where mu (resp. m_1u) is the smallest uniformity containing u that belongs to the respective class. In $[\mathrm{Fr}]_2$ and $[\mathrm{R}]_1$ it is shown that $mu = u/meu$. Finally, define the *locally sub-M-fine* operator m_0 by $m_0u = u/\lambda eu$. For separable uniform spaces (those with a basis of countable uniform covers), the locally sub-M-fine, sub-M-fine, and locally fine operators agree; in general each locally fine space is sub-M-fine ([GI]), each sub-M-fine space is locally sub-M-fine ($[\mathrm{R}]_2$), and the sub-M-fine spaces are precisely the subspaces of the M-fine spaces ($[\mathrm{R}]_1$).

A uniform cover of X is n-dimensional if each point of X belongs to at most $n + 1$ members of the cover. From [I], the finite dimensional, respectively point finite uniform covers relative uX form a basis for a uniformity denoted by fu, respectively $p_f u$. In this section we establish the degree to which finite dimensional covers determine the special uniformities previously defined.

Lemma 1.1. *For any three uniformities u, v and w on a set, $(u/v)/w = u/(v/w)$.*

The proof of 1.1 is routine and we shall omit it.

Lemma 1.2. *For each uniformity u on a set X, $p_f u \subset fu/eu$.*

From $[\mathrm{I}]_1$, if $\mathscr{U}$ is a point finite uniform cover, there exist uniform covers $\mathscr{U}_0$ and $\mathscr{W}$ with $\mathscr{U}_0 < \mathscr{U}$ and each member of $\mathscr{W}$ meeting only finitely many members of $\mathscr{U}_0$. Hence $\mathscr{U}_0$ is point finite and we may define $C_n = \{x : x \text{ belongs to at most } n \text{ members of } \mathscr{U}_0\}$, $n = 1, 2, \ldots$. Clearly $\mathscr{W} < \{C_n\}$, so $\{C_n\}$ is a countable uniform cover. Since $\mathscr{U}_0|_{C_n}$ is a finite dimensional cover of C_n there exists (from $[\mathrm{I}]_1$) a finite dimensional uniform cover $\mathscr{V}_n$ of X such that $\mathscr{V}_n|_{C_n} = \mathscr{U}_0|_{C_n}$; hence $\mathscr{U}_0 \in fu/eu$, which completes the proof.

Theorem 1.3. (i) *uX is locally sub-M-fine with a point finite basis if and only if $u = m_0(fu)$; hence if uX is sub-M-fine, $u = m_1(fu)$.*

(ii) *If uX is locally fine, then $u = (fu)^{(1)}/cu$.*

To establish (i), we first note that m_0 preserves the property of having a point finite basis ($[\mathrm{R}]_2$); hence it suffices to establish the equality $u = m_0(fu)$ from the assumptions on uX. $m_0(fu) \subset u$ is immediate since $m_0(fu)$ is the smallest locally sub-M-fine uniformity containing fu. For the reverse inclusion, first note that $u \subset$ $\subset fu/eu$ from 1.2. One easily shows that euX is locally fine; hence by 4.8 and 4.9 of [GI] $eu = (cu)^{(1)}$. Thus $fu/eu = fu/(cu)^{(1)} = (fu/cu)/cu$ from 1.1 and $(fu/cu)/cu \subset$ $\subset m_0(fu)/cu = m_0(fu)$ since $cu \subset efu$, so one obtains $u \subset m_0(fu)$. From $[\mathrm{R}]_2$ each sub-M-fine space is locally sub-M-fine and has a point finite basis, so the second part of (i) follows from the first. The proof of (ii) is similar to the proof of (i).

Theorem 1.3 may be summarized by saying that sub-M-fine spaces are determined by their finite dimensional uniform covers or, alternately, that the functor f is an isomorphism on the category of sub-M-fine spaces and uniform mappings. The latter formulation is analogous to the statement that the functor c is an isomorphism on the category of separable subfine spaces (from [CI] the separable locally fine spaces are precisely the separable subfine spaces and the equation $u = (cu)^{(1)}$ (4.9 of [GI]) shows that c is an isomorphism on such spaces; a different proof of this fact is found in $[\mathrm{Ha}]_4$, as well as a description of the image of c: the uX for which $u = cu$ and $C(uX)$ is closed under countable composition (if $uX \xrightarrow{f} \mathbf{R}^{\aleph_0}$ is a uniform mapping to a countable product of real lines and $\mathbf{R}^{\aleph_0} \xrightarrow{g} \mathbf{R}$ is continuous, then $g \circ f \in C(uX)$). The following characterization of the image of f is found in $[\mathrm{R}]_3$: the uX for which $u = fu$ and which (i) have the generalized composition property (replace $\mathbf{R}^{\aleph_0}$ by the class of complete metric spaces) and (ii) are two-dimensionally locally sub-M-fine (if $\{A_n\} \in eu$ is one-dimensional and $\{C_s^n\} \in u$ is two-dimensional, $n = 1, 2, \ldots$, then $\{A_n \cap C_s^n\} \in u$).

Finally, note that 1.3 is an optimal result in the sense that there exist fine spaces with no basis of finite dimensional covers. From 3.2 of [CI], a complete locally fine space has such a basis if and only if there exists a compact subset K such that each closed set disjoint from K has finite dimension; hence the fine uniformity on $\mathbf{R}^{\aleph_0}$ does not have such a basis.

