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ON THE LATTICES OF KERNELS OF ISOTONIC MAPPINGS II*)

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The present paper is a continuation of the papers [5–8]; it is particulary in a direct connection to the paper [5]. Given a mapping $f : X \to Y$; then the equivalence $f^{-1}f$ is called the kernel of f. If A is a set with a partial ordering u, then $F(A; u)$ denotes the set of all kernels of isotonic mappings, the domains of which are u-ordered subsets in A. $G(A; u)$ is the set of all kernels of isotonic mappings with u-ordered domain A. In the first part we investigate interrelations between the complete lattices $(F(A; u); \subseteq)$ and $(G(A; u); \subseteq)$. Particulary, we show that $(F(A; u); \subseteq)$ is determined by its principal filter $G(A; u)$ (sections 8 and 10). Furthermore, the relationship between posets $(A; u)$ and $(B; v)$, which is logically equivalent to the isomorphism of the lattices $(F(A; u); \subseteq)$ and $(F(B; v); \subseteq)$ is characterized (section 22). In the second part compact elements in $(G(A; u); \subseteq)$ and in $(F(B; v); \subseteq)$ are characterized (sections 28 and 32). It follows from this characterization that the lattices $(G(A; u); \subseteq)$ and $(F(A; u); \subseteq)$ are algebraic (sections 30 and 33). Let $\sigma \in G(A; u)$, let $\rangle, \sigma\rangle$ be the principal ideal in $(G(A; u); \subseteq)$ determined by the element σ and let Δ_σ be the set of all dual atoms in $(\rangle, \sigma\rangle; \subseteq)$. In the third part a certain Galois' correspondence between $(\rangle, p\rangle; \subseteq)$ and $(\exp \Delta_\sigma; \subseteq)$ is investigated (sections 41 and 44) and particulary, all elements of $\rangle, \sigma\rangle$ are proved to be closed in this correspondence. Finally, for $\varrho, \sigma \in G(A; u)$, $\varrho \subseteq \sigma$ the interval $\langle \varrho, \sigma\rangle$, ordered by inclusion, is proved to be reducible into a complete direct product of some complete lattices $(G(X_i; u_i); \subseteq)$, $i \in I$ (section 51).

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INTRODUCTORY REMARKS. ON CERTAIN RELATIONS BETWEEN THE LATTICES $(F(A; u); \subseteq)$, $(G(A; u); \subseteq)$ AND THE POSET $(A; u)$

1. A short account of symbols and terminology. This paper is a direct continuation of the paper [5], from which we take the symbols and terminology. In [6], [7] and [8], some modifications of this terminology have been made. We shall recall these modifications as well as some frequent symbols (see [5], section 1).

(x, y) denotes an ordered pair of elements x, y. If X is a set and τ is a relation (= binary relation), then (X, τ) denotes the relational structure with X as the underlying set, on which the considered relation is $\tau \cap (X \times X)$. The standard notation for the system of all subsets of X is $\exp X$. The composition of relations ϱ, σ is denoted by $\varrho\sigma$. For any set X we put $X^2 =_{\mathrm{Df}} X \times X$ (careful!, this symbol has nothing to do with $\sigma^2 =_{\mathrm{Df}} \sigma\sigma$ for a relation σ). Let us define

$$D(X) =_{\mathrm{Df}} \{\sigma \mid \sigma \subseteq X^2,\ \sigma^{-1} = \sigma,\ \sigma^2 \subseteq \sigma\}\,,$$

$$E(X) =_{\mathrm{Df}} \{\sigma \mid \sigma \in D(X),\ \mathrm{id}_X \subseteq \sigma\}\,,$$

$$\mathscr{U}(X) =_{\mathrm{Df}} \{u \mid u \subseteq X^2,\ u \cap u^{-1} = \mathrm{id}_X,\ u^2 \subseteq u\}\,;$$

the elements of $D(X)$ are called equivalences *in* X, the elements of $E(X)$ are called equivalences *on* X, and the elements of $\mathscr{U}(X)$ are called partial orderings on X. If $X \neq \emptyset$ and $\varrho \in E(X)$, then X/ϱ is the quotient set of X factorized by ϱ; for $\sigma \in D(X)$ and $\sigma \neq \emptyset$ put $X/\sigma =_{\mathrm{Df}} \operatorname{dom} \sigma/\sigma$; let us define $X/\emptyset =_{\mathrm{Df}} \{\emptyset\}$. If X is an arbitrary set and τ is an arbitrary equivalence, then we define $X/\tau =_{\mathrm{Df}} X/(\tau \cap X^2)$ (it is $\tau \cap X^2 \in D(X)$).

For $u \in \mathscr{U}(X)$ and $\sigma \in D(X)$ let us define

$$u_\sigma =_{\mathrm{Df}} \bigcup_{n=0}^{\infty} \sigma(u\sigma)^n\,, \quad \sigma_u =_{\mathrm{Df}} u_\sigma \cap (u_\sigma)^{-1}\,;$$

for $Y, Z \in X/\sigma$ put $(Y, Z) \in u_{X/\sigma}$ iff either $Y = Z = \emptyset$ or $Y \neq \emptyset \neq Z$ and, for every $y \in Y$ and $z \in Z$, $(y, z) \in u_\sigma$; for $U, V \in \exp X$ put $(U, V) \in \dot{u}$ iff either $U = V = \emptyset$ or there exist $y \in U$, $z \in V$ with $(y, z) \in u$; finally, we define

$$\boldsymbol{u}_{X/\sigma} =_{\mathrm{Df}} \bigcup_{n=1}^{\infty} (\dot{u} \cap (X/\sigma)^2)^n\,.$$

(According to [5], section 17 it is $\boldsymbol{u}_{X/\sigma} = u_{X/\sigma}$, and therefore we furthere use only symbol $u_{X/\sigma}$.) We put

$$F(X; u) =_{\mathrm{Df}} \{\varrho \mid \varrho \in D(X),\ \varrho_u = \varrho\},\ G(X; u) =_{\mathrm{Df}} F(X; u) \cap E(X)\,.$$

The notation for intervals in a poset $(X; u)$ can be found in [7], section 1. E.g. given $a, b \in X$

$$\langle a, b\rangle =_{\mathrm{Df}} \{x \mid x \in X,\ (a, x) \in u,\ (x, b) \in u\}\ ,$$

$$\langle a, b\langle\ =_{\mathrm{Df}} \{x \mid x \in X,\ (a, x) \in u,\ (x, b) \in u - \mathrm{id}_X\}\ ,$$

$$\rangle, a\rangle\ =_{\mathrm{Df}} \{x \mid x \in X,\ (x, a) \in u\}\ ,$$

and so on. For $a, b \in X$ in $(X; u)$ we define

$$[a, b] =_{\mathrm{Df}} \langle a, b\rangle \cup \langle b, a\rangle \cup \{a, b\}\ .$$

If we want to stress that $\langle a, b\rangle$ or $[a, b]$ are considered in $(X; u)$, we write: $\langle a,b\rangle_{(X;u)}$, $[a, b]_{(X;u)}$ and so on. The relation of covering in $(X; u)$ is denoted by $-\langle_{(X;u)}$ or, shorter, by $-\langle$; thus $x -\langle_{(X;u)} y$ iff card $\langle x, y\rangle_{(X;u)} = 2$. For $Y \in \exp X$ in a poset $(X; u)$ we define

$$k_u(Y) =_{\mathrm{Df}} \bigcup\{[x, y]_{(X;u)} \mid x, y \in Y\}\ ;$$

$k_u(Y)$ is the u-convex cover of the subset Y (in X). An equivalence σ in X is called u-convex (in X) if and only if all $Y \in X/\sigma$ are u-convex subsets in dom σ. The symbol $K(X; u)$ denotes the set of all u-convex equivalences in X and further we define $\bar{K}(X; u) =_{\mathrm{Df}} E(X) \cap K(X; u)$. For $\sigma \in D(X)$ we define

$$\bar{\sigma}_{(X;u)} =_{\mathrm{Df}} \bigcap\{\varrho \mid \varrho \in \bar{K}(X; u),\ \sigma \subseteq \varrho\}$$

(according to [7], section 5, $\bar{K}(X; u)$ is an algebraic system of closed elements in $(E(A); \subseteq)$). According to [5], section 36, $F(X; u) \subseteq K(X; u)$, and according to [5], section 41, for $\sigma \in F(X; u)$ we have

$$\bar{\sigma}_{(X;u)} = (\mathrm{id}_X \cup \bigcup\{(k_u(Y))^2 \mid Y \in X/\sigma\}) \in G(X; u)\ ,$$

and for $Y, Z \in X/\sigma$, $Y \neq Z$ we have $k_u(Y) \cap k_u(Z) = \emptyset$. If $(X; u)$ is fixed, then instead of $\bar{\sigma}_{(X;u)}$ we simply write symbol $\bar{\sigma}$.

In the whole paper, A is a given set and u is a given ordering on A.

The most frequented proof technique used in [5–8] (and also the present paper) is contained in the following statement, which characterizes elements of $F(A; u)$.

2. Lemma. *Let $\sigma \in D(A)$. Then the following statements are equivalent:*

(i) $\sigma \in F(A; u)$.

(ii) *If $n \geqq 1$ is a natural number, if $X_0, X_1, \ldots, X_n \in A/\sigma$ and if for all $i = 0, \ldots, n - 1$ the relations $(X_i, X_{i+1}) \in \dot{u}$ and $(X_n, X_0) \in \dot{u}$ hold, then $X_0 = X_1 = \ldots = X_n$.*

(ii′) *The relational structure* $(A/\sigma; u_{A/\sigma})$ *is a poset.*

(iii) *There exist a poset* $(B; v)$ *and an isotonic mapping* $f : (\text{dom}\,\sigma; u) \nearrow (B; v)$ *) *such that* $\sigma = \ker f$.

