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THE IDEAL STRUCTURE OF C-SEMIGROUPS

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Let S be a semigroup with zero 0. We say that $e \neq 0$ is *a categorical left unit of* S if ex is either x or 0 for any $x \in S$. An element $f \neq 0$ is *a categorical right unit of* S if xf is either x or 0 for every $x \in S$.

A semigroup is called *categorical at zero* if $abc = 0$ implies either $ab = 0$ or $bc = 0$.

In the following we always suppose card $S \geqq 2$.

Definition. A semigroup with 0 is called *a C-semigroup* if it satisfies the following conditions:

1. Every non-zero $a \in S$ has a categorical right unit $e_r(a)$ and a categorical left unit $e_l(a)$ such that $e_l(a) \,.\, a = a \,.\, e_r(a) = a$.

2. S is categorical at zero.

The following lemma is easy to prove. (See [1], Vol 2, pp. 78–79.)

Lemma 0,1. *In any C-semigroup*

a) *$e_r(a)$ and $e_l(a)$ are uniquely determined;*

b) *we have $a \in Sa \cap aS$, in particular $S^2 = S$;*

c) *any categorical left unit of S is a categorical right unit of S.*

With respect to Lemma 0,1 we may speak about the set of all categorical units (cat. idempotents). This set will be denoted by E. It is a subset of the set of all non-zero idempotents E_0. Simple examples show that E can be a proper subset of E_0.

C-semigroups (under another name) have been extensively studied by HOEHNKE ([2]–[5]). In [1] the name "small category with zero" is suggested. ŠUTOV [13] and KOŽEVNIKOV [6] call our C-semigroups "categorical semigroups". The name "categorical semigroups" used by McMORRIS and SATYANARAYANA [9] and MONZO [10] has another meaning. To avoid misunderstanding I use the word C-semigroup.

The present paper deals with problems of another kind than those treated in the papers mentioned above. There are of course some connections, in particular with a part of [6]. One of our aims is to study the "position" of cat. units in a C-semigroup. It will turn out that for some classes of semigroups this "position" can be more or less satisfactorily described. 0-simple C-semigroups are studied in a greater detail.

1. PRELIMINARIES

Let e_1, e_2 be two elements $\in E$. Since e_2 is a cat. right unit, the element e_1e_2 is either e_1 or 0. Further, since e_1 is a cat. left unit, e_1e_2 is either e_2 or 0. Hence if $e_1e_2 \neq 0$, then $e_1 = e_2$. We have

Lemma 1,1. *If $e_1, e_2 \in E$ and $e_1 \neq e_2$, then $e_1e_2 = 0$.*

Any $a \in S$, $a \neq 0$, can be written in the form $a = ae_1$, $e_1 = e_r(a) \in E$, hence $a \in Se_1$. For $e_1 \neq e_2$, $e_1, e_2 \in E$, we have $Se_1 \cap Se_2 = 0$. Indeed, $x \in Se_1 \cap Se_2$ implies $x = b_1e_1 = b_2e_2$ with $b_1, b_2 \in S$. Multiplying by e_1 we have $x = xe_1 = b_2e_2e_1 = 0$. We have proved

Lemma 1,2. *Any C-semigroup can be written in the form of a union of left (right) ideals:*

$$S = \bigcup_{e_\alpha \in E} Se_\alpha = \bigcup_{e_\alpha \in E} e_\alpha S,$$

where $Se_\alpha \cap Se_\beta = e_\alpha S \cap e_\beta S = 0$ for $e_\alpha \neq e_\beta$.

Note that Se_α contains a unique element $\in E$, namely e_α itself.

Lemma 1,3. *In a C-semigroup any non-zero nilpotent element a satisfies $a^2 = 0$.*

Proof. Suppose that $a^n = 0$ and $n > 2$. Then $aa^{n-2} \, . \, a = 0$ implies $a^{n-2} \, . \, a = 0$, i.e. $a^{n-1} = 0$. Repeating this argument we obtain $a^2 = 0$.

We now deal with some trivial cases.

First note that if S is a C-semigroup and has a (two-sided) identity element e, then e is the only cat. unit of S. Indeed, if $e_\alpha \neq 0$ is a cat. unit of S, then $ee_\alpha = e_\alpha \neq 0$ (since e is the identity element of S) and ee_α is either e or 0 (since e_α is a cat. unit). Hence $e_\alpha = e$.

Definition. We shall say that in a semigroup S with zero 0 the zero element is *externally adjoined* if $S - \{0\}$ is a semigroup.

Lemma 1,4. *A C-semigroup with a unique cat. unit is a semigroup having an identity element and its zero is externally adjoined.*

Proof. If e is the cat. unit, then for any $a \in S$ we have $e_r(a) = e_l(a) = e$, hence e is the identity element of S. Suppose that $ab = aeb = 0$. Since S is categorical at zero, it follows that either $ae = 0$ or $eb = 0$, i.e. either $a = 0$ or $b = 0$. Hence $S - \{0\}$ is a semigroup.

Conversely:

Lemma 1,5. *Any C-semigroup in which* 0 *is externally adjoined has an identity element* (and card $E = 1$).

Proof. By Lemma 1,1 the semigroup $S - \{0\}$ cannot have two different cat. units. Hence it contains exactly one cat. unit. The rest of the proof follows by Lemma 1,4.

Corollary 1. *Any semigroup with an identity element and without zero can be embedded in a C-semigroup by adjoining externally a zero element.*

More generally:

Corollary 2. *Any semigroup without zero can be embedded in a C-semigroup.*

Proof. If S has no identity element, adjoin an identity element 1 and denote the new semigroup by S^1. Next adjoin a zero element 0. The semigroup $S^1 \cup \{0\}$ is a C-semigroup.

We next treat the commutative case.

If S is a commutative C-semigroup, the decomposition of Lemma 1,2 implies that every Se_α is a semigroup with the identity element e_α. It is categorical at zero since S is categorical at zero. By Lemma 1,4 Se_α is a semigroup with an identity element and its zero is externally adjoined. Further, $Se_\alpha \,.\, Se_\beta = Se_\alpha e_\beta S = 0$.

In accordance with [1] we shall say that a semigroup is a 0-*direct union* of subsemigroups S_α, $\alpha \in \Lambda$, if $S = \bigcup_{\alpha \in \Lambda} S_\alpha$ and $S_\alpha S_\beta = S_\alpha \cap S_\beta = 0$ for $\alpha \neq \beta$.

Hence a commutative C-semigroup is a 0-direct union of semigroups having identity elements and, moreover, its zero is externally adjoined.

It can be easily verified that the construction of all such semigroups is described by the following

Theorem 1,1. *Let S_α, $\alpha \in \Lambda$ be a collection of disjoint commutative semigroups each of which has an identity element. Adjoin a zero element* 0 *and define $S_\alpha \,.\, S_\beta = 0$ for $\alpha \neq \beta$ and $0 \,.\, S_\alpha = S_\alpha \,.\, 0 = 0 \,.\, 0 = 0$. Then the* 0-*direct union $S = \{0\} \cup [\bigcup_{\alpha \in \Lambda} S_\alpha]$ is a commutative C-semigroup and any commutative C-semigroup can be obtained in this manner.*

We now return to the general case.

Lemma 1,6. *Let S be a C-semigroup and $e_\alpha, e_\beta, e_\gamma, e_\delta \in E$. Then*

a) $e_\alpha S \cap S e_\beta = e_\alpha S e_\beta$;

b) *two non-zero sets $e_\alpha S e_\beta$ and $e_\gamma S e_\delta$ have a non-zero element in common iff $e_\alpha = e_\gamma$ and $e_\beta = e_\delta$.*

Proof. a) First, we clearly have $e_\alpha S e_\beta \subset S e_\beta \cap e_\alpha S$. Next if $x \in S e_\beta \cap e_\alpha S$ and $x \neq 0$, then $x = x e_\beta$, $x = e_\alpha x$, hence $x = e_\alpha x = e_\alpha x e_\beta \in e_\sigma S e_\beta$. This proves part a).

b) Let $x \in e_\alpha S e_\beta \cap e_\gamma S e_\delta$ and let both the sets $\neq 0$. Then $x = e_\alpha x e_\beta$ and $x = e_\gamma x e_\delta$. Hence $x = e_\alpha e_\gamma x e_\delta e_\beta$. If $e_\alpha \neq e_\gamma$ or $e_\delta \neq e_\beta$, then $x = 0$. Hence both the sets may have a non-zero element in common iff $e_\alpha = e_\gamma$ and $e_\delta = e_\beta$. This proves the second assertion.

Now (with respect to Lemma 1,2) we may write

$$S = S^2 = [\bigcup_{e_\alpha \in E} e_\alpha S] \cdot [\bigcup_{e_\beta \in E} S e_\beta] = \bigcup_{e_\alpha, e_\beta \in E} e_\alpha S e_\beta .$$

Definition. Two subsets $A \subset S$ and $B \subset S$ will be called *quasidisjoint* if $A \cap B = 0$. In this terminology we have

Lemma 1,7. *Any C-semigroup can be written as a union of quasidisjoint sets: $S = \bigcup_{e_\alpha, e_\beta \in E} e_\alpha S e_\beta$.*

Example 1,1 below shows that some of the sets $e_\alpha S e_\beta$, $e_\alpha \neq e_\beta$, may reduce to zero. Further, since $(e_\alpha S e_\beta)^2 = (e_\alpha S e_\beta)(e_\alpha S e_\beta) = 0$, all idempotents $\in S$ (even those which are not cat. units) are contained in the sets $e_\alpha S e_\alpha$ and each of these sets is a non-zero subsemigroup of S.

In the following we denote $T_{\alpha\beta} = e_\alpha S e_\beta$ while $\Lambda = \{\alpha, \beta, \ldots\}$ will denote the index set of all cat. units.

Lemma 1,8. *Suppose that $T_{\alpha\beta} \neq 0$ and $T_{\beta\delta} \neq 0$. Then for any $u \in T_{\alpha\beta} - \{0\}$, $v \in T_{\beta\delta} - \{0\}$ we have $uv \neq 0$.*

Proof. Since $u \in T_{\alpha\beta}$, we have $u = u e_\beta$ and analogously $v = e_\beta v$. Now $uv = u e_\beta v = 0$ would imply either $u e_\beta = 0$ or $e_\beta v = 0$, contrary to the assumption.

Summarizing: If $T_{\alpha\beta} \neq 0$ and $T_{\gamma\delta} \neq 0$, then

$$T_{\alpha\beta} T_{\gamma\delta} = \begin{cases} 0 & \text{if } \beta \neq \gamma , \\ \neq 0 & \text{if } \beta = \gamma . \end{cases}$$

In this latter case we have $T_{\alpha\beta} T_{\beta\delta} \subset T_{\alpha\delta}$.

Introduce the set $\mathfrak{S} = \{\Lambda \times \Lambda\} \cup \{z\}$. For $\alpha, \beta, \gamma, \delta \in \Lambda$ define

$$(\alpha, \beta)(\gamma, \delta) = \begin{cases} z & \text{if } \beta \neq \gamma , \\ (\alpha, \delta) & \text{if } \beta = \gamma , \end{cases}$$

and $z . (\alpha, \beta) = (\alpha, \beta) . z = z . z = z$. It is well known that $\mathfrak{S}$ is a completely 0-simple semigroup (called the semigroup of $\Lambda \times \Lambda$ matrix units).