Define $\Delta duX = n$ for a uniform space uX if n is the least integer for which uX has a basis of n-dimensional uniform covers; if no such integer exists define $\Delta duX = \infty$. Define δduX analogously with respect to the finite uniform covers; then one easily shows that $\delta duX \leqq \Delta duX$.

Theorem 2.1. *Δd and δd coincide on locally sub-M-fine spaces with a point finite basis.*

From $[I]_3$, the dimension functions agree on spaces with a basis of finite dimensional covers; hence for any uX, $\Delta defuX = \delta defuX = \delta duX$. If uX is locally sub-M-fine with a point finite basis, then $u = m_0(fu) = fu/\lambda efu$ by 1.3 (i). Suppose $\delta duX = n$; then from $[\mathrm{I}]_1$ $\Delta d\lambda efuX \leqq \Delta defuX = n$, while $\Delta dfuX = n$ by the result cited from $[\mathrm{I}]_3$, so one easily shows that $\Delta duX \leqq (n + 1)^2 - 1$; once again using the result from $[\mathrm{I}]_3$ we conclude that $\Delta duX = n$, which completes the proof.

Theorem 2.2. *For each uniform space uX, $\Delta dmuX \leqq \Delta dm_1uX \leqq \Delta dm_0uX \leqq \leqq \Delta duX$.*

We first establish that $\Delta dm_0uX \leqq \Delta duX$. Suppose $\Delta duX < \infty$; then uX has a point finite basis, so from $[\mathrm{R}]_2$ m_0uX has a point finite basis and hence $\Delta dm_0uX = = \delta dm_0uX$ by 2.1. Now $\delta dm_0uX = \delta dem_0uX = \delta dm_0euX$ (by $[\mathrm{R}]_3$ the operators m_0 and e commute) $= \delta d\lambda euX$. By the result from $[\mathrm{I}]_1$ cited in 2.1, $\delta d\lambda euX \leqq \leqq \Delta deuX$, and one easily establishes $\Delta deuX \leqq \Delta duX$, so one obtains $\Delta dm_0uX \leqq \leqq \Delta duX$.

To establish $\Delta dm_1uX \leqq \Delta duX$, suppose that $\Delta duX = n$. From $[\mathrm{R}]_1$, if $\mathscr{U} \in m_1u$ there exists a uniform mapping $uX \xrightarrow{f} \varrho M$ to a complete metric space and open cover $\mathscr{O}$ of M such that $f^{-1}(\mathscr{O}) < \mathscr{U}$. There exists $\mathscr{V}_k \in u$, $k = 1, 2, \ldots$, each n-dimensional, such that $\mathscr{V}_k < f^{-1}\mathscr{S}_\varrho(1/k)$ and $\mathscr{V}_{k+1*} < \mathscr{V}_k$. The family $\{\mathscr{V}_k\}$ generates a u-uniform pseudometric ϱ_0 with $\Delta d\varrho_0X \leqq n$. Let $\bar{\varrho}_0\bar{X}$ be the metric space of equivalence classes determined by ϱ_0; then $\Delta d\bar{\varrho}_0\bar{X} \leqq n$ and f may be factored as $g \circ h$, where $uX \xrightarrow{h} \bar{\varrho}_0\bar{X}$ and $\bar{\varrho}_0\bar{X} \xrightarrow{g} \varrho M$ are uniform mappings. Let $\pi\bar{\varrho}_0\bar{X} \xrightarrow{g'} \varrho M$ be the extension of g to the completion $\pi\bar{\varrho}_0\bar{X}$. Then $m_1uX \xrightarrow{h} \alpha\pi\bar{\varrho}_0\bar{X}$ is a uniform mapping and $\Delta d\alpha\pi\bar{\varrho}_0\bar{X} \leqq n$ since from $[\mathrm{I}]_1$ the dimension of the fine uniformity on a metric space does not exceed the uniform dimension of any compatible metric uniformity and π does not alter uniform dimension. Consequently, there exists a n-dimensional open cover $\mathscr{V}$ that refines $g'^{-1}(\mathscr{O})$, so $h^{-1}(\mathscr{V})$ is an n-dimensional member of m_1u that refines $\mathscr{U}$. Therefore $\Delta dm_1uX \leqq n$.

A proof analogous to the one given above shows that $\Delta dmuX \leqq \Delta duX$, where ϱM is not necessarily complete and the completion need not be used. Finally, since $mm_1 = m$ and $m_1 m_0 = m_1$, the preceding work establishes that $\Delta dmuX \leqq \leqq \Delta dm_1 uX \leqq \Delta dm_0 uX$.

Corollary 2.3. *For each uniform space uX, $\Delta dmuX \leqq \delta duX$.*

From 2.1 $\Delta dmuX = \delta dmuX = \Delta dpmuX$. One easily shows that $\Delta dpmuX \leqq \leqq \Delta demuX$ and $\Delta demuX = \Delta dmeuX = \Delta dmpuX$ since the operators m and e commute ([R]$_1$) and $mp = me$ ([H]$_1$). Finally, $\Delta dmpuX \leqq \Delta dpuX = \delta duX$ by 2.2, which establishes the result.

The covering dimension of a Tychonoff space X is defined by $\dim X = \delta d\alpha X$ ($= \Delta d\alpha X$ by 1.4), where α is the fine uniformity with respect to the given topology. One may establish the following result using 2.3. Assume that the uniform and topological cozero sets of uX coincide. Then $\dim X \leqq \delta duX$. In particular, this hypothesis is satisfied if the uniform topology is Lindelöf; in fact for a Lindelöf space X it is known that $\dim X = \min \delta duX$, taken over all compatible uniformities u.

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