Proof. See [5], section 17 and 19 (the equivalence (i) $\Leftrightarrow$ (ii); the same concerns the equivalence (i) $\Leftrightarrow$ (ii′); if we consider [5], sections 16) and [5] sections 45 and 47 (equivalence (i) $\Leftrightarrow$ (iii)).

3. Remark. In [5–8] we have consequently supposed, that the set A is non-void. This assumption is unnecessary in [5] as well as in the present paper, because for $A = \emptyset$ the results in [5] are mostly trivial, or evidently false (e.g., theorem 52 in [5]; in this case, the trouble is, that for $X \neq \emptyset$ and $Y = \emptyset$ there does exists no mapping $h : X \to Y$). The necessary revision of the results in case that $A = \emptyset$ is left to the reader. It is very easy, anyway, when we consider the following statements. (The section number, where symbols in [5] are introduced, is sometimes written in brackets ( ). See also section 1 above.)

Let $A = \emptyset$, $u = \emptyset$, $\varrho = \emptyset$ and let σ be a relation. Then:

$$\exp A = \{\emptyset\};\ D(A) = \{\emptyset\};\ E(A) = \{\emptyset\};\ \mathscr{U}(A) = \{\emptyset\};\ A/\varrho = \{\emptyset\}\ (4/\text{a});$$

$\text{id}_A = \emptyset$; $\varrho\sigma = \sigma\varrho = \emptyset$; $\text{dom}\,\varrho = \text{cod}\,\varrho = \emptyset$ (1; $\text{cod}\,\sigma =_{\text{Df}} \text{dom}\,\sigma^{-1}$); $\varrho^0 = \emptyset$ (1); for $n = 1, 2, \ldots$ it is $\varrho^n = \emptyset$ (1); $\varrho^{-1} = \emptyset$ (1);

$$u_\varrho = \emptyset\ ;\quad \varrho_u = \emptyset\ ;\quad \dot{u} = \{(\emptyset, \emptyset)\}\ ;\quad u_{A/\varrho} = \{(\emptyset, \emptyset)\}\ ;$$

$F(A; u) = G(A; u) = \{\emptyset\}$; $-\!\!<_{(A;u)} = \emptyset$; for $X \subseteq A$ we have $k_u(X) = \emptyset$; for $\tau \in D(A)$ we have $\bar{\tau}_{(A;u)} = \emptyset$; $K(A; u) = \bar{K}(A; u) = \{\emptyset\}$;

if $f : A \to B$, then $f = \emptyset$ and $\ker f = \emptyset$; if $f : B \to A$ then $B = \emptyset$ and $f = \emptyset$; if $f : (A; u) \nearrow (B; v)$ or $f : (A; u) \searrow (B; v)$ then $f = \emptyset$ (44, where we define $f : (A; u) \searrow (B; v)$ iff $f : (A; u) \nearrow (B; v^{-1})$ – an antitonic mapping); for $\text{nat}\,\varrho : A \to A/\varrho$ we have $\text{nat}\,\varrho = \emptyset$ (44); if also $\sigma = \emptyset$ the $\sigma/\varrho = \{(\emptyset, \emptyset)\}$ (54), $(A/\varrho)/(\sigma/\varrho) = \{\emptyset\}/\{(\emptyset, \emptyset)\} = \{\{\emptyset\}\}$, and for $\text{nat}\,(\sigma/\varrho) : A/\varrho \to (A/\varrho)/(\sigma/\varrho)$ we have $\text{nat}\,(\sigma/\varrho) = \{(\emptyset, \{\emptyset\})\}$.

In the present paper we assume in all proofs that the set A is non-void, unless explicitly stated otherwise; for $A = \emptyset$ the statements are trivial.

4. Remark. In section 5–10 some interrelations between the systems $F(A; u)$ and $G(A; u)$ are studied. If $F(A; u)$ is given, then we clearly know the system $G(A; u)$ because $G(A; u) = \langle \text{id}_A, \langle_{(F(A;u);\subseteq)}$. It is rather interesting that also the converse holds. (See section 8). Nevertheless, the complete lattices $(G(A; u); \subseteq)$ have a number of properties, which do not take place in the complete lattices $(F(A; u); \subseteq)$ (e.g. see [7], section 24/a). On the other hand, the proof of theorem 22 is substantially based on some particular properties of the system $F(A; u)$.

*) For the notation [5], section 44 (page 140); recall that for $f: X \to Y$ we define the equivalence $\ker f$ by $\ker f =_{\text{Df}} f^{-1} f$ (i.e. for $x, y \in X$ we have $(x, y) \in \ker f$ iff $f(x) = f(y)$).

5. Lemma. *Let σ be an equivalence in A. Then*

$$u_{A/\sigma} = (u \cap (\mathrm{dom}\,\sigma)^2)_{\mathrm{dom}\sigma/\sigma}\,.$$

Proof. Assume at first that $\sigma \neq \emptyset$. Put $B =_{\mathrm{Df}} \mathrm{dom}\,\sigma$ and $v = u \cap B^2$. According to the convention mentioned in section 1 (or in [5], section 4/a) we have $B/\sigma = A/\sigma$. Let $X, Y \in A/\sigma$ and $(X, Y) \in \dot{u}$. Then there exist elements $x \in X$ and $y \in Y$ for which $(x, y) \in u$. Also $X, Y \subseteq B$ and therefore $(x, y) \in u \cap B^2 = v$. If, conversely, $X_1, Y_1 \in B/\sigma$ and $(X_1, Y_1) \in \dot{v}$, then there exist elements $x_1 \in X_1$ and $y_1 \in Y_1$, for which $(x_1, y_1) \in v$. It is $v = u \cap B^2$, and therefore also $(x_1, y_1) \in u$. Then we get relation $(X_1, Y_1) \in \dot{u}$. Considering $A/\sigma = B/\sigma$ we finally obtain

$$\dot{u} \cap (A/\sigma)^2 = \dot{v} \cap (B/\sigma)^2\,. \tag{1}$$

The quasiordering $\boldsymbol{u}_{A/\sigma}$ is the transitive closure of the relation $\dot{u} \cap (A/\sigma)^2$ on (A/σ) (see section 1) and $\boldsymbol{v}_{B/\sigma}$ is the transitive closure of $\dot{v} \cap (B/\sigma)^2$ on B/σ. Thus from (1) and from the fact that $A/\sigma = B/\sigma$ we get the proof of our proposition in case that $\sigma \neq \emptyset$.

If $\sigma = \emptyset$, then $\mathrm{dom}\,\sigma = \emptyset$, $A/\sigma = \{\emptyset\} = \mathrm{dom}\,\sigma/\sigma$, $\dot{u} \cap (A/\sigma)^2 = \{(\emptyset, \emptyset)\}$ and hence

$$u_{A/\sigma} = \bigcup_{n=1}^{\infty} (\dot{u} \cap (A/\sigma)^2)^n = \{(\emptyset, \emptyset)\}\,,$$
$$(u \cap (\mathrm{dom}\,\sigma)^2_{\mathrm{dom}\sigma/\sigma} = (u \cap \emptyset^2)_{\{\emptyset\}} = \{(\emptyset, \emptyset)\}$$

(see section 3). Hence, our proposition holds for $\sigma = \emptyset$ too.

6. Theorem. *Let $X \subseteq A$. Then*

$$G(X; u) = \{\sigma \cap X^2 \mid \sigma \in G(A; u)\}\,.$$

Proof. Let $X \neq \emptyset$ (for $X = \emptyset$ is the theorem trivial – see section 3). According to our convention from section 1 we have $(X; u) = (X; u \cap X^2)$ and $u \cap X^2 \in \mathscr{U}(X)$, so that the symbol $G(X; u)$ makes sense.

Let $\sigma \in G(X; u)$. Then $\mathrm{dom}\,\sigma = X$. By lemma 5 we have $(u \cap X^2)_{X/\sigma} = u_{A/\sigma}$, and according to section 2 the relation $(u \cap X^2)_{X/\sigma}$ is an ordering on X/σ. Since $X/\sigma = A/\sigma$, $(A/\sigma, u_{A/\sigma})$ is a poset. Therefore, via section 2, we see that $\sigma \in F(A; u)$. According to [5], section 41, $\bar{\sigma}_{(A;u)} \in G(A; u)$, and, for $x, y \in X$, $(x, y) \in \sigma$ holds if and only if $(x, y) \in \bar{\sigma}_{(A;u)}$. Hence $\sigma = \bar{\sigma}_{(A;u)} \cap X^2$ and $\bar{\sigma}_{(A;u)} \in G(A; u)$ and we get inclusion

$$G(X; u) \subseteq \{\sigma \cap X^2 \mid \sigma \in G(A; u)\}\,.$$

Let us derive the converse inclusion. Let $\sigma \in G(A; u)$. Then $\mathrm{dom}\,\sigma = A$ and so $\sigma \cap X^2$ is an equivalence on X: We denote $\varrho =_{\mathrm{Df}} \sigma \cap X^2$ and $v =_{\mathrm{Df}} u \cap X^2$. Let $n \geqq 1$ be a natural number, let $X_0, \ldots, X_n \in X/\varrho$ and let $(X_i, X_{i+1}) \in \dot{v}$, $(X_n, X_0) \in \dot{v}$

for all $i = 0, \ldots, n - 1$. For $Y \in X/\varrho$ there exists exactly one element $\bar{Y} \in A/\sigma$ with $Y \subseteq \bar{Y}$; and for all $Z \in X/\varrho$ the inclusion $Z \subseteq \bar{Y}$ implies $Y = Z$. From the definition $v =_{\mathrm{Df}} u \cap X^2$ we get relations $(\bar{X}_i, \bar{X}_{i+1}) \in \dot{u}$, $(\bar{X}_n, \bar{X}_0) \in \dot{u}$; then $\sigma \in G(A; u)$ and hence, via section 2, $\bar{X}_0 = \ldots = \bar{X}_n$. So we get that also $X_0 = \ldots = X_n$. Therefore, it follows from lemma 2 that $\varrho \in G(X; v) = G(X; u)$. Thus we see that the converse inclusion holds:

$$\{\sigma \cap X^2 \mid \sigma \in G(A; u)\} \subseteq G(X; u) .$$

7. Theorem. *We have*

$$F(A; u) = \bigcup\{G(X; u) \mid X \subseteq A\} ,$$

*and the union on the right side of the equality is disjoint.**)

Proof. Let $\varrho \in F(A; u)$. Then, according to [5], section 41, $\bar{\varrho}_{(A;u)} = \bar{\varrho} \in G(A; u)$ and $\bar{\varrho} \cap (\operatorname{dom} \varrho)^2 = \varrho$. From theorem 6 we get $\varrho \in G(\operatorname{dom} \varrho; u)$ and hence $F(A; u) \subseteq$ $\subseteq \bigcup\{G(X; u) \mid X \subseteq A\}$, because $\operatorname{dom} \varrho \subseteq A$.