Let now S be a C-semigroup: $S = \bigcup_{\alpha\in\Lambda,\beta\in\Lambda} \bigcup T_{\alpha\beta}$. Consider the mapping φ of S into $\mathfrak{S}$ defined as follows:

$$\varphi(0) = z\,, \quad \varphi(T_{\alpha\beta} - \{0\}) = (\alpha, \beta) \quad \text{if} \quad T_{\alpha\beta} \neq 0\,.$$

Then φ is a homomorphism of S into $\mathfrak{S}$. Indeed: a) If $T_{\alpha\beta} \neq 0$, $T_{\beta\delta} \neq 0$ and $x \in T_{\alpha\beta} - \{0\}$, $y \in T_{\beta\delta} - \{0\}$, we have $xy \in T_{\alpha\delta} - \{0\}$, hence $\varphi(xy) = (\alpha, \delta) = (\alpha, \beta) \,.\, (\beta, \delta) = \varphi(x) \,.\, \varphi(y)$. b) If $T_{\alpha\beta} \neq 0$, $T_{\gamma\delta} \neq 0$, $\beta \neq \gamma$ and $x \in T_{\alpha\beta} - \{0\}$, $y \in T_{\gamma\delta} - \{0\}$, then $xy = 0$, hence $\varphi(xy) = z = (\alpha, \beta)(\gamma, \delta) = \varphi(x) \,.\, \varphi(y)$. c) If $x \in T_{\alpha\beta}$ and $y = 0$, then $\varphi(xy) = \varphi(0) = z = \varphi(x) \,.\, z = \varphi(x) \,.\, \varphi(y)$.

We have

Theorem 1,2. *Any C-semigroup possesses a homomorphic mapping into the completely 0-simple semigroup of $\Lambda \times \Lambda$ matrix units.*

Remark. In the case of a 0-simple C-semigroup we shall obtain a stronger result.

Example 1,1. To show that some of the sets $e_\alpha S e_\beta$ may be zero consider the following example. Let S be a set consisting of an element z and all ordered pairs (i, j), where i, j are integers such that $i \geqq j$. Define a product in S by the rules

$$(i, j)(r, s) = \begin{cases} (i, s) & \text{if} \quad j = r\,, \\ z & \text{if} \quad j \neq r\,, \end{cases}$$

and $zx = z = xz$ for all $x \in S$. S is a C-semigroup. The set of all cat. units is $E = \{(i, i) \mid -\infty < i < \infty\}$. The left ideal generated by (i, j) is the "horizontal half-line" $\{z\} \cup \{(r, j) \mid r \geqq i\}$, the right ideal generated by (i, j) is the "vertical half-line" $\{z\} \cup \{(i, s) \mid s \leqq i\}$. The two sided ideal generated by (i, j) is the "rectangle" $\{z\} \cup \{(r, s) \mid r \leqq i,\, s \geqq j\}$. In this case we have

$$(i, i)\, S(j, j) = \begin{cases} z & \text{if} \quad i < j\,, \\ (i, j) & \text{if} \quad i \geqq j\,. \end{cases}$$

2. 0-SIMPLE C-SEMIGROUPS

In the following 0-simple C-semigroups will play an important role. Therefore we treat them first.

Lemma 2,1. *If S is a 0-simple C-semigroup, then for any $e_\alpha, e_\beta \in E$ we have $e_\alpha S e_\beta \neq 0$.*

Proof. $S(e_\alpha S e_\beta) = (S e_\alpha S)\, e_\beta = S e_\beta$. If $e_\alpha S e_\beta$ were 0, we would have $S(e_\alpha S e_\beta) = S \,.\, 0 = 0$, hence $S e_\beta = 0$, a contradiction to $e_\beta \in S e_\beta$.

Corollary. *If S is a 0-simple C-semigroup, then to any couple e_α, e_β there is an $a \in S$, $a \neq 0$ such that $a = e_\alpha a e_\beta$.*

Lemma 2,2. *If S is a 0-simple C-semigroup, then*

$$T_{\alpha\beta}T_{\gamma\delta} = \begin{cases} 0 & \text{if } \beta \neq \gamma, \\ T_{\alpha\delta} & \text{if } \beta = \gamma. \end{cases}$$

Proof. $T_{\alpha\beta}T_{\beta\delta} = e_\alpha S e_\beta e_\beta S e_\delta = e_\alpha(Se_\beta S)\, e_\delta = e_\alpha S e_\delta = T_{\alpha\delta}$.
By the same argument as in the proof of Theorem 1,2 we deduce

Theorem 2,1. *Any 0-simple C-semigroup possesses a homomorphic mapping onto the completely 0-simple semigroup of $\Lambda \times \Lambda$ matrix units.*

Consider now the subsemigroup $T_{\alpha\alpha} = e_\alpha S e_\alpha$. This semigroup contains a unique cat. unit, namely e_α which is the identity element of $T_{\alpha\alpha}$. For any $x \in T_{\alpha\alpha}$, $x \neq 0$ we have $e_\alpha x e_\alpha = x$. Now

$$T_{\alpha\alpha} x T_{\alpha\alpha} = e_\alpha S e_\alpha \,.\, x \,.\, e_\alpha S e_\alpha = e_\alpha(SxS)\, e_\alpha = T_{\alpha\alpha}\,.$$

Hence $T_{\alpha\alpha}$ is a 0-simple semigroup. $T_{\alpha\alpha}$ is categorical at zero since S is categorical at zero. Hence $T_{\alpha\alpha}$ is a C-semigroup. By Lemma 1,4 the zero 0 is externally adjoined.

We have proved

Theorem 2,2. *If S is a 0-simple C-semigroup, then each of the subsemigroups $e_\alpha S e_\alpha$ is a 0-simple semigroup containing an identity element and the zero 0 is externally adjoined.*

This theorem suggests a method how to construct 0-simple C-semigroups.

Construction. Let $T = \{1, t, u, \ldots\}$ be a simple semigroup with the identity element 1 and without zero. Let $\Lambda = \{\alpha, \beta, \ldots\}$ be a set of symbols. Consider the set S consisting of $\{0\}$ and all triples (t, α, β), where $t \in T$, $\alpha, \beta \in \Lambda$. Define in S a multiplication by the rules

$$(t, \alpha, \beta)(u, \gamma, \delta) = \begin{cases} 0 & \text{if } \beta \neq \gamma, \\ (tu, \alpha, \delta) & \text{if } \beta = \gamma, \end{cases}$$

and $0 \,.\, (t, \alpha, \beta) = (t, \alpha, \beta) \,.\, 0 = 0 \,.\, 0 = 0$. Then S is a C-semigroup. It is clearly categorical at zero. The cat. right and cat. left units of (t, α, β) are $(1, \beta, \beta)$ and $(1, \alpha, \alpha)$ respectively. Finally, it is easy to see that S is 0-simple since for any triple (t, α, β) we have $S \,.\, (t, \alpha, \beta) \,.\, S = S$.

Remark. We emphasize that we do not assert to obtain in this way all 0-simple C-semigroups. In our construction the subsemigroups $e_\alpha S e_\alpha$ of Theorem 2,2 are of the form $T_{\alpha\alpha} = \bigcup_{t \in T} \{(t, \alpha, \alpha)\} \cup \{0\}$ and all are isomorphic semigroups. At this moment I am unable to prove or disprove whether the subsemigroups $e_\alpha S e_\alpha$ in Theorem 2,2 are necessarily isomorphic or not. We shall return to this problem in Theorem 2.6, In the proofs of the following theorems we shall use the following well known statement ([1], Theorem 2,54): If S is a 0-simple semigroup which is not completely 0-simple, and S contains an idempotent, then S contains an infinite number of idempotents.

Theorem 2,3. *A* 0-*simple C-semigroup S is completely* 0-*simple iff all non-zero idempotents* $\in S$ *are cat. units.*

Proof. a) Suppose that S is 0-simple but not completely 0-simple and all non-zero idempotents are cat. units. Then S contains non-primitive idempotents, i.e. there is a couple of non-zero idempotents $e_1, e_2, e_1 \neq e_2$ such that $e_1 e_2 = e_2 e_1 = e_1 \neq 0$. The idempotent e_1 is not a cat. unit since otherwise $e_2 e_1$ would be either e_2 or 0. The existence of an idempotent which is not a cat. unit constitutes a contradiction to the supposition.

b) Suppose conversely that the C-semigroup S is completely 0-simple. Let $e \neq 0$ be any idempotent $\in S$. Since S is a C-semigroup, there is a cat. unit $e_r \in S$ such that $ee_r = e$. This implies $ee_re = e$, hence $e_re \neq 0$. Therefore $e_re = e$. Now since S is completely 0-simple, $ee_r = e_re = e$ implies $e_r = e$. Any non-zero idempotent $\in S$ is a cat. unit. This proves our theorem.

Note that we have not used the assumption that S is categorical at zero. Indeed we have proved the following somewhat stronger result:

Theorem 2,3a. *Let S be a* 0-*simple semigroup in which to every* $a \in S$ *there is a cat. left unit* $e_l(a)$ *and a cat. right unit* $e_r(a)$ *such that* $e_l(a) \,.\, a = a \,.\, e_r(a) = a$. *Then S is a completely* 0-*simple semigroup iff each non-zero idempotent* $\in S$ *is a cat. unit.*

The following lemma is known and has been proved in [1] (Lemma 8,23, p. 98).

Lemma 2,3. *Let S be a* 0-*simple semigroup containing a* 0-*minimal left ideal (in particular, a completely* 0-*simple semigroup). Then S is categorical at zero.*

Hence we may state part of our results in the following form which will be needed later.

Theorem 2,4. *A completely* 0-*simple semigroup S in which to every a there are cat. units* $e_l(a), e_r(a)$ *such that* $a = a \,.\, e_r(a) = e_l(a) \,.\, a$ *is a C-semigroup. In this case all non-zero idempotents* $\in S$ *are cat. units.*

Remark. It should be emphasized that we cannot prove that a 0-simple semigroup (which is not completely 0-simple) satisfying the conditions of Theorem 2,4 is a C-semigroup. In particular: A simple semigroup with zero and an identity element need not be categorical at zero. An example of such a semigroup has been given by Munn [11], p. 156. This example will be reproduced below (see Example 6,1).

In addition to our theorems we prove

Theorem 2,4a. *A* 0*-simple semigroup containing non-zero idempotents in which all non-zero idempotents are cat. units is a completely* 0*-simple C-semigroup.*

Proof. Suppose that S is not completely 0-simple. Then there exists a couple of non-zero idempotents $e_\alpha \neq e_\beta$ such that $e_\alpha e_\beta = e_\beta e_\alpha = e_\beta \neq 0$. Since e_β is a cat. unit and $e_\alpha e_\beta \neq 0$, we have $e_\alpha e_\beta = e_\alpha$. But then $e_\alpha = e_\beta$, a contradiction. Now S being completely 0-simple it can be written in the form of unions of left (right) 0-minimal ideals: $S = \bigcup_{e_\alpha \in E} Se_\alpha = \bigcup_{e_\alpha \in E} e_\alpha S$, so that to any $a \in S$ there are cat. units e_γ, e_δ with $a = ae_\gamma = e_\delta a$. The rest of the proof follows by Lemma 2,3.

Now a completely 0-simple C-semigroup is known to be a 0-simple inverse semigroup (i.e. a Brandt semigroup). It can be characterized also as a simple dual semigroup [12]. All such semigroups can be obtained if in the construction discussed above, T is taken a group. In this case it is of course well known that all the $e_\alpha Se_\alpha$ are isomorphic.

We have

Theorem 2,5. *Any completely* 0*-simple C-semigroup is isomorphic to a semigroup obtained by the construction described above when taking a suitably chosen group for T and a set with a suitably chosen cardinal number for* Λ.

We now return to Theorem 2,2. We have seen that any one of the subsets $e_\alpha Se_\alpha - \{0\}$ is a simple semigroup with a unit element. We also remarked that at present we are unable to prove whether all $e_\alpha Se_\alpha$, $\alpha \in \Lambda$, are isomorphic to each other. We prove a weaker statement.

Theorem 2,6. *In a* 0*-simple C-semigroup any semigroup* $e_\alpha Se_\alpha$ *can be isomorphically mapped into any other* $e_\beta Se_\beta$.

Remark. This theorem is formulated for C-semigroups and cat. units. An analogous statement holds mutatis mutandis for any 0-simple semigroup and non-necessarily cat. idempotents.

Proof. Since $e_\alpha \in e_\alpha Se_\alpha = e_\alpha Se_\beta \,.\, e_\beta Se_\alpha$, there are two elements $v \in e_\alpha Se_\beta$, $u \in e_\beta Se_\alpha$ such that $e_\alpha = v \,.\, u$. The element $e' = uv = ue_\alpha v \in e_\beta Se_\beta$ is an idempotent since $e'^2 = u(vu)\, v = ue_\alpha v = uv = e'$.

Consider now the mapping $\varphi : e_\alpha Se_\alpha \to e_\beta Se_\beta$ defined by $x \mapsto uxv$.

a) This is a homomorphic mapping of $e_\alpha Se_\alpha$ into $e_\beta Se_\beta$. Indeed, if $y_1 = ux_1v$, $y_2 = ux_2v (x_1, x_2 \in e_\alpha Se_\alpha)$, then

$$\varphi(x_1)\,\varphi(x_2) = y_1y_2 = ux_1(vu)\,x_2v = ux_1e_\alpha x_2v = ux_1x_2v = \varphi(x_1x_2)\,.$$

b) If $x_1 \neq x_2$, then $\varphi(x_1) \neq \varphi(x_2)$. Indeed, suppose $ux_1v = ux_2v$. Multiply by v from the left and by u from the right. We have $vux_1vu = vux_2vu$, i.e. $e_\alpha x_1 e_\alpha = e_\alpha x_2 e_\alpha$, hence $x_1 = x_2$.

c) Note that $\varphi(e_\alpha) = ue_\alpha v = uv = e' \in e_\beta Se_\beta$.

d) If $x \in e_\alpha S e_\alpha$, then $y = \varphi(x) = uxv$ has e' for an identity element. Indeed, we have $ye' = ux(vue_\alpha v) = uxe_\alpha v = uxv = y$, and analogously $e'y = y$.

e) Finally, we show that φ carries $e_\alpha S e_\alpha$ onto the semigroup $e'Se'$. To this end it is sufficient to show that to any $y \in e'Se'$ there is an $x_1 \in e_\alpha S e_\alpha$ such that $\varphi(x_1) = y$. [By b), x_1 is uniquely determined.] Consider the element $x_1 = vyu$. Then $\varphi(x_1) = u(vyu)v = e'ye' = y$. Hence $e_\alpha S e_\alpha$ is isomorphically mapped onto $e'Se' \subset e_\beta S e_\beta$. This proves Theorem 2,6.

Remark. It follows immediately from our proof that if u, v can be chosen so that $vu = e_\alpha$ and $uv = e_\beta$, then $e_\alpha S e_\alpha$, $e_\beta S e_\beta$ are isomorphic semigroups.

This is certainly the case if e.g. $e_\beta S e_\beta$ contains a unique non-zero idempotent. Then uv is necessarily equal to e_β. In this case it is well known that $e_\beta S e_\beta - \{0\}$ is a group. Since any $e_\alpha S e_\alpha$ can be isomorphically mapped onto $e_\beta S e_\beta$ it is clear that all $e_\alpha S e_\alpha - \{0\}$, $\alpha \in \Lambda$ are groups and all are isomorphic to one another. In this case all idempotents $\in S$ are cat. units and S is completely 0-simple. This is the case treated in Theorems 2,3 and 2,5.

We now proceed to a more general situation.

Let $\alpha \in \Lambda$ be chosen fixed. Since S is 0-simple, there exist to any cat. unit $e_\mu \in S = Se_\alpha S$ two elements $u_\mu \in Se_\alpha$, $v_\mu \in e_\alpha S$ such that $e_\mu = u_\mu e_\alpha v_\mu = u_\mu v_\mu$. Suppose that it is possible to choose for any $\mu \in \Lambda$ the elements u_μ, v_μ such that the idempotents $v_\mu u_\mu$ are equal to e_α. (Note that all idempotents $v_\mu u_\mu$ are necessarily contained in $e_\alpha S e_\alpha$.) For convenience, define $v_\alpha = u_\alpha = e_\alpha$.

Denote $T = e_\alpha S e_\alpha$. Consider the mapping $\varphi_{\lambda\varrho} : T \to u_\lambda T v_\varrho$ defined by $x \mapsto u_\lambda x v_\varrho$, $x \in T$.

$\varphi_{\lambda\varrho}$ is onto. Indeed, let $y \neq 0$ be any element $\in u_\lambda T v_\varrho$. Then for the element $x = e_\alpha v_\lambda y u_\varrho e_\alpha \in T$ we have $\varphi_{\lambda\varrho}(x) = u_\lambda x v_\varrho = (u_\lambda e_\alpha v_\lambda)\, y (u_\varrho e_\alpha v_\varrho) = e_\lambda y e_\varrho = y$. If $y \in u_\lambda T v_\varrho$, $y \neq 0$ is given, there is a unique $x \in T$ such that $\varphi_{\lambda\varrho}(x) = y$. Indeed, $u_\lambda x_1 v_\varrho = u_\lambda x_2 v_\varrho (x_1, x_2 \in T)$ implies $(v_\lambda u_\lambda)\, x_1 (v_\varrho u_\varrho) = (v_\lambda u_\lambda)\, x_2 (v_\varrho u_\varrho)$, i.e. $e_\alpha x_1 e_\alpha = e_\alpha x_2 e_\alpha$ and $x_1 = x_2$. Hence $\varphi_{\lambda\varrho}$ is a one-to-one mapping of T onto $u_\lambda T v_\varrho$.

We next prove that $u_\lambda T v_\varrho = e_\lambda S e_\varrho$. Firstly, we have $u_\lambda T v_\varrho = e_\lambda u_\lambda T v_\varrho e_\varrho \subset e_\lambda S e_\varrho$. Secondly, $e_\lambda S e_\varrho = u_\lambda v_\lambda S u_\varrho v_\varrho = u_\lambda e_\alpha v_\lambda S u_\varrho e_\alpha v_\varrho \subset u_\lambda (e_\alpha S e_\alpha)\, v_\varrho = u_\lambda T v_\varrho$. Hence $u_\lambda T v_\varrho = e_\lambda S e_\varrho$.

The semigroup S can be written as a union of quasidisjoint semigroups $S = \bigcup_{\lambda,\varrho \in \Lambda} u_\lambda T v_\varrho$. The importance of this representation of S is due to the fact that to any $z \in S$, $z \neq 0$, there is a unique couple λ, ϱ and a unique $x \in T$ such that $z = u_\lambda x v_\varrho$.

Consider now the semigroup S_1 consisting of a zero 0_1 and all triples $(u_\lambda, x, v_\varrho)$, $\lambda, \varrho \in \Lambda$, $x \in T_0 = T - \{0\}$, with the multiplication defined by

$$(u_\lambda, x, v_\varrho)(u_\varkappa, y, v_\mu) = \begin{cases} 0_1 & \text{if } \varrho \neq \varkappa, \\ (u_\lambda, xy, v_\mu) & \text{if } \varrho = \varkappa, \end{cases}$$

and 0_1 having the usual properties of a zero element.

Consider the mapping $\psi : S \to S_1$, where

$$\psi(z) = \begin{cases} 0_1 & \text{for } z = 0\,, \\ (u_\lambda, x, v_\varrho) & \text{for } z = u_\lambda x v_\varrho \neq 0\,. \end{cases}$$

This is a one-to-one mapping of S into S_1. It is onto, since for any $(u_\lambda, y, v_\varrho)$, $y \in T_0$ there is an element $z \in S$ (namely $z = u_\lambda y v_\varrho$) such that $\psi(z) = (u_\lambda, y, v_\varrho)$.

We show that ψ is an isomorphism. Let $z_1, z_2 \in S$ and $z_1 \in u_\lambda T v_\varrho$, $z_2 \in u_\varkappa T v_\mu$, $z_1 \neq 0$, $z_2 \neq 0$. Write $z_1 = u_\lambda x_1 v_\varrho$, $z_2 = u_\varkappa x_2 v_\mu$, where $x_1, x_2 \in T_0$ are uniquely determined. We have

$$z_1 z_2 = (u_\lambda x_1 v_\varrho e_\varrho)(e_\varkappa u_\varkappa x_2 v_\mu) = \begin{cases} 0 & \text{if } \varrho \neq \varkappa\,, \\ u_\varkappa x_1 (v_\varrho u_\varrho) x_2 v_\mu = u_\lambda x_1 x_2 v_\mu & \text{if } \varrho = \varkappa\,. \end{cases}$$

The images satisfy $\psi(0) = 0_1$, $\psi(z_1) = (u_\lambda, x_1, v_\varrho)$, $\psi(z_2) = (u_\varkappa, x_2, v_\mu)$ for $z_1 \neq 0, z_2 \neq 0$, and

$$\psi(z_1 z_2) = \begin{cases} 0_1 & \text{if } z_1 z_2 = 0\,, \\ (u_\lambda, x_1 x_2, v_\mu) & \text{if } z_1 z_2 \neq 0\,. \end{cases}$$

In the last case we may write $\psi(z_1 z_2) = (u_\lambda, x_1 x_2, v_\mu) = (u_\lambda, x_1, v_\varrho)(u_\varrho, x_2, v_\mu) = \psi(z_1) \,.\, \psi(z_2)$. This proves our statement.

In particular: All simple semigroups $u_\lambda T_0 u_\lambda = e_\lambda S e_\lambda - \{0\}$, $\lambda \in \Lambda$, are isomorphic to one another.