Let us derive the converse inclusion. Let $\varrho \in \bigcup\{G(X; u) \mid X \subseteq A\}$. Then there exist a subset $Y \subseteq A$, with $\varrho \in G(Y; u)$, especially $\operatorname{dom} \varrho = Y$. From section 2 we know that the relation $(u \cap Y^2)_{Y/\varrho}$ is an ordering on Y/ϱ. It is clear that $\varrho \in D(A)$, and therefore, according to section 5, we have $(u \cap Y^2)_{Y/\varrho} = u_{A/\varrho}$. Thus $(A/\varrho, u_{A/\varrho})$ is a poset and from section 2 we get that $\varrho \in F(A; u)$ and the inclusion $\bigcup\{G(X; u) \mid X \subseteq$ $\subseteq A\} \subseteq F(A; u)$ is proved.

We will show, finally, that the union $\bigcup\{G(X; u) \mid X \subseteq A\}$ is disjoint. If $X, Y \subseteq A$ and $X \neq Y$ then for $\varrho \in G(X; u)$ and $\sigma \in G(Y; u)$ we get $\operatorname{dom} \varrho = X \neq Y = \operatorname{dom} \sigma$. Therefore $\varrho \neq \sigma$.

8. Corollary. *We have*

$$F(A; u) = \{\sigma \cap X^2 \mid \sigma \in G(A; u),\ X \subseteq A\} .$$

Proof. Direct from sections 6 and 7.

9. Corollary. *We have*

$$D(A) = \bigcup\{E(X) \mid X \subseteq A\} .$$

Proof. Direct from the section 7 if we consider that, by lemma 2, $D(A) = F(A; \mathrm{id}_A)$, $E(X) = G(X; \mathrm{id}_A)$ for all $X \subseteq A$ (see also [7], section 30)

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*) This means that for $X_1, X_2 \in \exp A$, $X_1 \neq X_2$ we have $G(X_1; u) \cap G(X_2; u) = \emptyset$.

10. Corollary. *Let $u, v \in \mathscr{U}(A)$. Then the following hold:*

a) $F(A; u) \subseteq F(A; v)$ *iff* $G(A; u) \subseteq G(A; v)$.
b) $F(A; u) = F(A; v)$ *iff* $G(A; u) = G(A; v)$.
c) $F(A; u) \subset F(A; v)$ *iff* $G(A; U) \subset G(A; v)$.

Proof. a) Let $F(A; u) \subseteq F(A; v)$. Then, by [5] section 21, we have

$$G(A; u) = \{\sigma \mid \sigma \in F(A; u), \mathrm{id}_A \subseteq \sigma\} \subseteq \{\sigma \mid \sigma \in F(A; v), \mathrm{id}_A \subseteq \sigma\} = G(A; v).$$

If $G(A; u) \subseteq G(A; v)$ then – according to theorem 6 – for all $X \subseteq A$,

$$G(X; u) = \{\sigma \cap X^2 \mid \sigma \in G(A; u)\} \subseteq \{\sigma \cap X^2 \mid \sigma \in G(A; v)\} = G(X; v).$$

Therefore, according to theorem 7,

$$F(A; u) = \bigcup\{G(X; u) \mid X \subseteq A\} \subseteq \bigcup\{G(X; v) \mid X \subseteq A\} = F(A; v).$$

b) From (a) we get that,

$$(F(A; u) = F(A; v)) \Leftrightarrow (F(A; u) \subseteq F(A; v) \subseteq F(A; u)) \Leftrightarrow$$
$$\Leftrightarrow (G(A; u) \subseteq G(A; v) \subseteq G(A; u)) \Leftrightarrow (G(A; u) = G(A; v)).$$

c) This statement is a direct consequence of (a) and (b).

11. Remark. We can see almost immediately, that from the existence of an isotone isomorphism of posets $(A; u)$ and $(B; v)$ there follows the existence of an isomorphism of the complete lattices $(F(A; u); \subseteq)$ and $(F(B; v); \subseteq)$; analogously for the complete lattices $(G(A; u); \subseteq)$ and $(G(B; v); \subseteq)$ (see sections 19 and 20). In section 22 we investigate one of the converse questions: what is the relation between posets $(A; u)$ and $(B; v)$ if the lattices $(F(A; u); \subseteq)$ and $(F(B; v); \subseteq)$ are isomorphic. There remains an open problem:

Characterize the relation between posets $(A; u)$ and $(B; v)$ which is equivalent to the fact that the lattices $(G(A; u) \subseteq)$ and $(G(B; v); \subseteq)$ are isomorphic.

12. Lemma. *Let $(A; u)$ and $(B; v)$ be posets and let $\varphi : F(A; u) \to F(B; v)$ be an isomorphism from the complete lattice $(F(A; u); \subseteq)$ onto the complete lattice $(F(B; v); \subseteq)$. For every $x \in A$ we define*

$$\varphi^*(x) = y \Leftrightarrow_{\mathrm{Df}} \varphi(\{x\}^2) = \{y\}^2. \tag{2}$$

Then $\varphi^* : A \to B$ is a bijection.

Proof. Due to section 2 we have $\{x\}^2 \in F(A; u)$ and $\{y\}^2 \in F(B; v)$ for all $x \in A$ and $y \in B$. An equivalence ϱ is an atom in $(F(A; u); \subseteq)$ if and only if there exist an element $x \in A$ with $\varrho = \{x\}^2$; analogously for $(F(B; v) \subseteq)$. The mapping φ is an isomorphism from $(F(A; u); \subseteq)$ onto $(F(B; v) \subseteq)$ and hence both the φ-image of an atom in $(F(A; u); \subseteq)$ is an atom in $(F(B; v); \subseteq)$ and the φ – preimage of an atom in $(F(B; v); \subseteq)$ is an atom in $(F(A; u); \subseteq)$. Since moreover $\varphi : F(A; u) \to F(B; v)$ is an injection, the proposition follows.

13. Lemma. *Let $(A; u)$ and $(B; v)$ be posets; let $\varphi : F(A; u) \to F(B; v)$ be an isomorphism from the complete lattice $(F(A; u); \subseteq)$ onto the complete lattice $(F(B; v); \subseteq)$. Define*

$$(3) \qquad (x, y) \in w(\varphi) \Leftrightarrow_{\mathrm{Df}} (\varphi^*(x), \varphi^*(y)) \in v\,.$$

for all $x, y \in A$. Then the mapping $\varphi^* : A \to B$ is an isotone isomorphism from the poset $(A; w(\varphi))$ onto $(B; v)$.

Proof. Due to section 12 we see that $\varphi^* : A \to B$ is a bijection. The relational structure $(B; v)$ is a poset and hence it follows directly from the definition (3) that $(A; w(\varphi))$ is a poset, which is φ^* – isotone isomorphic to $(B; v)$.

14. Lemma. *Let $(A; u)$ and $(B; v)$ be posets and let $\varphi : F(A; u) \to F(B; v)$ be an isomorphism from the complete lattice $(F(A; u); \subseteq)$ onto the complete lattice $(F(B; v); \subseteq)$. Then*

$$\varphi(\sigma) = \{(\varphi^*(x), \varphi^*(y)) \mid (x, y) \in \sigma\}$$

for all $\sigma \in F(A; u)$. *In particular,*

$$\varphi(\mathrm{id}_A) = \mathrm{id}_B\,.$$

Proof. Let us denote

$$\psi(\sigma) =_{\mathrm{Df}} \{(\varphi^*(x), \varphi^*(y)) \mid (x, y) \in \sigma\}$$

for $\sigma \in F(A; u)$. Let $x, y \in A$ with $x \neq y$. Applying lemma 2 we conclude that $\emptyset$, $\{x\}^2, \{y\}^2, \{x\}^2 \cup \{y\}^2$ and $\{x, y\}^2$ are elements of $F(A; u)$. The diagram of the poset $(\rangle, \{x, y\}^2\rangle_{(F(A;u);\subseteq)}; \subseteq)$ is shown in fig. 1. a. Since $\varphi : F(A; u) \to F(B; v)$ is a lattice – isomorphism, we get from (2) (via section 12) that $\varphi(\{x\}^2) = \{\varphi^*(x)\}^2$ and $\varphi(\{y\}^2) = \{\varphi^*(y)\}^2$, therefore $\varphi(\{x, y\}^2) = \{\varphi^*(x), \varphi^*(y)\}^2 = \psi(\{x, y\}^2)$: The element $\{x\}^2 \cup \{y\}^2$ in $(F(A; u) \subseteq)$ is covered both by $\{x, y\}^2$ and by all elements of the form $\{x\}^2 \cup \{y\}^2 \cup \{z\}^2$, where $z \in A - \{x, y\}$ (these equivalences are elements of $F(A; u)$ according to section 2; the situation in $(F(A; u); \subseteq)$ is shown on the diagram in figure 1. b).