When replacing $u_\lambda, v_\mu, \ldots$ by their indices $\lambda, \mu, \ldots \in \Lambda$ it is easy to see that S_1 is isomorphic to the semigroup S_2 consisting of 0_1 and all triples (λ, x, ϱ), $x \in T_0$, $\lambda, \varrho \in \Lambda$ with the multiplication

$$(1) \qquad (\lambda, x, \varrho)(\varkappa, y, \mu) = \begin{cases} 0_1 & \text{if } \varrho \neq \varkappa\,, \\ (\lambda, xy, \mu) & \text{if } \varrho = \varkappa\,, \end{cases}$$

and 0_1 having the usual properties of a zero element.

Summarizing, we have proved

Theorem 2,7. *Let S be a 0-simple C-semigroup and e_α a fixed chosen cat. unit of S. For any $e_\mu \in E$ let $e_\mu = u_\mu v_\mu$, $u_\mu \in S e_\alpha$, $v_\mu \in e_\alpha S$. Suppose that it is possible to choose u_μ, v_μ such that all idempotents $v_\mu u_\mu$ are equal to e_α. Then S is isomorphic to a semigroup S_2 consisting of a zero 0_1 and the set of all triples (λ, x, μ), where λ, μ run independently through a set Λ and $x \in T_0$, T_0 being a simple semigroup with a unit element. Hereby the multiplication in S_2 is given by the rules* (1).

Remark. In the next section we shall show that the suppositions of Theorem 2,7 are, in particular, satisfied in any 0-bisimple C-semigroup.

3. 0-BISIMPLE C-SEMIGROUPS

We recall: If $a \in S$, we denote by $L^{(a)}$ the set of all generators of the left ideal $\{a, Sa\}$. The set $L^{(a)}$ is called the $\mathscr{L}$-class containing a. In the case of a C-semigroup we may write $L^{(a)} = \{x \mid Sx = Sa\}$.

Analogously, in a C-semigroup the $\mathscr{R}$-class containing a is defined by $R^{(a)} = \{x \mid xS = aS\}$. An $\mathscr{J}$-class containing a is defined as the set $J^{(a)} = \{x \mid SxS = SaS\}$. $\mathscr{L}$ and $\mathscr{R}$ are equivalence relations. The $\mathscr{D}$-relation is the smallest equivalence relation containing both $\mathscr{L}$ and $\mathscr{R}$.

Note: If e is an idempotent, $L^{(e)}$, $R^{(e)}$ the $\mathscr{L}$-and $\mathscr{R}$-classes containing e, then $D^{(e)} = L^{(e)}R^{(e)}$ is a $\mathscr{D}$-class.

The next lemma concerns general semigroups. It is known in one or another form. For completeness we prove it in the form needed here.

Lemma 3,1. *Let e be an idempotent $\in S$. Let e' be any idempotents $\in L^{(e)}R^{(e)}$.*

a) *If $e' = \xi\eta$, $\xi \in L^{(e)}$, $\eta \in R^{(e)}$, then $\eta\xi = e$.*

b) *The subsemigroups eSe and $e'Se'$ are isomorphic semigroups.*

Proof. a) Since $\xi \in L^{(e)}$, we have $\{\xi, S\xi\} = \{e, Se\} = Se$. Hence either $e = \xi$ or there is an $x \in S$ such that $e = x\xi$. If $e = \xi$, we may write $e = x\xi$ with $x = \xi = e$, so that in both cases we may write $e = x\xi$. Analogously, since $\eta \in R^{(e)}$, we have $\{\eta, \eta S\} = eS$ and again there is an $y \in S$ such that $e = \eta y$. Note further that $\xi \in L^{(e)} \subset Se$ implies $\xi e = \xi$ and $\eta \in R^{(e)}$ implies $e\eta = \eta$.

Now, we have successively: $\eta\xi = (e\eta)(\xi e) = e(\eta\xi)e = (x\xi)(\eta\xi)(\eta y) = x(\xi\eta)^2 y = x(\xi\eta)y = (x\xi)(\eta y) = e \, . \, e = e$.

b) Consider the mapping $\varphi : eSe \to e'Se'$ defined by $x \mapsto \xi x\eta$ for $x \in eSe$.

α) $\varphi(e) = \xi e\eta = \xi\eta = e'$.

β) Note that $e'\xi = (\xi\eta)\xi = \xi(\eta\xi) = \xi e = \xi$ and $\eta e' = \eta(\xi\eta) = (\eta\xi)\eta = e\eta = \eta$. Hence $\varphi(x) = \xi x\eta = e'\xi x\eta e' \subset e'Se'$.

γ) φ is a mapping of eSe onto $e'Se'$. Indeed, let $z = e'ze'$ be any element $\in e'Se'$. Consider the element $x_1 = \eta z\xi = e\eta z\xi e \in eSe$. Then $\varphi(x_1) = \xi(\eta z\xi)\eta = (\xi\eta)z(\xi\eta) = e'ze' = z$.

δ) φ is a homomorphism since for any $u, v \in eSe$ we have $\varphi(u) \, . \, \varphi(v) = \xi u\eta \, . \, \xi v\eta = \xi uev\eta = \xi uv\eta = \varphi(uv)$.

ε) Finally, φ is one-to-one since $\varphi(u) = \varphi(v)$, i.e. $\xi u\eta = \xi v\eta$, implies successively $\eta\xi u\eta\xi = \eta\xi v\eta\xi$, $eue = eve$, $u = v$. This proves Lemma 3,1.

Lemma 3,2. *Let S be a semigroup, e an idempotent. Then the complement of $L^{(e)}$ in Se, i.e. the set $K_l^{(e)} = Se - L^{(e)}$ is either empty or a left ideal of S.*

Proof. Suppose that $K_l^{(e)}$ is non-empty and $y \in K_l^{(e)} \subset Se$. Then $\{y, Sy\} \subset Se$. It is sufficient to prove that $Sy \cap L^{(e)} = \emptyset$. Suppose for an indirect proof that there is an element $z \in Sy$ and $z \in L^{(e)}$. The first inclusion implies $Sz \subset Sy$, the other one $\{z, Sz\} = Se$. Hence $Se = \{z, Sz\} \subset Sy$. This together with $Sy \subset Se$ implies $Se = Sy$ and $Se = \{y, Sy\}$. This is equivalent to $y \in L^{(e)}$, contrary to the assumption.

Lemma 3,3. *Let S be a 0-simple semigroup and e a fixed chosen non-zero idempotent $\in S$. Then for any non-zero idempotent $e' \in S$ we have $e'Se \cap L^{(e)} \neq \emptyset$.*

Proof. Suppose for an indirect proof that there is an idempotent $e'' \neq 0$ such that $e''Se \cap L^{(e)} = \emptyset$, i.e. $e''Se \subset K_l^{(e)}$. By Lemma 3,2 $Se''Se \subset SK_l^{(e)} \subset K_l^{(e)} \neq Se$. On the other hand (since S is simple), $Se''Se = (Se''S)\,e = Se$. This is an obvious contradiction.

Let now S be a 0-*simple C-semigroup*. Let $E = \{e_\alpha, e_\beta, \ldots\}$ be the set of all cat. units $\in S$ and $\Lambda = \{\alpha, \beta, \ldots\}$.

By Theorem 2,2, $e_\alpha S e_\alpha$ is a 0-simple semigroup containing an identity element and the zero is externally adjoined. By Lemma 3,1, for any idempotent $e' \in L^{(e_\alpha)}R^{(e_\alpha)}$ (independently of whether e' is a cat. unit or not) $e'Se'$ is isomorphic with $e_\alpha S e_\alpha$.

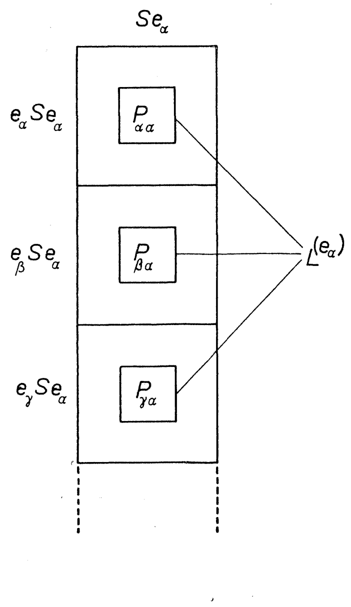
Denote $L^{(e_\alpha)} \cap e_\gamma S e_\alpha = P_{\gamma\alpha}$, so that $L^{(e_\alpha)} = \bigcup_{\gamma \in \Lambda} P_{\gamma\alpha}$. Denote analogously $R^{(e_\alpha)} \cap \cap\, e_\alpha S e_\delta = Q_{\alpha\delta}$, so that $R^{(e_\alpha)} = \bigcup_{\delta \in \Lambda} Q_{\alpha\delta}$. We have

$$P_{\alpha\beta} Q_{\gamma\delta} \begin{cases} = 0 & \text{if} \quad \beta \neq \gamma\,, \\ \subset e_\alpha S e_\delta & \text{if} \quad \beta = \gamma\,. \end{cases}$$

Finally, the $\mathscr{D}$-class containing e_α can be written in the form

$$D^{(e_\alpha)} = L^{(e_\alpha)}R^{(e_\alpha)} = [P_{\alpha\alpha} \cup P_{\beta\alpha} \cup P_{\gamma\alpha} \cup \ldots]\,[Q_{\alpha\alpha} \cup Q_{\alpha\beta} \cup Q_{\alpha\gamma} \cup \ldots]\,.$$

Remark 1. Lemma 3,3 when applied to the case of a 0-simple C-semigroup and cat. units says that $L^{(e_\alpha)}$ is scattered through all $e_\gamma S e_\alpha$ $(\gamma \in \Lambda)$. The whole semigroup $e_\alpha S e_\alpha$ need not belong to $L^{(e_\alpha)}$. Further, $P_{\alpha\alpha}$ is a semigroup. Indeed, if $a, b \in L^{(e_\alpha)} \cap e_\alpha S e_\alpha$, we have $Sa = Sb = Se_\alpha$, hence $Sab = (Sa)\,b = (Se_\alpha)\,b = S(e_\alpha b) = Sb = Se_\alpha$; since further $ab \in e_\alpha S e_\alpha$, we have $ab \in P_{\alpha\alpha}$. The situation can be visualised by the following figure:



Remark 2. Though $L^{(e_\alpha)}$ itself need not be a semigroup we show that $L^{(e_\alpha)} \cup \{0\}$ is a semigroup. Clearly $[P_{\beta\alpha}]^2 = 0$ for $\beta \neq \alpha$. Further,

$$[L^{(e_\alpha)}]^2 = (P_{\alpha\alpha} \cup P_{\beta\alpha} \cup \ldots)(P_{\alpha\alpha} \cup P_{\beta\alpha} \cup \ldots) =$$
$$= (P_{\alpha\alpha} \cup P_{\beta\alpha} \cup \ldots) P_{\alpha\alpha} \cup \{0\} .$$

For any β we have $P_{\beta\alpha}P_{\alpha\alpha} \subset P_{\beta\alpha}$, but since $e_\alpha \in P_{\alpha\alpha}$, we have $P_{\beta\alpha}P_{\alpha\alpha} = P_{\beta\alpha}$. Therefore $[L^{(e_\alpha)}]^2 = L^{(e_\alpha)} \cup \{0\}$.

Lemma 3,4. *$P_{\alpha\alpha}$ is exactly the $\mathscr{L}$-class of the 0-simple semigroup $e_\alpha S e_\alpha$ containing e_α. $Q_{\alpha\alpha}$ is exactly the $\mathscr{R}$-class of the 0-simple semigroup $e_\alpha S e_\alpha$ containing e_α.*

Proof. Denote the $\mathscr{L}$-class of $e_\alpha S e_\alpha$ containing e_α by $L_0^{(e_\alpha)}$. For any $x \in P_{\alpha\alpha}$ we have $Sx = Se_\alpha$, hence $e_\alpha Sx = e_\alpha Se_\alpha$ and (since $x = xe_\alpha = e_\alpha x$) $(e_\alpha Se_\alpha)\, x = e_\alpha Se_\alpha$. Hence $x \in L_0^{(e_\alpha)}$ and $P_{\alpha\alpha} \subset L_0^{(e_\alpha)}$.

Let on the other hand y be any element $\in L_0^{(e_\alpha)} \subset e_\alpha Se_\alpha$, i.e. $(e_\alpha Se_\alpha)\, y = e_\alpha Se_\alpha$. Multiplying by S from the left we have $(Se_\alpha S)\, e_\alpha y = Se_\alpha Se_\alpha$, i.e. $Se_\alpha y = Se_\alpha$, $Sy = Se_\alpha$, hence $y \in L^{(e_\alpha)}$ and $y \in L^{(e_\alpha)} \cap e_\alpha Se_\alpha = P_{\alpha\alpha}$, i.e. $L_0^{(e_\alpha)} \subset P_{\alpha\alpha}$. Therefore $P_{\alpha\alpha} = L_0^{(e_\alpha)}$.

The second statement can be proved analogously.

Suppose now that S is a 0-*bisimple semigroup*. Then $D^{(e_\alpha)} = L^{(e_\alpha)}R^{(e_\alpha)}$ is a bisimple subsemigroup of S and $S = D^{(e_\alpha)} \cup \{0\}$. Further,

$$S - 0 = L^{(e_\alpha)}R^{(e_\alpha)} = \bigcup_{e_\alpha, e_\beta \in E} e_\alpha Se_\beta - \{0\} = [P_{\alpha\alpha} \cup P_{\beta\alpha} \cup \ldots]\,[Q_{\alpha\alpha} \cup Q_{\alpha\beta} \cup \ldots] .$$

Since none of the products $P_{\lambda\mu}Q_{\sigma\tau}$ wit the exception of $P_{\alpha\alpha}Q_{\alpha\alpha}$ is contained in $e_\alpha Se_\alpha$, we conclude that $P_{\alpha\alpha}Q_{\alpha\alpha} = e_\alpha Se_\alpha - \{0\}$, i.e. $e_\alpha Se_\alpha - \{0\} = L_0^{(e_\alpha)}R_0^{(e_\alpha)}$. Hence $e_\alpha Se_\alpha - \{0\}$ is a bisimple semigroup.

We have proved

Theorem 3,1. *Let S be a 0-bisimple C-semigroup. Then for any cat. unit $e_\alpha \in E$ the subsemigroup $e_\alpha Se_\alpha - \{0\}$ is a bisimple semigroup with a unit element. All such subsemigroups are isomorphic to one another.*

Remark. It follows from Lemma 3,1 that even for any idempotent $e \in e_\alpha Se_\alpha - \{0\}$ the subsemigroup $eSe - \{0\}$ is isomorphic with $e_\alpha Se_\alpha - \{0\}$.

Now in the bisimple case Lemma 3,1 says that for any $e_\mu \in E$ there are elements $u_\mu \in Se_\alpha$, $v_\mu \in e_\alpha S$ such that $v_\mu u_\mu = e_\alpha$. Hence the suppositions of Theorem 2,7 are satisfied and T_0 is a bisimple semigroup with an identity element.

We finally obtain

Theorem 3,2. *Let T be a bisimple semigroup with an identity element. Let $\Lambda = \{\alpha, \beta, \ldots\}$ be an index set. Consider the set S consisting of an element 0 and all triples $\{(t, \alpha, \beta)\}$, $t \in T$, $\alpha, \beta \in \Lambda$. Define*

$$(t_1, \alpha, \beta)(t_2, \gamma, \delta) = \begin{cases} 0 & \text{if } \beta \neq \gamma, \\ (t_1 t_2, \alpha, \delta) & \text{if } \beta = \gamma, \end{cases}$$

the element 0 having the usual properties of a zero element. Then S is a 0-bisimple C-semigroup. Conversely, every 0-bisimple C-semigroup is obtained (up to an isomorphism) in this manner by choosing suitably the bisimple semigroup T with an identity element and an index set Λ.

4. MAXIMAL ONE-SIDED IDEALS

We shall now study the existence of maximal left (right) ideals.

Let us first recall that the set $\{L^{(a)}\}$ of all $\mathscr{L}$-classes can be partially ordered by defining $L^{(b)} \leqq L^{(a)}$ iff $(b, Sb) \subset (a, Sa)$. It is clear what we shall mean by a maximal $\mathscr{L}$-class in this ordering. The ordering of $\mathscr{R}$-classes and $\mathscr{I}$-classes is defined analogously. In particular, in a C-semigroup we have $I^{(b)} \leqq I^{(a)}$ iff $SbS \subset SaS$.

Theorem 4,1. *Any C-semigroup contains maximal left and maximal right ideals.*

Proof. Let $e_\alpha \in E$. Consider the union L_α of all left ideals of S which do not contain e_α. If card $E \geqq 2$, then L_α contains $\{\bigcup_e Se \mid e \in E, e \neq e_\alpha\}$ (but L_α may be larger). We state that L_α is a maximal left ideal of S. If L'_α is a left ideal of S which is larger than L_α, then L'_α contains e_α, hence it contains Se_α and, in the case card $E \geqq 2$, we have $L'_\alpha = \{\bigcup_e Se \mid e \in E\} = S$. If card $E = 1$, we have $L'_\alpha \supset Se_\alpha = S$, hence $L'_\alpha = S$. This proves Theorem 4,1 for left ideals. The existence of maximal right ideals is proved analogously.

To describe more precisely the set of all maximal left ideals and maximal $\mathscr{L}$-classes we need the following

Lemma 4,1. *A left ideal L of a semigroup S is a maximal left ideal of S iff $S - L$ is a maximal $\mathscr{L}$-class.*

Remark. Generalizations of Lemma 4,1 to unary algebras can be found in the paper [15].

Proof. a) If L is a maximal left ideal of S, card $(S - L) \geqq 2$ and $x, y \in S - L$, $x \neq y$, then the left ideals $L \cup \{x, Sx\}$ and $\{y, Sy\} \cup L$ are larger than L, hence $L \cup \{x, Sx\} = S = L \cup \{y, Sy\}$. This implies $y \in Sx$, $x \in Sy$, whence $Sx = Sy$ and $\{x, Sx\} = \{y, Sy\}$. Hence all elements $\in S - L$ belong to the same $\mathscr{L}$-class, say $L^{(x)}$.

The set $L^{(x)}$ cannot meet L, since $z \in L^{(x)} \cap L$ would imply $\{z, Sz\} = \{x, Sx\}$ and $\{z, Sz\} \subset L$, hence $\{x, Sx\} \subset L$, contrary to the assumption that $x \notin L$. We have proved that $S - L$ is an $\mathscr{L}$-class. The same argument can be applied in the case when card $(S - L) = 1$, i.e. $L^{(x)} = \{x\}$.

To prove that $L^{(x)} = S - L$ is a maximal $\mathscr{L}$-class, suppose for an indirect proof that there is $z \in S$ such that $\{z, Sz\} \supsetneqq \{x, Sx\}$. Then $z \notin L^{(x)}$, hence $z \in L$ and $\{x, Sx\} \subsetneqq (z, Sz) \subset L$. This implies $x \in L$, a contradiction with the assumption.

b) Let conversely $L^{(x)}$ be a maximal $\mathscr{L}$-class. We first show that $S - L^{(x)}$ is a left ideal of S. Let $y \in S - L^{(x)}$. It is sufficient to show that $Sy \subset S - L^{(x)}$. Suppose for an indirect proof that this is not the case, i.e. there is an element $z \in Sy \cap L^{(x)}$. Then we have $Sz \subset Sy$ and (since $z \in L^{(x)}$) $\{z, Sz\} = \{x, Sx\}$. Therefore $\{x, Sx\} = \{z, Sz\} \subset \{y, Sy\}$. Since $L^{(x)}$ is maximal this implies $\{x, Sx\} = \{y, Sy\}$ and $y \in L^{(x)}$, contrary to our assumption.

To prove that $S - L^{(x)}$ is a maximal left ideal take any $t \in L^{(x)}$. Then $(S - L^{(x)}) \cup \{t, St\}$ is a left ideal of S. Since $\{t, St\} = \{u, Su\}$ for any $u \in L^{(x)}$, we have $\{t, St\} = \bigcup_{u \in L^{(x)}} \{u, Su\} \supset L^{(x)}$. Hence $(S - L^{(x)}) \cup \{t, St\} = S$. This completes the proof of Lemma 4,1.

Theorem 4,2. *Let e_α be a cat. unit of a C-semigroup. Then the $\mathscr{L}$-class $L^{(e_\alpha)}$ is maximal $\mathscr{L}$-class of S. Conversely, every maximal $\mathscr{L}$-class of S is of the form $L^{(e_\beta)}$ with a suitably chosen $e_\beta \in E$.*

Proof. a) By definition $L^{(e_\alpha)} = \{a \mid Sa = Se_\alpha\}$. Suppose that there is $b \in S$ such that $L^{(e_\alpha)} \lneqq L^{(b)}$, i.e. $Se_\alpha \subsetneqq Sb$. There is a cat. unit e_β such that $b = be_\beta$. We have $Sb = Sbe_\beta \subset Se_\beta$, hence $0 \neq Se_\alpha \subsetneqq Se_\beta$. This is a contradiction to Lemma 1,2.

b) Let $L^{(b)}$ be a maximal $\mathscr{L}$-class of S. Writing again $b = be_\beta$ with $e_\beta \in E$ we have $Sb = Sbe_\beta \subset Se_\beta$, or otherwise $L^{(b)} \leqq L^{(e_\beta)}$. Since $L^{(b)}$ is a maximal $\mathscr{L}$-class, we have $L^{(b)} = L^{(e_\beta)}$ which proves our statement.

Lemma 4,2. *The (maximal) $\mathscr{L}$-class $L^{(e_\alpha)}$ contains a unique idempotent (namely e_α).*

Proof. Suppose that e is an idempotent contained in $L^{(e_\alpha)}$. Then, since $Se_\alpha = Se$, e is a right identity for all $x \in Se_\alpha$. In particular $e_\alpha e = e_\alpha \neq 0$. Since e_α is a cat. unit we have either $e_\alpha e = e$ or $e_\alpha e = 0$. Hence $e_\alpha = e$.

Clearly $L^{(e_\alpha)} \subset Se_\alpha$ so that we can write $Se_\alpha = L^{(e_\alpha)} \cup K_l^{(e_\alpha)}$ with $L^{(e_\alpha)} \cap K_l^{(e_\alpha)} = \emptyset$. The set $K_l^{(e_\alpha)}$ is a left ideal of S, since $K_l^{(e_\alpha)} = Se_\alpha \cap L_\alpha$. (It may occur that $K_l^{(e_\alpha)} = 0$.)