Moreover

$$\varphi(\{x\}^2 \cup \{y\}^2 \cup \{z\}^2) = \varphi(\sup_{(F(A;u);\subseteq)} \{\{x\}^2, \{y\}^2, \{z\}^2) =$$

$$= \sup_{(F(B;v);\subseteq)} \{\varphi(\{x\}^2), \varphi(\{y\}^2), \varphi(\{z\}^2)\} = \{\varphi^*(x)\}^2 \cup \{\varphi^*(y)\}^2 \cup \{\varphi^*(z)\}^2,$$

because $\varphi : F(A; u) \to F(B; v)$ is an isomorphism. Taking into account that φ is bijection we get $\varphi(\{x, y\}^2) = \{\varphi^*(x), \varphi^*(y)\}^2$.

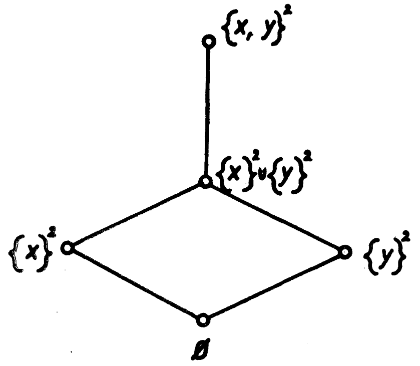

Fig. 1. a

For $x, y \in A$ and $x = y$ the equivality $\varphi(\{x, y\}^2) = \psi(\{x, y\}^2)$ follows directly from (2). Thus for all $x, y \in A$ we have

$$\{x, y\}^2 \in F(A; u), \quad \varphi(\{x, y\}^2) = \psi(\{x, y\}^2). \tag{4}$$

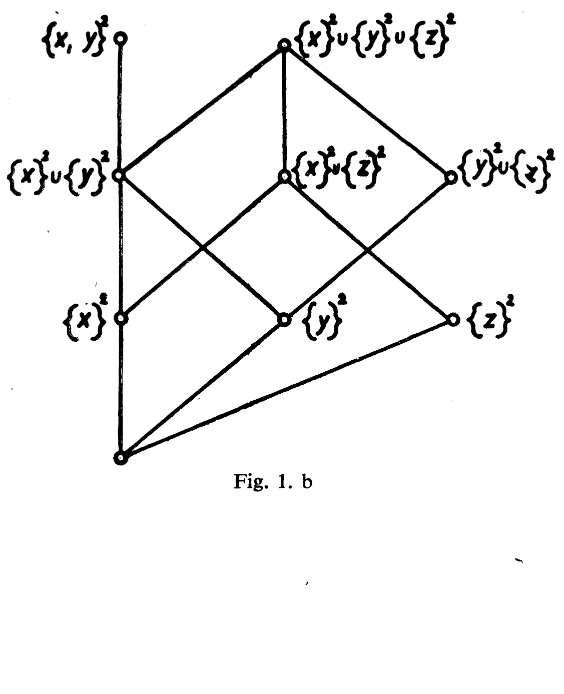

Fig. 1. b

Let $\sigma \in F(A; u)$. Then

$$(5) \qquad \sigma = \bigcup\{\{x, y\}^2 \mid (x, y) \in \sigma\} = \sup_{(F(A;u;\subseteq)} \{\{x, y\}^2 \mid (x, y) \in \sigma\} .$$

An equequivality analogous to (5) holds in the complete lattice $(F(B; v); \subseteq)$. Since φ is an isomorphism from the complete lattice $(F(A; u); \subseteq)$ onto the complete lattice $(F(B; v); \subseteq)$, it follows from (4) and (5) that

$$\varphi(\sigma) = \varphi(\sup_{(F(A;u);\subseteq)} \{\{x, y\}^2 \mid (x, y) \in \sigma\}) = \sup_{(F(B;v);\subseteq)} \{\varphi(\{x, y\}^2) \mid (x, y) \in \sigma\} =$$
$$= \sup_{(F(B;v);\subseteq)} \{\{\varphi^*(x), \varphi^*(y)\}^2 \mid (x, y) \in \sigma\} .$$

Since $\{\varphi^*(x), \varphi^*(y)\}^2 \in F(B; v)$ for all $(x, y) \in \sigma$ we get from the definition of supremum

$$\psi(\sigma) = \{(\varphi^*(x), \varphi^*(y)) \mid (x, y) \in \sigma\} \subseteq$$
$$\subseteq \sup_{(F(B;v);\subseteq)} \{\{\varphi^*(x), \varphi^*(y)\}^2 \mid (x, y) \in \sigma\} = \varphi(\sigma) .$$

Let us suppose, to the contrary, that $(r, s) \in \varphi(\sigma)$; then $\{r, s\}^2 \subseteq \varphi(\sigma)$. The mapping $\varphi^* : A \to B$ is a bijection and therefore there exist $x_1, y_1 \in A$, with $r = \varphi^*(x_1)$ and $s = \varphi^*(y_1)$. As $\varphi : F(A; u) \to F(B; v)$ is an isomorphism, we have

$$\varphi^{-1}(\{r, s\}^2) = \{x_1, y_1\}^2 \subseteq \sigma$$

(consider, that $\varphi(\{x_1, y_1\}^2) = \{r, s\}^2 \subseteq \varphi(\sigma)$ and that $\varphi : F(A; u) \to F(B; v)$ is an isomorphism). So we get, that $(x_1, y_1) \in \sigma$ and

$$(r, s) = (\varphi^*(x_1), \varphi^*(y_1)) \in \{(\varphi^*(x), \varphi^*(y)) \mid (x, y) \in \sigma\} = \psi(\sigma)$$

and the converse inclusion is proved:

$$\varphi(\sigma) \subseteq \{(\varphi^*(x), \varphi^*(y)) \mid (x, y) \in \sigma\} = \psi(\sigma) ,$$

and so the equality $\varphi = \psi$ holds.

From $\varphi = \psi$ and from the fact that $\mathrm{id}_A \in F(A; u)$ and $\varphi^* : A \to B$ is a bijection there follows:

$$\varphi(\mathrm{id}_A) = \{(\varphi^*(x), \varphi^*(x)) \mid x \in A\} = \mathrm{id}_B .$$

15. Corollary. *Let $(A; u)$ and $(B; v)$ be posets and let $\varphi : F(A; u) \to F(B; v)$ be an isomorphism from the complete lattice $(F(A; u); \subseteq)$ onto the complete lattice $(F(B; v); \subseteq)$. Then the partial mapping*

$$\varphi \mid G(A; u) : G(A; u) \to G(B; v)$$

is an isomorphism from the complete lattice $(G(A; u); \subseteq)$ onto the complete lattice $(G(B; v); \subseteq)$.

Proof. Due to section 14 we get: $\varphi(\mathrm{id}_A) = \mathrm{id}_B$ and hence $\varphi^{-1}(\mathrm{id}_B) = \mathrm{id}_A$. Furthermore $\sigma \in G(A; u)$ iff $\sigma \in F(A; u)$ and $\mathrm{id}_A \subseteq \sigma$; the mapping $\varphi : F(A; u) \to F(B; v)$ is an isomorphism therefore $\varphi(\sigma) \in F(B; v)$ and $\mathrm{id}_B = \varphi(\mathrm{id}_A) \subseteq \varphi(\sigma)$, i.e. $\varphi(\sigma) \in G(B; v)$. Thus we have $\varphi \mid G(A; u) : G(A; u) \to G(B; v)$. As $\varphi : F(A; u) \to F(B; v)$ is a bijection, the mapping $\varphi \mid G(A; u)$ is an injection. If $\varrho \in G(B; v)$, then analogously $\varphi^{-1}(\varrho) \in G(A; u)$ (also φ^{-1} is an isomorphism) and there exist an element $\sigma = \varphi^{-1}(\varrho) \in G(A; u)$, for which $\varphi(\sigma) = \varrho$. So $\varphi \mid G(A; u) : G(A; u) \to G(B; v)$ is a bijection. This concludes the proof because φ is a surjective isomorphism.

16. Lemma. *Let $(A; u)$ and $(B; v)$ be posets and let $\varphi : F(A; u) \to F(B; v)$ be an isomorphism from the complete lattice $(F(A; u); \subseteq)$ onto the complete lattice $(F(B; v); \subseteq)$. Then*

$$\varphi^{*-1} = \varphi^{-1*} \text{ *)}.$$

Proof. Due to section 12 we get that $\varphi^{-1*} : B \to A$ is a bijection and

$$\varphi^{-1*}(y) = x \Leftrightarrow \varphi^{-1}(\{y\}^2) = \{x\}^2$$

for all $y \in B$, i.e., following (2) (section 12) we get

$$\varphi^{-1*}(y) = x \Leftrightarrow \varphi^*(x) = y .$$

So $\varphi^{-1*}(y) = \varphi^{*-1}(y)$ for all $y \in B$.

17. Corollary. *Let $(A; u)$ and $(B; v)$ be posets and let a mapping $\varphi : F(A; u) \to F(B; v)$ be an isomorphism from the complete lattice $(F(A; u); \subseteq)$ onto the complete lattice $(F(B; v); \subseteq)$. Then*

$$\varphi^{-1}(\sigma) = \{(\varphi^{*-1}(x), \varphi^{*-1}(y)) \mid (x, y) \in \sigma\}$$

for all $\sigma \in F(B; v)$.

Proof. The mapping $\varphi^{-1} : F(B; v) \to F(A; u)$ is an isomorphism from $(F(B; v); \subseteq)$ onto $(F(A; u); \subseteq)$ and the corollary follows directly from sections 14 and 16.

18. Notation. Let X, Y be sets and let $f : X \to Y$ be a mapping. Let us define

$$f_2(\alpha) =_{\mathrm{Df}} \{(f(x), f(y)) \mid (x, y) \in \alpha\} \quad \text{for} \quad \alpha \subseteq X^2 .$$

This defines a mapping f_2: $\exp X^2 \to \exp Y^2$. We recall once more that for $Z \subseteq X$ that $f \mid Z : Z \to Y$ denotes the partial mapping $f \mid Z = f \cap (Z \times Y)$.

---

*) Inverse mapping $\varphi^{-1} : F(B; v) \to F(A; u)$ is an isomorphism from $(F(B; v); \subseteq)$ onto $(F(A; v); \subseteq)$. Therefore, formula (2), applied to the mapping φ^{-1}, defines a bijection $(\varphi^{-1})^* : B \to A$ (see section 12).

19. Lemma. *Let $(A; u)$ and $(B; v)$ be posets and let $f : A \to B$ be an isotonic isomorphism from $(A; u)$ onto $(B; v)$. Then*

$$f_2 \mid F(A; u) : F(A; u) \to F(B; v)$$

is an isomorphism from the complete lattice $(F(A; u); \subseteq)$ onto the complete lattice $(F(B; v); \subseteq)$.

Proof. Since $f : A \to B$ is an injection, thus for every equivalence σ in A

$$f_2(\sigma) = \bigcup\{(f(X))^2 \mid X \in A/\sigma\} \tag{6}$$

is an equivalence in B. Since $F(A; u) \subseteq D(A), f_2 \mid F(A; u)$ is a mapping from $F(A; u)$ into $D(B)$.

$f : A \to B$ is an isotonic isomorphism from $(A; u)$ onto $(B; v)$ and so $(X, Y) \in \dot{u}$ iff $(f(X), f(Y)) \in \dot{v}$ for $X, Y \in \exp A$. From this it follows that for $\sigma \in D(A)$ and $X, Y \in A/\sigma$ the relation $(X, Y) \in u_{A/\sigma}$ holds if and only if $(f(X), f(Y)) \in v_{B/f_2(\sigma)}$ holds. (See the definition of $u_{A/\sigma}$ in section 1 or [5], section 17; from (6) we get that

$$B/f_2(\sigma) = \{f(X) \mid X \in A/\sigma\}.)$$

So for $\sigma \in D(A)$, $(A/\sigma; u_{A/\sigma})$ is a poset iff $(B/f_2(\sigma); v_{B/f_2(\sigma)})$ is a poset. Thus, via section 2, for $\sigma \in D(A)$

$$\sigma \in F(A; u) \Leftrightarrow f_2(\sigma) \in F(B; v)\,. \tag{7}$$

The mapping $f : A \to B$ is a bijection and, therefore, $f_2 : \exp A^2 \to \exp B^2$ is a bijection too. From (7) and from this fact it follows, that $f_2 \mid F(A; u)$ is a bijection from $F(A; u)$ onto $F(B; v)$. It also follows from the bijectivity of f that f_2 is an isomorphism from $(\exp A^2; \subseteq)$ onto $(\exp B^2; \subseteq)$. Hence $f_2 \mid F(A; u)$ is an isomorphism from the complete lattice $(F(A; u); \subseteq)$ onto the complete lattice $(F(B; v); \subseteq)$.

20. Corollary. *Let $(A; u)$ and $(B; v)$ be posets and let $f : A \to B$ be an isotonic isomorphism from $(A; u)$ onto $(B; v)$. Then*

$$f_2 \mid G(A; u) : G(A; u) \to G(B; v)$$

is an isomorphism from the complete lattice $(G(A; u); \subseteq)$ onto the complete lattice $(G(B; v); \subseteq)$.

Proof. It follows directly from lemma 19, because according to the definition of f_2 in section 18, for a bijection $f : A \to B$ it is $f_2(\mathrm{id}_A) = \mathrm{id}_B$.

21. Lemma. *Let $(A; u)$ and $(B; v)$ be poset and let $\varphi : F(A; u) \to F(B; v)$ be an isomorphism from the complete lattice $(F(A; u); \subseteq)$ onto the complete lattice $(F(B; v); \subseteq)$. Then*

$$G(A; u) = G(A; w(\varphi))\ ^*) .$$

Proof. Section 15 implies that $\sigma \in G(A; u)$ iff $\varphi(\sigma) \in G(B; v)$. According to sections 16 and 17 we get for $\sigma \in G(A; u)$ that

$$(8) \qquad (\varphi^{-1*})_2\,(\varphi(\sigma)) = (\varphi^{*-1})_2\,(\varphi(\sigma)) = \\ = \{(\varphi^{*-1}(x), \varphi^{*-1}(y)) \mid (x, y) \in \varphi(\sigma)\} = \varphi^{-1}(\varphi(\sigma)) = \sigma .$$

Via section 13, $\varphi^{*-1} : B \to A$ is an isotonic isomorphism from $(B; v)$ onto $(A; w(\varphi))$. Therefore, due to section 20, we have for $\varrho \in G(B; v)$: $(\varphi^{*-1})_2\,(\varrho) \in G(A; w(\varphi))$. Considering the beginning of the present proof and the equivality (8) we see that $\sigma = (\varphi^{*-1})_2\,(\varphi(\sigma)) \in G(A; w(\varphi))$ for $\sigma \in G(A; u)$. Thus the inclusion $G(A; u) \subseteq$ $\subseteq G(A; w(\varphi))$ is proved.

Conversely, let $\sigma \in G(A; w(\varphi))$. The mapping $\varphi^* : A \to B$ is an isotonic isomorphism from $(A; w(\varphi))$ onto $(B; v)$ (see section 13) and therefore, via section 20, $(\varphi^*)_2\,(\sigma) \in G(B; v)$. Then, according to section 15, $\varphi^{-1}((\varphi^*)_2\,(\sigma)) \in G(A; u)$. If we consider, that according to sections 18 and 14

$$(\varphi^*)_2\,(\sigma) = \{(\varphi^*(x), \varphi^*(y)) \mid (x, y) \in \sigma\} = \varphi(\sigma) ,$$

we see, finally, that

$$\sigma = \varphi^{-1}\,\varphi(\sigma) = \varphi^{-1}((\varphi^*)_2\,(\sigma)) \in G(A; u)$$

and the converse inclusion $G(A; w(\varphi)) \subseteq G(A; u)$ is proved.

22. Theorem. *Let $(A; u)$ and $(B; v)$ be posets. Then the lattices $(F(A; u); \subseteq)$ and $(F(B; v); \subseteq)$ are isomorphic iff there exist such an ordering w on A, for which the posets $(A; w)$ and $(B; v)$ are isotonic isomorphic and for which $G(A; w) = G(A; u)$.*

Proof. Let $\varphi : F(A; u) \to F(B; v)$ be an isomorphism from the complete lattice $(F(A; u); \subseteq)$ onto $(F(B; v); \subseteq)$. Then, via section 13, $\varphi^* : A \to B$ is an isotonic isomorphism from the poset $(A; w(\varphi))$ onto the poset $(B; v)$ and also $G(A; w(\varphi)) =$ $= G(A; u)$ (see lemma 21).

Conversely, let there exist an isotonic isomorphism $f : A \to B$ from the poset $(A; w)$ onto the poset $(B; v)$ and let $G(A; w) = G(A; u)$. Then, following section 19, the mapping $f_2 \mid F(A; w) : F(A; w) \to F(B; v)$ is a lattice-isomorphism from $(F(A; w); \subseteq)$ onto $(F(B; v); \subseteq)$. From $G(A; w) = G(A; u)$ (see section 10/b) it follows that $F(A; w) = F(A; u)$ and so $f_2 \mid F(A; w)$ is a lattice-isomorphism from $(F(A; u); \subseteq)$ onto $(F(B; v); \subseteq)$.

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*) The ordering $w(\varphi)$ on A is defined in (3), section 13.

23. Remark. Let us recall, that $\mathscr{U}(A)$ is the set of all orderings on A (see section 1). We define a relation A_G:

$$(9) \qquad (u, v) \in A_G \Leftrightarrow_{\mathrm{Df}} u, v \in \mathscr{U}(A) \quad \text{and} \quad G(A; u) = G(A; v) .$$

Then A_G is clearly an equivalence on $\mathscr{U}(A)$. The importance of this equivalence follows from theorem 22. In a paper "*On Some Equivalences on the Set of All Orderings of a Given Set*" which is now being prepared, this equivalence is completely characterized. But deriving of properties of A_G is executed by rather slow methods of a combination theory and therefore it has not appeared in this paper.

THE CHARACTERIZATION OF COMPACT ELEMENTS IN $(G(A; u); \subseteq)$ AND $(F(A; u); \subseteq)$; THE ALGEBRAICITY OF THESE LATTICES

24. Lemma. (Ward). *Let $\mathscr{L} = (L; \leqq)$ be a complete lattice; let $\varphi : L \to L$ be closure operator on $\mathscr{L}$. The following holds for $X \subseteq \varphi(L)$:*

$$\sup_{(\varphi(L); \leqq)} X = \varphi(\sup_{\mathscr{L}} X)$$

Particularly, for $X \subseteq F(A; u)$ we have

$$\sup_{(F(A;u); \subseteq)} X = (\sup_{(D(A); \subseteq)} X)_u .$$

For $X \subseteq G(A; u)$

$$\sup_{(G(A;u); \subseteq)} X = (\sup_{(E(A); \subseteq)} X)_u .$$

Proof. The first part of the theorem (due to Ward) is proved e.g. in [9] page 76, theorem 15. The consequence concerning $F(A; u)$ follows from the general part of the theorem, since $(D(A); \subseteq)$ is a complete lattice (see [5] section 9) and the mapping $\sigma \mapsto \sigma_u$ ($\sigma \in D(A)$) is a closure operator on $(D(A); \subseteq)$ such that $F(A; u)$ is the system of closed elements, corresponding to this operator (see [5] section 22'). The consequence concerning $G(A; u)$ follows directly from the above because $E(A)$ is the principal filter in $(D(A); \subseteq)$, determined by the element id_A (see [5] section 8') and $G(A; u) = F(A; u) \cap E(A)$ (see [5] section 18).