Theorem 4,3. *Any maximal left ideal of a C-semigroup can be written in the form $L_\alpha = [\bigcup_{e_\beta \in E, e_\beta \neq e_\alpha} Se_\beta] \cup K_l^{(e_\alpha)}$, where the left ideal $K_l^{(e_\alpha)}$ is the complement of $L^{(e_\alpha)}$ in Se_α.*

It should be emphasized that $K_l^{(e_\alpha)}$ may contain a number of idempotents none of them being, of course, a cat. unit.

So far we have identified all maximal $\mathscr{L}$-classes. Analogously, all maximal $\mathscr{R}$-classes are of the form $R^{(e_\alpha)}$, where e_α runs through all elements $\in E$.

It will turn out that the same problem concerning maximal $\mathscr{J}$-classes and maximal two-sided ideals is much more complicated. To explain where the difficulties arise, let us consider – just for a while – the case that S is a finite C-semigroup in which case the $\mathscr{D}$-classes and $\mathscr{J}$-classes coincide. Then the product of a maximal $\mathscr{L}$-class $L^{(e_\alpha)}$ and a maximal $\mathscr{R}$-class $R^{(e_\alpha)}$ is the $\mathscr{J}$-class $L^{(e_\alpha)}R^{(e_\alpha)}$. One may suspect that $L^{(e_\alpha)}R^{(e_\alpha)}$ is a maximal $\mathscr{J}$-class. Example 5,1 below shows that this need not be true.

Example 4,1. In Example 1,1 every cat. unit (i, i) is a maximal $\mathscr{L}$-class. All maximal left ideals are of the form $L_i = S - \{(i, i)\}$. Incidentally, these are at the same time maximal right and maximal two-sided ideals.

Example 4,2. To have a quite different example (and for further purposes), consider the bicyclic semigroup B with an identity 1, i.e. the semigroup generated by two symbols p, q subject to the single generating relation $pq = 1$. Adjoin to B a zero 0. Then $S = B \cup \{0\}$ is a C-semigroup. The identity 1 is the unique cat. unit of S. The $\mathscr{L}$-class containing 1 is $L^{(1)} = \{1, q, q^2, \ldots\}$, the unique maximal $\mathscr{L}$-class. Analogously $R^{(1)} = \{1, p, p^2, \ldots\}$ is the unique maximal $\mathscr{R}$-class. Further $L_1 = S - L^{(1)}$ and $R_1 = S - R^{(1)}$, are maximal left and right ideals of S respectively. Note that in this case $L^{(1)}R^{(1)} = B$ and the maximal two-sided ideal of S is $\{0\}$.

5. MAXIMAL TWO-SIDED IDEALS

We shall now study maximal two-sided ideals and maximal $\mathscr{J}$-classes of a C-semigroup.

The following general statement can be proved by an analogous argument as Lemma 4,1.

Lemma 5,1. *A two-sided ideal M of a semigroup S is a maximal two-sided ideal of S iff $S - M$ is a maximal $\mathscr{J}$-class.*

The existence of maximal two-sided ideals in a C-semigroup cannot be proved in the same way as Theorem 4,1 since the following statement holds:

Theorem 5,1. *There exist C-semigroups without maximal two-sided ideals.*

We postpone giving an example which proves this statement after we shall have proved Lemma 5,2.

Even in the finite case, an "undesired" situation may arise. It is natural to try to find maximal two-sided ideals by examining the largest two-sided ideal which does not contain a given cat. unit e_α. Such an ideal always exists but need not be a maximal ideal of S.

Example 5,1. Consider the semigroup $S = \{0, e_\alpha, e_\beta, u, v, e\}$ with the following multiplication table:

	e_α	e_β	u	v	e
e_α	e_α	0	0	v	e
e_β	0	e_β	u	0	0
u	u	0	0	e_β	u
v	0	v	e	0	0
e	e	0	0	v	e

This is a C-semigroup with two cat. units e_α. e_β. We have

$$Se_\alpha = \{0, e_\alpha, u, e\}, \quad L^{(e_\alpha)} = \{e_\alpha\},$$
$$Se_\beta = \{0, e_\beta, v\}, \quad L^{(e_\beta)} = \{e_\beta, v\},$$
$$e_\alpha S = \{0, e_\alpha, v, e\}, \quad R^{(e_\alpha)} = \{e_\alpha\},$$
$$e_\beta S = \{0, e_\beta, u\}, \quad R^{(e_\beta)} = \{e_\beta, u\}.$$

The maximal left and right ideals of S are

$$L_\alpha = \{0, e_\beta, u, v, e\}, \quad L_\beta = \{0, e_\alpha, u, e\},$$
$$R_\alpha = \{0, e_\beta, u, v, e\}, \quad R_\beta = \{0, e_\alpha, v, e\}.$$

There is a unique maximal two-sided ideal $M_\alpha = R_\alpha = L_\alpha = \{0, e_\beta, u, v, e\}$. This is the largest two-sided ideal of S which does not contain e_α. The largest two-sided ideal of S which does not contain e_β is $\{0\}$ and this is, of course, not a maximal two-sided ideal of S. [Otherwise expressed: The largest two-sided ideal of S contained in the maximal left ideal L_β is $\{0\}$.]

In this example we have three $\mathscr{I}$-classes:

$$I^{(e_\alpha)} = L^{(e_\alpha)}R^{(e_\alpha)} = \{e_\alpha\}, \quad I^{(e_\beta)} = L^{(e_\beta)}R^{(e_\beta)} = \{e_\beta, v, u, e\}, \quad I^{(0)} = \{0\},$$

and $I^{(0)} \lneqq I^{(e_\beta)} \lneqq I^{(e_\alpha)}$. The "undesired" situation is due to the fact that $Se_\beta S \subsetneqq Se_\alpha S$ though e_β is a cat. unit. Note also that the product of the maximal $\mathscr{L}$-class $L^{(e_\beta)}$ and the maximal $\mathscr{R}$-class $R^{(e_\beta)}$ is not a maximal $\mathscr{I}$-class.

In what follows, when speaking about maximal $\mathscr{I}$-classes we shall suppose, of course, that a maximal $\mathscr{I}$-class exists.

It should be remarked in advance: If M is a maximal two-sided ideal of a C-semigroup, M cannot contain all elements $\in E$, since this would imply $M \supset SM \supset SE = S$, i.e. $M = S$. There exists therefore at least one $e_\alpha \in E$ such that M does not contain e_α. In this case we have $M \subset L_\alpha \cap R_\alpha$, where $L_\alpha(R_\alpha)$ is the maximal left (right) ideal of S which does not contain e_α. There may exist several maximal left (right) ideals containing M. On the other hand, if a maximal left ideal L_α of S contains a maximal two-sided ideal M of S, then M is uniquely determined.

Lemma 5,2. *Any maximal $\mathcal{I}$-class of a C-semigroup contains at least one cat. unit.*

Proof. Let $I^{(a)}$ be a maximal $\mathcal{I}$-class. Since $a = a \,.\, e_r(a)$, we have $SaS = Sa \,.\, e_r(a)\, S \subset Se_r(a)\, S$. Hence, with respect to the maximality of $I^{(a)}$, we have $SaS = Se_r(a)\, S$, i.e. $e_r(a) \in I^{(a)}$.

Lemma 5,3. *Any two different maximal $\mathcal{I}$-classes of a C-semigroup satisfy $I^{(a)} \,.\, I^{(b)} = 0$.*

Proof. Any $x \in I^{(a)}$ can be written in the form $x = xe_r(x)$, $e_r(x) \in E$ and by the same argument as in the foregoing Lemma, $e_r(x) \in I^{(a)}$. Analogously, any $y \in I^{(b)}$ can be written in the form $y = e_l(y) \,.\, y$ with $e_l(y) \in E \cap I^{(b)}$. Since $I^{(a)} \cap I^{(b)} = \emptyset$, we have $e_r(x) \neq e_l(y)$, hence $xy = x\, e_r(x) \,.\, e_l(y)\, y = 0$.

We now give an example of a C-semigroup without maximal two-sided ideals.

Example 5,2. Let S be the set consisting of a zero $\{z\}$ and all $r \times s$ matrices A_{rs}, r, s running independently through the set $N = \{1, 2, 3, \ldots\}$, the entries of A_{rs} being non-negative integers.

We define $A_{rs} \,.\, A_{tu}$ to be z if $s \neq t$, and to be the ordinary matrix product if $s = t$.

It is immediately seen that S is a C-semigroup and the cat. units of S are the $n \times n$ unit matrices U_n $(n = 1, 2, 3, \ldots)$. [To avoid misunderstanding let us note explicitly that any rectangular zero matrix is merely an element $\in S$ and not z.]

We first show that U_{n+1} is not contained in the two-sided ideal generated by U_n, i.e. in SU_nS.

Suppose for an indirect proof that U_{n+1} is contained in SU_nS. Then there exist two matrices $A_{n+1,n} = (a_{ik})$ and $B_{n,n+1} = (b_{jl})$ such that $A_{n+1,n}B_{n,n+1} = U_{n+1}$. Consider the product

$$C = A_{n+1,n}B_{n,n+1} = \begin{pmatrix} a_{11}, & \ldots, a_{1n} \\ \ldots & \ldots \\ a_{n+1,1}, & \ldots, a_{n+1,n} \end{pmatrix} \begin{pmatrix} b_{11}, \ldots, b_{1,n+1} \\ \ldots \\ b_{n1}, \ldots, b_{n,n+1} \end{pmatrix}.$$

The elements in the diagonal of C are

$$c_{ii} = \sum_{j=1}^{n} a_{ij}b_{ji}, \quad i = 1, 2, \ldots, n + 1 .$$

If C were the unit matrix U_{n+1}, then for any i there would exist exactly one summand in c_{ii} equal to 1 (while the others are zeros). Hence there exist integers $j_1, j_2, \ldots, j_{n+1}$ such that

$$a_{1j_1}b_{j_11} = 1 , \quad a_{2j_2}b_{j_22} = 1, \ldots, a_{n+1,j_{n+1}}b_{j_{n+1},n+1} = 1 .$$

Now since $\{j_1, j_2, \ldots, j_{n+1}\} \subset \{1, 2. \ldots, n\}$ there exist at least two integers $k \neq l$ such that $j_k = j_l$. The equalities $a_{k,j_k} b_{j_k,k} = 1$ and $a_{l,j_l} b_{j_l,l} = 1$ imply $a_{k,j_k} = b_{j_k,k} = a_{l,j_l} = b_{j_l,l} = 1$. But then the element c_{lk} in the matrix C is

$$c_{lk} = \sum_{u=1}^{n} a_{lu} b_{uk} = \ldots + a_{l,j_l} b_{j_l,k} + \ldots = \ldots + a_{l,j_l} b_{j_k,k} + \ldots \geqq 1 .$$

In other words: If the product C contains one's along the whole main diagonal, then C contains necessarily at least one non-zero element outside of the main diagonal. Hence there cannot exist $A_{n+1,n}$, $B_{n,n+1}$ such that $C = U_{n+1}$. The cat. unit U_{n+1} is not contained in SU_nS.