25. Lemma. *Let $X \subseteq F(A; u)$ and let $(x, y) \in \sup_{(F(A;u); \subseteq)} X$. Then there exists a finite subset $X' \subseteq X$ with $(x, y) \in \sup_{(F(A;u); \subseteq)} X'$.*

(See also section 37).

Proof. Denote

$$\alpha =_{\mathrm{Df}} \sup_{(F(A;u); \subseteq)} X , \qquad \beta =_{\mathrm{Df}} \sup_{(D(A); \subseteq)} X .$$

By hypothesis, $(x, y) \in \alpha$ and therefore $\alpha \neq \emptyset$, and so $X \neq \emptyset$. Following section 24, $\alpha = \beta_u$, thus $(x, y) \in \beta_u$. According to the definition of the relation β_u (see section 1 or [5], section 12 and 14)

$$u_\beta = \bigcup_{m=0}^{\infty} \beta(u\beta)^m , \quad \beta_u = u_\beta \cap (u_\beta)^{-1}$$

and, due to [5], section 6 we have $\beta = \bigcup_{n=1}^{\infty} \{\beta_1 \dots \beta_n \mid \beta_1, \dots, \beta_n \in X\}$ so the following relations are valid:

$$(x, y) \in u_\beta = \bigcup_{m=0}^{\infty} \beta(u\beta)^m =$$

$$= \bigcup_{n=1}^{\infty} \{\beta_1 \dots \beta_n \mid \beta_1, \dots, \beta_n \in X\} \cup \bigcup_{m=1}^{\infty} ((\bigcup_{n=1}^{\infty} \{\beta_1 \dots \beta_n \mid \beta_1, \dots, \beta_n \in X\}) .$$

$$. (u \bigcup_{n_m=1}^{\infty} \{\beta_1^{(m)} \dots \beta_{n_m}^{(m)} \mid \beta_1^{(m)}, \dots, \beta_{n_m}^{(m)} \in X\})^m) =$$

$$= \bigcup_{n=1}^{\infty} \{\beta_1 \dots \beta_n \mid \beta_1, \dots, \beta_n \in X\} \cup \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{n_1=1}^{\infty} \dots$$

$$\dots \bigcup_{n_m=1}^{\infty} \{\beta_1 \dots \beta_n u \beta_1^{(1)} \dots \beta_{n_1}^{(1)} \dots u \beta_1^{(m)} \dots \beta_{n_m}^{(m)} \mid \beta_1, \dots, \beta_n, \beta_1^{(1)}, \dots, \beta_{n_1}^{(1)}. \dots, \beta_{n_m}^{(m)} \in X\} .$$

Therefore there exist a finite set

$$X'_1 = \{\beta_1, \dots, \beta_n, \beta_1^{(1)}, \dots, \beta_{n_1}^{(1)}, \dots, \beta_1^{(m)}, \dots, \beta_{n_m}^{(m)}\}$$

such that $X'_1 \subseteq X$ and that

(10) $$(x, y) \in \beta_1 \dots \beta_n u \beta_1^{(1)} \dots \beta_{n_1}^{(1)} \dots u \beta_1^{(m)} \dots \beta_{n_m}^{(m)} .$$

Further,

(10') $$(u_\beta)^{-1} = (\bigcup_{m=0}^{\infty} \beta(u\beta)^m)^{-1} = \bigcup_{m=0}^{\infty} (\beta u^{-1})^m \beta =$$

$$= \beta \cup \bigcup_{m=1}^{\infty} (\beta u^{-1})^m \beta = \bigcup_{m=0}^{\infty} \beta(u^{-1}\beta)^m = (u^{-1})_\beta .$$

From (10'), from the fact that also u^{-1} is an ordering and from the hypotheses that $(x, y) \in (u^{-1})_\beta$ (since $(x, y) \in \beta_u \subseteq (u_\beta)^{-1}$) it follows (by the first part of the present proof) that there is a finite set

$$X'_2 = \{\gamma_1, \dots, \gamma_r, \gamma_1^{(1)}, \dots, \gamma_{r_1}^{(1)}, \dots, \gamma_1^{(s)}, \dots, \gamma_{r_s}^{(s)}\} ,$$

such that $X'_2 \subseteq X$ and that

(10'') $$(x, y) \in \gamma_1 \dots \gamma_r u^{-1} \gamma_1^{(1)} \dots \gamma_{r_1}^{(1)} \dots u^{-1} \gamma_1^{(s)} \dots \gamma_{r_s}^{(s)} .$$

Put $X' =_{\mathrm{Df}} X'_1 \cup X'_2$, then X' is a finite subset of X. According to (10) and (10″) the following holds

$$(x, y) \in (\beta_1 \ldots \beta_n u \beta_1^{(1)} \ldots \beta_{n_1}^{(1)} \ldots u \beta_1^{(m)} \ldots \beta_{n_m}^{(m)}) \cap$$
$$\cap (\gamma_1 \ldots \gamma_r u^{-1} \gamma_1^{(1)} \ldots \gamma_{r_1}^{(1)} \ldots u^{-1} \gamma_1^{(s)} \ldots \gamma_{r_s}^{(s)}) \subseteq$$
$$\subseteq (u) \sup_{(D(A);\subseteq)} X' \cap (u^{-1}) \sup_{(D(A);\subseteq)} X' = (\sup_{(D(A);\subseteq)} X')_u = \sup_{(F(A;u);\subseteq)} X'$$

(see (10′) and corollary in section 24).

26. Corollary. *Let* $X \subseteq G(A; u)$ *and let* $(x, y) \in \sup_{(G(A;u);\subseteq)} X$. *Then there exist a finite subset* $X' \subseteq X$, *with* $(x, y) \in \sup_{(G(A;u);\subseteq)} X'$.

Proof. In case that $x = y$, it is possible to choose $X' = \emptyset$, because $\sup_{(G(A;u);\subseteq)} \emptyset = \mathrm{id}_A$. If $x \neq y$, then $X \neq \emptyset$; therefore in this case the proposition is a direct consequence of section 25, because $G(A; u) = \langle \mathrm{id}_A, \langle_{(F(A;u);\subseteq)}$.

27. Lemma. *Let X be a u-convex subset in A. Then*

$$X^2 \cup \mathrm{id}_A \in G(A; u).$$

Particulary if $a, b \in A$, *then* $[a, b]^2 \cup \mathrm{id}_A \in G(A; u)$.

Proof. The first part of the lemma is verified in [7], section 7. This directly implies the second statement, because $[a, b]_{(A;u)}$ is a u-convex subset in A.

28. Theorem. *Let* $\sigma \in G(A; u)$. *Then σ is a compact element in the complete lattice* $(G(A; u); \subseteq)$ *iff, the following conditions are satisfied:*

(i) *Let* $X \in A/\sigma$. *Then every maximal chain in* $(X; u)$ *has a lower and an upper bound in* $(X; u)$.

(ii) *Let* $X \in A/\sigma$. *Then the set of all maximal and minimal elements in* $(X; u)$ *is finite.*

(iii) *The subsystem of all non-singleton sets which are elements of the system* A/σ, *is finite.*

Proof. We divide the proof into several parts. We denote, for convenience,

$$B =_{\mathrm{Df}} \{X \mid X \in A/\sigma, \ \mathrm{card}\, X \geqq 2\}.$$

1. Let σ not satisfy (i). Then there exist $X \in A/\sigma$ and a maximal u-chain R in X, which is not bounded in $(X; u)$. Suppose, that the set R has not upper bound in $(X; u)$. There the following statement holds:

For every $x \in R$ there exist $x' \in R$ with

$$(11) \qquad (x, x') \in u - \mathrm{id}_A .$$

(The chain R is maximal in $(X; u)$.) For $x \in R$ we define

$$X(x) =_{\mathrm{Df}} \{y \mid y \in X,\ (x, y) \notin u - \mathrm{id}_A\}, \quad \varrho(x) =_{\mathrm{Df}} (X(x))^2 \cup \mathrm{id}_A .$$

By the definition of $X(x)$ we get, following proposition (11) that $x \in X(x) \subset X$.

Given $x \in R$, $r, t \in X(x)$ and $s \in A$ such that $(r, s) \in u$ and $(s, t) \in u$ then $s \in X$ (since X is u-convex in A, see [5] section 36). If, moreover, $s \notin X(x)$, then $(x, s) \in u\text{-}\mathrm{id}_A$ and therefore $(x, t) \in u\text{-}\mathrm{id}_A$ (we assume that $(s, t) \in u$). Therefore, under the considered hypothesis, also $s \in X(x)$.