We next show, on the other hand, that $U_{n-1} \in SU_nS$. Consider to this end the following $(n-1) \times n$ matrix A and the $n \times (n-1)$ matrix B:

$$A = \begin{pmatrix} 1, & 0, & \ldots, & 0, & 0 \\ 0, & 1, & \ldots, & 0, & 0 \\ \ldots & & & & \\ 0, & 0, & \ldots, & 1, & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1, & 0, & \ldots, & 0 \\ 0, & 1, & \ldots, & 0 \\ \ldots & & & \\ 0, & 0, & \ldots, & 0 \end{pmatrix}$$

We then have: $AU_nB = AB = U_{n-1}$.

Hence we have an increasing sequence of two-sided ideals

$$SU_1S \subset SU_2S \subset \ldots \subset SU_nS \subset SU_{n+1}S \subset \ldots ,$$

where all inclusions are proper. The $\mathscr{I}$-class containing U_n (for any n) cannot be maximal. This proves the statement of Theorem 5,1.

Theorem 5,2. *Let S be a C-semigroup. An $\mathscr{I}$-class which is not maximal and contains idempotents, contains at least one idempotent which is not a cat. unit of S.*

Proof. Let $I^{(b)}$ be an $\mathscr{I}$-class which is not maximal and contains idempotents. If none of the idempotents $\in I^{(b)}$ is a cat. unit there is nothing to prove. Let e_β be a cat. unit contained in $I^{(b)} = I^{(e_\beta)}$. Since $I^{(e_\beta)}$ is not maximal, there is an $\mathscr{I}$-class $I^{(a)} \underset{\neq}{>} I^{(e_\beta)}$, i.e. $Se_\beta S \subsetneqq SaS$. Write a in the form $a = ae_\alpha$, $e_\alpha \in E$. Then $Se_\beta S \subsetneqq SaS = Sae_\alpha S \subset Se_\alpha S$. Hence $e_\beta \in Se_\alpha S$ and $e_\alpha \neq e_\beta$.

There exist therefore two elements $u, v \in S$ such that $e_\beta = ue_\alpha v$. We have $u = ue_\alpha = e_\beta u$, $v = e_\alpha v = ve_\beta$ and $e_\beta = uv$. Denote $e_\gamma = vu$. Then e_γ is an idempotent since $e_\gamma^2 = v(uv)u = ve_\beta u = vu = e_\gamma$.

Now $e_\beta = uv = u(vu)v = ue_\gamma v$ implies $Se_\beta S \subset Se_\gamma S$ and $e_\gamma = vu = v(uv)u = ve_\beta u$ implies $Se_\gamma S \subset Se_\beta S$. Hence $Se_\beta S = Se_\gamma S$ and $I^{(e_\beta)} = I^{(e_\gamma)}$. (This proves also that $e_\gamma \neq 0$.)

Further, $e_\alpha e_\gamma = e_\alpha vu = vu = e_\gamma$ and $e_\gamma e_\alpha = vue_\alpha = vu = e_\gamma$. Since $I^{(e_\gamma)} \neq I^{(e_\alpha)}$, we have $e_\gamma \neq e_\alpha$ and the equality $e_\gamma e_\alpha = e_\alpha e_\gamma = e_\gamma$ shows that e_γ is not a cat. unit of S. This proves Theorem 5,2.

Example 5,3. In Example 5,1 we have $I^{(0)} \lneqq I^{(e_\beta)} \lneqq I^{(e_\alpha)}$. The class $I^{(e_\beta)}$ contains the cat. unit e_β. Since it is not maximal, it contains an idempotent which is not a cat. unit, namely the idempotent e.

Theorem 5,2 immediately implies

Theorem 5,3. *Let S be a C-semigroup and I an $\mathcal{I}$-class containing idempotents. If each of these idempotents is a cat. unit of S, then I is a maximal $\mathcal{I}$-class of S.*

Remark. The converse need not hold. This can be seen from Example 4,2. Here B is the unique maximal $\mathcal{I}$-class of S. It contains the cat. unit 1 but also an infinite sequence of idempotents $\{qp, q^2p^2, q^3p^3, \ldots\}$.

6. THEOREMS ON THE FACTOR SEMIGROUP S/M

Let S be a C-semigroup and M a maximal two-sided ideal of S. Then S/M is a 0-simple semigroup. We shal study conditions under which S/M is a C-semigroup.

Theorem 2,3a implies

Theorem 6,1. *Let S be a C-semigroup and $M_\alpha = S - I^{(e_\alpha)}$ a maximal two-sided ideal of S. Then S/M_α is a completely 0-simple C-semigroup iff all idempotents $\in I^{(e_\alpha)}$ are cat. units.*

This Theorem will be strengthened in Theorem 6,2.

In order to find some relations between $I^{(e_\alpha)}$ and the ideal $M_\alpha = S - I^{(e_\alpha)}$ we introduce in accordance with [5] and [6] the following general notion.

Definition. Let S be a semigroup with zero and M a two-sided ideal of S. The ideal M is called 0-*isolated* if for any $a, b \in S - M$, $ab \in M$ implies $ab = 0$.

Lemma 6,1. *Let S be a C-semigroup and M a two-sided ideal of S. The factor semigroup S/M is a C-semigroup iff M is 0-isolated.*

Proof. Denote $S - M = K$. Adjoin to K a zero element $\bar{0}$ and denote $\bar{K} = K \cup \{\bar{0}\}$. Then $\bar{K}$ (with the obvious multiplication) is isomorphic to S/M.

a) Suppose that S/M is a C-semigroup. Take two elements $c \in K$, $d \in K$ such that $cd \in M$. Suppose for an indirect proof that $cd \neq 0$. Since S is a C-semigroup, there is an $e_r \in E$ such that $ce_r = c$. [Here $e_r \in K$, since $e_r \in M$ would imply $c \in M$.] Since $cd = ce_rd \neq 0$, we also have $e_rd \neq 0$, therefore $e_rd = d$. Hence $\bar{K}$ contains three elements c, d, e_r such that $ce_rd = \bar{0}$, while $ce_r \neq \bar{0}$ and $e_rd \neq \bar{0}$. Hence $\bar{K}$ is not categorical at $\bar{0}$ so that S/M is not a C-semigroup. We have therefore $cd = 0$ for any pair $c, d \in K$ such that $cd \in M$. This means that M is 0-isolated.

b) Let conversely S be a C-semigroup and M a 0-isolated two-sided ideal of S. Then $S/M \approx \bar{K} = K \cup \{0\}$ is a semigroup in which the first condition of Definition 0,1 is satisfied. [Indeed, if $c \in K$ and $ce_r = c$, $e_l c = c$, $e_r, e_l \in E$, the cat. units e_r, e_l are contained in K.] We next prove that $\bar{K}$ is categorical at $\bar{0}$. This means: We shall prove that if $a, b, c \in K$ and $abc \in M$ then either $ab \in M$ or $bc \in M$. Suppose for an indirect proof that $ab \in S - M$ and $bc \in S - M$. Since $bc \in S - M$, we have $c \notin M$, hence $c \in S - M$. Now since M is 0-isolated, $ab \in S - M$, $c \in S - M$ and $(ab)\,c \in M$ imply $abc = 0$. Since S is a C-semigroup this implies either $ab = 0$ or $bc = 0$, hence either $ab \in M$ or $bc \in M$, a contradiction to the supposition. This completes the proof of Lemma 6,1.

We now apply the last lemma to the case when M is a maximal two-sided ideal of a C-semigroup such that all idempotents $\in S - M$ are cat. units of S. By Theorem 2,4a the 0-simple semigroup S/M is a completely 0-simple C-semigroup. By Lemma 6,1 M is 0-isolated. This implies

Theorem 6,2. *Let S be a C-semigroup and $I^{(e_\alpha)}$ a maximal $\mathscr{I}$-class of S in which all idempotents are cat. units of S. Then $I^{(e_\alpha)} \cup \{0\}$ is both a subsemigroup of S and a 0-simple inverse semigroup. Moreover, $M_\alpha = S - I^{(e_\alpha)}$ is a (maximal) two-sided ideal of S which is 0-isolated.*

The suppositions of Theorem 6,2 are, in particular, satisfied if $I^{(e_\alpha)}$ is any maximal $\mathscr{I}$-class of any finite C-semigroup.

Note that in this case all idempotents in any maximal $\mathscr{I}$-class are automatically cat. units. We have therefore the following special result:

Theorem 6,3. *Let S be a finite C-semigroup and $I^{(e_\alpha)}$ a maximal $\mathscr{I}$-class of S. Then $I^{(e_\alpha)} \cup \{0\}$ is a completely 0-simple C-semigroup (i.e. a 0-simple inverse semigroup).*

The following pertinent question arises. Let S be a C-semigroup and $I^{(e_\alpha)}$ a maximal $\mathscr{I}$-class in which not all idempotents are cat. units of S. Denote again $M_\alpha = S - I^{(e_\alpha)}$. Then S/M_α is a 0-simple semigroup containing cat. units. In contradistinction to Theorem 6,2 it is "in general" not true that M_α is 0-isolated (hence S/M_α a C-semigroup).

To show this we reproduce here a slightly modified example due to Munn which was mentioned above.

Example 6,1. Let X be the set of all positive integers, T_X the full transformation semigroup on X and ε the identical mapping. If $\alpha \in T_X$ we call card $X\alpha$ the rank of α. The set of all elements $\alpha \in T_X$ with a finite rank is the maximal two-sided ideal of M. The factor semigroup T_X/M is a 0-simple semigroup.

Now adjoin to T_X a zero element 0. Then $T_X^0 = T_X \cup \{0\}$ is a C-semigroup with the identity element ε (and no other cat. idempotent). $M^0 = M \cup \{0\}$ is the maximal two-sided ideal of T_X^0 and $S = T_X^0/M^0$ is a 0-simple semigroup containing ε. To show that S is not a C-semigroup it is sufficient to show that there are two elements α, β such that $\alpha\varepsilon\beta \in M$ while $\alpha = \alpha\varepsilon \notin M$ and $\beta = \varepsilon\beta \notin M$. Define α, β as follows:

$$n\alpha = \begin{cases} n & \text{if } n \text{ is odd}, \\ 1 & \text{if } n \text{ is even}. \end{cases} \qquad n\beta = \begin{cases} 2 & \text{if } n \text{ is odd}, \\ n & \text{if } n \text{ is even}. \end{cases}$$

Then $n\alpha\beta = 2$ for any $n \in X$, hence $\alpha\beta \in M$, while $\alpha \notin M$, $\beta \notin M$. (Hence M^0 is not 0-isolated.)

Remark. In the semigroup T_X we have $\mathscr{J} = \mathscr{D}$. Hence Example 6,1 shows that even if $I^{(e_\alpha)}$ is a $\mathscr{D}$-class, $S - I^{(e_\alpha)}$ need not be 0-isolated.

We can slightly modify the general result of Lemma 6,1 in the following way. If $x \in I^{(e_\alpha)}$, $y \in I^{(e_\alpha)}$, then $SxS = SyS = Se_\alpha S$. Hence $xy \in SxyS \subset SxS = Se_\alpha S$. Denote $M_\alpha \cap Se_\alpha S = M^{(\alpha)}$. Then $M^{(\alpha)}$ is a two-sided ideal of S and the largest two-sided ideal of S contained in $Se_\alpha S$ which does not contain e_α. Hereby $Se_\alpha S = I^{(e_\alpha)} \cup M^{(\alpha)}$. Hence we have

Theorem 6,4. *$I^{(e_\alpha)} \cup \{0\}$ is a C-semigroup iff the largest two-sided ideal of S which is properly contained in $Se_\alpha S$ is 0-isolated in $Se_\alpha S$.*

Remark. If S is a C-semigroup and I a maximal $\mathscr{J}$-class, then the maximal ideal $M = S - I$ need not be itself a C-semigroup. This is shown by the following example.

Example 6,2. Consider the semigroup $S = \{e_\alpha, e_\beta, v, e, 0\}$ with the multiplication table

	e_α	e_β	v	e
e_α	e_α	0	v	e
e_β	0	e_β	0	0
v	0	v	0	0
e	e	0	v	e

This is a C-semigroup in which $\{e_\alpha\}$, $\{e_\beta\}$ are maximal $\mathscr{J}$-classes. $M_\alpha = S - \{e_\alpha\}$ is a maximal ideal which is itself a C-semigroup, while $M_\beta = S - \{e_\beta\}$ is a maximal two-sided ideal which is itself not a C-semigroup. The reason for this is easily to be understood by an inspection of the corresponding graph. [Note that e which is not a cat. unit of S is a cat. unit of M_α.]

Let now S be a C-semigroup containing at least one maximal two-sided ideal. Let $\{M_\lambda \mid \lambda \in H\}$ be the set of all maximal two-sided ideals of S. Denote $I^{(\lambda)} = S - M_\lambda$ and $M^* = \bigcap_{\lambda \in H} M_\lambda$ (the intersection of all maximal two-sided ideals of S). Then S can be written in the form of a union of disjoint subsets:

(2) $$S = [\bigcup_{\lambda \in H} I^{(\lambda)}] \cup M^* .$$

By Lemma 5,3, if card $H \geqq 2$, we have $I^{(\lambda)} . I^{(\mu)} = 0$ for $\lambda \neq \mu$. [This is, of course, not sufficient to assert that M^* is 0-isolated, since $a \in I^{(\lambda)}$, $b \in I^{(\lambda)}$, $ab \in M^*$ do not necessarily imply $ab = 0$.] In any case S/M^* is a 0-direct union of 0-simple semigroups each of which contains at least one cat. unit.

If every maximal two-sided ideal is 0-isolated, then so is M^*. To prove this, suppose that $a, b \in S - M^*$ and $ab \in M^*$. We have to show that $ab = 0$. If $a \in I^{(\lambda)}$, $b \in I^{(\mu)}$ and $\lambda \neq \mu$ we have $ab = 0$ (independently of whether M_λ, M_μ are 0-isolated or not). Suppose next $\lambda = \mu$, hence $a, b \in S - M_\lambda$. Since $ab \in M^* \subset M_\lambda$ and M_λ is 0-isolated, we have $ab = 0$.

Conversely, suppose that M^* is 0-isolated. Let $a, b \in I^{(\lambda)}$ and $ab \in S - I^{(\lambda)}$. Then ab cannot be contained in a $I^{(\mu)}$, $\mu \neq \lambda$. Indeed, $ab \in I^{(\mu)}$ would imply the existence of a cat. unit $e_\mu \in I^{(\mu)}$ such that $abe_\mu = ab$. Further, since $b = be_\lambda$ with some $e_\lambda \in I^{(\lambda)}$, we have $abe_\mu e_\lambda = abe_\lambda$, i.e. $ab = 0 \notin I^{(\mu)}$, a contradiction. Hence $a \in I^{(\lambda)}$, $b \in I^{(\lambda)}$ and $ab \in S - I^{(\lambda)}$ imply $ab \in M^*$. Since M^* is 0-isolated, we have $ab = 0$. Hence $M_\lambda = S - I^{(\lambda)}$ is 0-isolated. Summarizing: M^* is 0-isolated iff each M_λ is 0-isolated.

Applying the foregoing results we derive

Theorem 6,5. *Let S be a C-semigroup containing maximal two-sided ideals. Then S/M^* is a 0-direct union of 0-simple C-semigroups iff M^* is 0-isolated.*

Remark. If S is a finite C-semigroup, then M^* is 0-isolated. Write $P^{(\lambda)} = I^{(\lambda)} \cup \{0\}$ and $T = \bigcup_{\lambda \in H} P^{(\lambda)}$. We then have a decomposition of S into two quasidisjoint subsemigroups: $S = T \cup M^*$. Here "in general" the union need not be 0-direct (see the Remark after Theorem 6,8 below) while T is either a completely 0-simple C-semigroup or a 0-direct union of completely 0-simple C-semigroups.

As a special case of Theorem 6,5 we have

Theorem 6,6. *Let S be a finite C-semigroup. Then S/M^* is a 0-direct union of 0-simple inverse semigroups.*

Now M^* may contain cat. units of S. We shall find conditions under which this cannot occur.

Definition. We shall say that *a C-semigroup S satisfies Condition A* if every maximal left ideal of S contains a maximal two-sided ideal of S.

Example 5,1 shows that this condition need not be satisfied even in the finite case.

Proposition 6,1. *A C-semigroup satisfies Condition A iff every $\mathcal{J}$-class containing a cat. unit is a maximal $\mathcal{J}$-class.*

Proof. a) Suppose that Condition A is satisfied. Let e_α be any element $\in E$. We have to show that $I^{(e_\alpha)}$ is a maximal $\mathscr{I}$-class. We know (see Theorem 4,1) that there is a maximal left ideal L_α which does not contain e_α. By the supposition there is a maximal two-sided ideal M_α of S such that $M_\alpha \subset L_\alpha$. Since $S - M_\alpha$ is a maximal $\mathscr{I}$-class and $e_\alpha \in S - M_\alpha$, $I^{(e_\alpha)}$ is a maximal $\mathscr{I}$-class.

b) Suppose conversely that every $\mathscr{I}$-class containing a cat. unit is a maximal $\mathscr{I}$-class. Let L_α be the maximal left ideal which does not contain e_α. Since $I^{(e_\alpha)}$ is maximal, $M_\alpha = S - I^{(e_\alpha)}$ is a maximal two-sided ideal of S (which does not contain e_α). Since M_α is also a left ideal, we have $M_\alpha \subset L_\alpha$ which completes the proof of our statement.

Proposition 6,2. *A C-semigroup satisfies Condition A iff for any pair of cat. units e_α, e_β we have: $e_\beta \in Se_\alpha S$ implies $Se_\beta S = Se_\alpha S$.*

Proof. a) Suppose that $e_\beta \in Se_\alpha S$ implies $Se_\beta S = Se_\alpha S$. Let L_α be the maximal left ideal of S which does not contain e_α. Let further M_α be the largest two-sided ideal of S which does not contain e_α. Clearly $M_\alpha \subset L_\alpha$. Let $a \in S - M_\alpha$. Then SaS is a two-sided ideal containing a, hence $M_\alpha \cup SaS$ is larger than M_α so that $e_\alpha \in M_\alpha \cup$ $\cup\, SaS$. It follows $e_\alpha \in SaS$, whence $Se_\alpha S \subset SaS$. On the other hand, since S is a C-semigroup, there is a cat. unit e_ν such that $a = ae_\nu$, hence $Se_\alpha S \subset SaS =$ $= Sae_\nu S \subset Se_\nu S$. By the supposition $Se_\alpha S = Se_\nu S$. Hence $Se_\alpha S = SaS$. We have proved: For every $a \in S - M_\alpha$ we have $I^{(a)} = I^{(e_\alpha)}$. Hence $S = M_\alpha \cup I^{(e_\alpha)}$, so that M_α is a maximal two-sided ideal of S (contained in L_α).

b) Suppose that there is a couple of cat. units e_α, e_β such that $e_\beta \in Se_\alpha S$ and $Se_\beta S \subsetneqq Se_\alpha S$. The maximal left ideal L_β cannot contain a maximal two-sided ideal M_β of S, since then $I^{(e_\beta)}$ would be a maximal $\mathscr{I}$-class, a contradiction to the fact that $I^{(e_\beta)} \lneqq I^{(e_\alpha)}$. The proof of Proposition 6,2 is complete.

Theorem 6,7. *Let S be a C-semigroup containing at least one maximal two-sided ideal. Then M^* does not contain a cat. unit of S iff S satisfies Condition A.*

Proof. a) Suppose that Condition A is satisfied and suppose for an indirect proof that M^* contains a cat. unit e^*. By Theorem 4,1 there is a maximal left ideal L_0 which does not contain e^*. The maximal two-sided ideal M_0 contained in L_0 (which exists by the supposition) does not contain e^*. This is an apparent contradiction, since M^* is the intersection of all maximal two-sided ideals.

b) Suppose that M^* does not contain a cat. unit. It follows from the decomposition $S = [\bigcup_{e_\alpha \in A} I^{(e_\alpha)}] \cup M^*$ that every $\mathscr{I}$-class containing a cat. unit is a maximal $\mathscr{I}$-class. By Proposition 6,1 S satisfies Condition A.

We have seen above that a C-semigroup containing maximal two-sided ideals can be written in the form

$$S = [\bigcup_{\lambda \in H} I^{(\lambda)}] \cup M^* . \tag{3}$$

Suppose now that S satisfies Condition A. Then we always have $M^* . [\bigcup_{\lambda \in H} I^{(\lambda)}] = M^*$. Indeed, to any $a \in M^*$ there is a cat. unit $e_\alpha \in S$ such that $ae_\alpha = a$. Since M^* does not have cat. units, we have $e_\alpha \in S - M^*$. Therefore the decomposition (3) is not 0-direct unless $M^* = 0$. In this last case M^* is, of course, 0-isolated and each $I^{(\lambda)} \cup \{0\}$ is a 0-simple C-semigroup. We have proved

Theorem 6,8. *Let S be a C-semigroup satisfying Condition A. Then S is a 0-direct union of 0-simple C-semigroups iff $M^* = 0$.*

Remark. It may occur (independently of whether S satisfies Condition A or not) that in the decomposition $S = [\bigcup_{\lambda \in H} P^{(\lambda)}] \cup M^*$ all summands are 0-simple C-semigroups. But even in such a decomposition the union need not be 0-direct. Consider, e.g., Example 5,1. Here $M^* = \{e_\beta, v, u, e, 0\}$. Denote $P^{(e_\alpha)} = I^{(e_\alpha)} \cup \{0\} = \{0, e_\alpha\}$. Then $S = P^{(e_\alpha)} \cup M^*$ is a decomposition into quasidisjoint completely 0-simple C-semigroups, but this decomposition is not 0-direct, since $P^{(e_\alpha)} . M^* = \{0, v, e\} \neq 0$, $M^* P^{(e_\alpha)} = \{0, u, e\} \neq 0$. Note that e, which is not a cat. unit of S, is a cat. unit of M^*. Note also that in this example Condition A is not satisfied. We have $P^{(e_\alpha)} M^* \neq 0$, but not $P^{(e_\alpha)} . M^* = M^*$.

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