We have derived, that for every $x \in R$, $X(x)$ a u-convex subset of A. Therefore, according to lemma 27, for every $x \in R$ we get $\varrho(x) \in G(A; u)$; according to the definition of $\varrho(x)$ and according to (11) evidently the proper inclusion $\varrho(x) \subset \sigma$ holds. For $x, y \in R$ we have $(x, y) \in u$ iff $X(x) \subseteq X(y)$, and therefore $(x, y) \in u$ iff $\varrho(x) \subseteq \varrho(y)$. $(R; u)$ is a chain, and therefore so is $(\{\varrho(x) \mid x \in R\}; \subseteq)$. Thus according to [5] section 22 we get $\bigcup\{\varrho(x) \mid x \in R\} \in G(A; u)$. Denote $\varrho =_{\mathrm{Df}} \bigcup\{\varrho(x) \mid x \in R\}$. If $z \in X$, then either $z \in R$ or $z \in X - R$. If $z \in R$, then $z \in X(z)$, hence clearly $z \in \bigcup\{X(x) \mid x \in R\}$. If $z \in X - R$, then (since R is a maximal u-chain in X) there exists such $y \in R$, for which $(y, z) \notin u$; then $z \in X(y)$ and so $z \in \bigcup\{X(x) \mid x \in R\}$. Thus we get that $X \subseteq \bigcup\{X(x) \mid x \in R\}$; the converse inclusion is evident and therefore

$$X = \bigcup\{X(x) \mid x \in R\} .$$

Since

$$\varrho = \mathrm{id}_A \cup \bigcup\{(X(x)^2 \mid x \in R\} = \mathrm{id}_A \cup (\bigcup\{X(x) \mid x \in R\})^2$$

also

$$(12) \qquad \varrho = X^2 \cup \mathrm{id}_A .$$

Finally, we denote

$$\varrho' =_{\mathrm{Df}} \bigcup\{Y^2 \mid Y \in A/\sigma,\ Y \neq X\} \cup \mathrm{id}_A .$$

Evidently $\varrho' \in E(A)$, $\varrho' \subset \sigma$ (because $X \in B$) and according to [5] section 23 we get $\varrho' \in G(A; u)$ (because $\sigma \in G(A; u)$, $X \in A/\sigma$, $\mathrm{id}_X \in G(X; u)$ and $\varrho' = (\sigma \cap (A - X)^2) \cup \mathrm{id}_X$).

Denote $Y =_{\mathrm{Df}} \{\varrho'\} \cup \{\varrho(x) \mid x \in R\}$. Then $\varrho' \subset \sigma$, $\varrho(x) \subset \sigma$ for every $x \in R$ and hence $\sup\limits_{(G(A;u);\subseteq)} Y \subseteq \sigma$. From (12) and from the definition of ϱ' the converse inclusion follows, because it is

$$\sigma = \varrho' \cup (\bigcup\{\varrho(x) \mid x \in R\}) ;$$

and so $\sigma = \sup_{(G(A;u);\subseteq)} Y$. We shall show that $\sigma = \sup_{(G(A;u);\subseteq)} Y'$ does not hold for any finite non-empty subset Y' of Y. Let $Y' \subseteq Y$, $0 < \operatorname{card} Y' < \aleph_0$. Denote by R' the set of those $x \in R$, for which $\varrho(x) \in Y'$. If $R' = \emptyset$ then $Y' = \{\varrho'\}$ and so $\sup_{(G(A;u);\subseteq)} Y' = \varrho' \subset \sigma$. If $R' \neq \emptyset$ then the finiteness of R' implies that there exists the greatest element a in $(R'; u)$. According to our hypothesis about R (see (11)) there exists $b \in R$ with $(a, b) \in u\text{-id}_A$. Then certainly $(a, b) \notin \varrho(a)$; since $a, b \in X$, also $(a, b) \notin \varrho'$. We have

$$\sup_{(G(A;u);\subseteq)} Y' = \sup_{(G(A;u);\subseteq)} \{\varrho', \sup_{(G(A;u);\subseteq)} \{\varrho(x) \mid x \in R'\}\} =$$

$$= \sup_{(G(A;u);\subseteq)} \{\varrho', \varrho(a)\} = \varrho' \cup \varrho(a),$$

and so $(a, b) \notin \sup_{(G(A;u);\subseteq)} Y$. Moreover $(a, b) \in X^2 \subseteq \sigma$; thus, we proved the proper inclusion $\sup_{(G(A;u);\subseteq)} Y' \subset \sigma$ in case $R' \neq \emptyset$. We have demonstrated that from the covering Y of the element σ in $(G(A; u); \subseteq)$ no finite subcovering can be chosen.

If the chain R has no lower bound, the proof proceeds dually.

Thus we have verified that an element $\sigma \in G(A; u)$, which does not satisfy (i) is not a compact element of the complete lattice $(G(A; u); \subseteq)$.

2. Let us assume, that an equivalence σ satisfies (i), but not (ii); we shall show also in this case σ is not a compact element in $(G(A; u); \subseteq)$. For $X \in A/\sigma$ denote by $M(X)$ the set of all maximal and minimal elements in $(X; u)$. From the non-validity of (ii) there follows that for some $Y \in A/\sigma$ the set $M(Y)$ is infinite; certainly, $Y \in B$. Suppose that the set M_1 of all maximal elements in $(Y; u)$ is infinite (if M_1 is finite then, since $M(Y)$ is infinite, the set M_2 of all minimal elements in $(Y; u)$ is also infinite and the proof then proceeds dually). For $x, y \in M_1$ we define

$$\varrho(x, y) =_{\mathrm{Df}} ((\rangle, x\rangle_{(A;u)} \cup \rangle, y\rangle_{(A;u)}) \cap Y)^2 \cup \mathrm{id}_A .$$

Let

$$r, t \in (\rangle, x\rangle \cup \rangle, y\rangle) \cap Y, \quad s \in A, \quad (r, s) \in u, \quad (s, t) \in u .$$

Then $s \in \rangle, x\rangle \cup \rangle, y\rangle$, and $s \in Y$, because, following [5] section 36, Y is a u-convex subset in A. Thus $(\rangle, x\rangle \cup \rangle, y\rangle) \cap Y$ is a u-convex subset in A, and hence, according to lemma 27, $\varrho(x, y) \in G(A; u)$. Evidently $\varrho(x, y) \cap (A - Y)^2 = \mathrm{id}_{A-Y}$ and from the hypothesis that M_1 is infinite it follows that $\varrho(x, y) \subset \sigma$. The equivalence σ satisfies (i) and therefore for all $r, s \in Y$ there exist such elements $x, y \in M_1$, that $(r, x) \in u$ and $(s, y) \in u$; then $(r, s) \in \varrho(x, y)$ and hence

$$Y^2 \cup \mathrm{id}_A \subseteq \bigcup\{\varrho(x, y) \mid x, y \in M_1\} \subseteq \sup_{(G(A;u);\subseteq)} \{\varrho(x, y) \mid x, y \in M_1\} .$$

From the definition of $\varrho(x, y)$ we get also the converse inclusion and therefore

$$Y^2 \cup \mathrm{id}_A = \sup_{(G(A;u);\subseteq)} \{\varrho(x, y) \mid x, y \in M_1\} .$$

Let us denote

$$\varrho' =_{\mathrm{Df}} (\sigma \cap (A - Y)^2) \cup \mathrm{id}_A , \quad Z =_{\mathrm{Df}} \{\varrho'\} \cup \{\varrho(x, y) \mid x, y \in M_1\} ,$$

then by [5], section 23, $\varrho' \in G(A; u)$ and the following holds:

$$\sup_{(G(A;u);\subseteq)} Z = \sup_{(G(A;u);\subseteq)} \{\varrho', \sup_{(G(A;u);\subseteq)} \{\varrho(x, y) \mid x, y \in M_1\}\} =$$
$$= \sup_{(G(A;u);\subseteq)} \{\varrho', Y^2 \cup \mathrm{id}_A\} = \sigma .$$

We shall show that there exists no finite non-empty subset Z which covers σ in $(G(A; u); \subseteq)$. Let Z_1 be a finite non-empty subset of Z. If $Z_1 = \{\varrho'\}$, then

$$\sup_{(G(A;u);\subseteq)} Z_1 = \varrho' = (\sigma \cap (A - Y)^2) \cup \mathrm{id}_A \subset (\sigma \cap (A - Y)^2) \cup Y^2 = \sigma ,$$

and so Z_1 does not cover σ in this case. Let $Z_1' = Z_1 - \{\varrho'\}$ be a non-empty set. Then there exists a finite number of elements $x_1, x_2, \ldots, x_{2n-1}, x_{2n} \in M_1$ with $Z_1' = \{\varrho(x_{2i-1}, x_{2i}) \mid i = 1, \ldots, n\}$. The set M_1 is infinite and therefore there exists $y \in M_1 - \{x_1, x_2, \ldots, x_{2n}\}$. The elements from M_1 are maximal in $(Y; u)$ and therefore

(13) $$y \notin k_u((\bigcup_{i=1}^{2n} \rangle, x_i\rangle) \cap Y)$$

(for the notation see section 1, page 260). Via section 27 we get

$$\varrho =_{\mathrm{Df}} (k_u((\bigcup_{i=1}^{2n} \rangle, x_i\rangle) \cap Y))^2 \cup \mathrm{id}_A \in G(A; u) ;$$

and from (13) it follows that

$$(x_1, y) \notin \varrho \subseteq \sup_{(G(A;u);\subseteq)} \{\varrho(x_{2i-1}, x_{2i}) \mid i = 1, \ldots, n\} .$$

By the definition of ϱ' also $(x_1, y) \notin \varrho'$ and therefore

(14) $$(x_1, y) \notin \varrho' \cup \varrho = \sup_{(G(A;u);\subseteq)} \{\varrho, \varrho'\} \supseteq$$
$$\supseteq \sup_{(G(A;u);\subseteq)} \{\varrho', \sup_{(G(A;u);\subseteq)} \{\varrho(x_{2i-1}, x_{2i}) \mid i = 1, \ldots, n\}\} = \sup_{(G(A;u);\subseteq)} Z_1$$

(the first equality follows directly from [5] section 23:

$$\sigma \in G(A; u) , \quad Y \in A/\sigma , \quad \varrho' = (\sigma \cap (A - Y)^2 \cup \mathrm{id}_A , \quad \varrho \cap Y^2 \in G(Y; u)) .$$

We have $x_1, y \in M_1 \subseteq Y$, $Y \in A/\sigma$ and therefore $(x_1, y) \in \sigma$. From this fact and from (14) we get

$$(x_1, y) \in \sigma - \sup_{(G(A;u);\subseteq)} Z_1 ,$$

i.e. $\sup_{(G(A;u);\subseteq)} Z_1 \subset \sigma$.

We have shown that no finite non-empty subset of Z covers the element σ in the complete lattice $(G(A; u); \subseteq)$ i.e. that the element σ is not compact in $(G(A; u); \subseteq)$.

3. Let us suppose that the system B is infinite, i.e. that σ does not satisfy (iii). We shall show that σ is not compact in $(G(A; u); \subseteq)$ in this case, too. For $X \in B$ we define $\varrho(X) =_{\mathrm{Df}} X^2 \cup \mathrm{id}_A$. According to [5] section 36, X is a u-convex subset in A, because $X \in A/\sigma$ and $\sigma \in G(A; u)$. Via lemma 27 $\varrho(X) \in G(A; u)$. Denote $Y =_{\mathrm{Df}} \{\varrho(X) \mid X \in B\}$. We have

$$\sigma = \bigcup Y \subseteq \sup_{(G(A;u);\subseteq)} Y$$

and, since $\varrho(X) \subseteq \sigma$ holds for all $X \in B$, the oposite inclusion is also valid and we get

$$\sigma = \sup_{(G(A;u);\subseteq)} Y .$$

We shall verify that there does not exist any finite non-empty subset Y' of Y which covers the element σ in the complete lattice $(G(A; u); \subseteq)$. From lemma 2 and from the fact that $\bigcup Y'$ is an equivalence on A it follows, that

$$\sup_{(G(A;u);\subseteq)} Y' = \bigcup Y' \tag{15}$$

(indeed this holds for every non-empty subset in Y – see also section 39). The set B is infinite and hence there exists $X_0 \in B - Y'$. For arbitrary distinct elements a, b in X_0 we have

$$(a, b) \in X_0^2 \subset \sigma , \quad (a, b) \notin \bigcup Y' .$$

From (15) it follows that Y' does no cover the equivalence σ in $(G(A; u); \subseteq)$ and so σ is not compact in $(G(A; u); \subseteq)$.

4. Let an equivalence $\sigma \in G(A; u)$ satisfy (i), (ii), (iii): then we shall prove, that σ is compact in $(G(A; u); \subseteq)$. For $X \in B$ let $M_1(X)$ be the set of all maximal elements in $(X; u)$, and let $M_2(X)$ be the set of all minimal elements in $(X; u)$. Let $Y \subseteq G(A; u)$ and let $\sigma \subseteq \sup_{(G(A;u);\subseteq)} Y$. For $X \in B$, $x \in M_1(X)$ and $y \in M_2(X)$ we have $(x, y) \in$ $\in \sup_{(G(A;u);\subseteq)} Y$ and hence, according to section 26, there exists a finite subset $Y(x, y)$ in Y with $(x, y) \in \sup_{(G(A;u);\subseteq)} Y(x, y)$. Via (ii) and (iii) also the set

$$Y' =_{\mathrm{Df}} \bigcup \{ Y(x, y) \mid X \in B , \quad x \in M_1(X) , \quad y \in M_2(X)\}$$

is finite, and

$$(x, y) \in \sup_{(G(A;u);\subseteq)} Y(x, y) \subseteq \sup_{(G(A;u);\subseteq)} Y'$$

for every $X \in B$; $x \in M_1(X)$, $y \in M_2(X)$. We shall show that $\sigma \subseteq \sup_{(G(A;u);\subseteq)} Y'$.

Let $(r, s) \in \sigma$. If $r = s$ then evidently $(r, s) \in \sup_{(G(A;u);\subseteq)} Y'$; therefore we suppose, that $r \neq s$. Then there exists $X \in B$ with $r, s \in X$. In the poset $(X; u)$ every chain is a subset of some maximal chain and every upper (or lower) bound of the maximal chain in $(X; u)$ is the element of $M_1(X)$ (or of $M_2(X)$); from this fact and from (i), that there are elements $r_1, s_1 \in M_1(X)$ and $r_2, s_2 \in M_2(X)$ such that

$$(16) \qquad (r_2, r) \in u\,, \quad (r, r_1) \in u\,, \quad (s_2, s) \in u\,, \quad (s, s_1) \in u\,.$$

Furthermore,

$$(17) \qquad (r_2, r_1),\ (s_2, s_1),\ (r_2, s_1) \in \sup_{(G(A;u);\subseteq)} (Y(r_1, r_2) \cup$$

$$\cup\, Y(s_1, s_2) \cup Y(s_1, r_2)) \subseteq \sup_{(G(A;u);\subseteq)} Y'\,.$$

Following [5], section 36 we get that $\sup_{(G(A;u);\subseteq)} Y'$ is a u-convex equivalence on A and therefore according to (16) and (17), also

$$(r_1, r),\ (s, s_2),\ (r_1, s_2) \in \sup_{(G(A;u);\subseteq)} Y'.$$

This proves that $(r, s) \in \sup_{(G(A;u);\subseteq)} Y'$ and thus the inclusion

$$\sigma \subseteq \sup_{(G(A;u);\subseteq)} Y'$$

holds. We have verified that every covering of the element σ in $(G(A; u); \subseteq)$ has finite subcovering; thus σ is a compact element in $(G(A; u); \subseteq)$.

5. Via parts 1 – 3 of the present proof, the conditions (i) (ii) and (iii) are necessary for compactness in $(G(A; u); \subseteq)$; via part 4, their conjunction is also sufficient. This proves the theorem.

29. Corollary. *Let $a, b \in A$. Then $[a, b]^2 \cup \mathrm{id}_A$ is a compact element in the complete lattice $(G(A; u); \subseteq)$.*

Proof. According to lemma 27 $[a, b]^2 \cup \mathrm{id}_A \in G(A; u)$, and this equivalence satisfies evidently the conditions (i), (ii) and (iii) of theorem 28.

30. Theorem. *$(G(A; u); \subseteq)$ is an algebraic lattice.*

Proof. Let us denote $\sigma_{ab} =_{\mathrm{Df}} [a, b]^2 \cup \mathrm{id}_A$ for $a, b \in A$. The poset $(G(A; u); \subseteq)$ is a complete lattice (see [5] section 21). Let $\sigma \in G(A; u)$. If $(x, y) \in \sigma$, then evidently $\sigma_{xy} \subseteq \sigma$ (see [5], section 36; the equivalence $\sigma \in G(A; u)$ is u-convex on A) and so we get that

$$\sup_{(G(A;u);\subseteq)} \{\sigma_{xy} \mid (x, y) \in \sigma\} \subseteq \sigma .$$

If $(x, y) \in \sigma$, then $(x, y) \in \sigma_{xy}$ and therefore the converse inclusion

$$\sigma \subseteq \bigcup\{\sigma_{xy} \mid (x, y) \in \sigma\} \subseteq \sup_{(G(A;u);\subseteq)} \{\sigma_{xy} \mid (x, y) \in \sigma\}$$

holds, too. Thus $\sigma = \sup_{(G(A;u);\subseteq)} \{\sigma_{xy} \mid (x, y) \in \sigma\}$ and, according to section 29 the elements σ_{xy} are compact in $(G(A; u); \subseteq)$. Therefore, every element in the complete lattice $(G(A; u); \subseteq)$ can be expressed as a supremum of compact elements in $(G(A; u); \subseteq)$.

31. Lemma. *Let $X \subseteq F(A; u)$. Then*

$$\operatorname{dom} \sup_{(F(A;u);\subseteq)} X = \bigcup\{\operatorname{dom} \sigma \mid \sigma \in X\} .$$

Proof. According to [5], section 6 $\sup_{(D(A);\subseteq)} X = \bigcup_{n=1}^{\infty} \{\sigma_1 \dots \sigma_n \mid \sigma_1, \dots, \sigma_n \in X\}$, where for $\sigma_1 \dots \sigma_n \in X$ we have $\operatorname{dom}(\sigma_1 \dots \sigma_n) \subseteq \operatorname{dom} \sigma_1$.

So we get that

$$\bigcup\{\operatorname{dom} \sigma \mid \sigma \in X\} = \bigcup_{n=1}^{\infty} \{\operatorname{dom}(\sigma_1 \dots \sigma_n) \mid \sigma_1, \dots, \sigma_n \in X\} =$$

$$= \operatorname{dom}\left(\bigcup_{n=1}^{\infty} \{\sigma_1 \dots \sigma_n \mid \sigma_1, \dots, \sigma_n \in X\}\right) = \operatorname{dom} \sup_{(D(A);\subseteq)} X .$$

By [5], section 14 $\operatorname{dom} \sup_{(D(A);\subseteq)} X = \operatorname{dom}(\sup_{(D(A);\subseteq)} X)_u$ and the equality $(\sup_{(D(A);\subseteq)} X)_u = \sup_{(F(A;u);\subseteq)} X$ holds (see lemma 24). From all these facts the proof directly follows.

32. Theorem. *Let $\sigma \in F(A; u)$. Then σ is a compact element in the complete lattice $(F(A; u); \subseteq)$ iff σ is a finite set.*

Proof. Let us define $\varrho_{xy} =_{\mathrm{Df}} \{x, y\}^2$ for $x, y \in A$. Then $A/\varrho_{xy} = \{x, y\}$ and therefore, via lemma 2, $\varrho_{xy} \in F(A; u)$. Let us assume first, that σ is an infinite set. Then

$$\sigma = \bigcup\{\varrho_{xy} \mid (x, y) \in \sigma\} = \sup_{(F(A;u);\subseteq)} \{\varrho_{xy} \mid (x, y) \in \sigma\} .$$

If X is a finite subset of $\{\varrho_{xy} \mid (x, y) \in \sigma\}$, then, by section 31,

$$\operatorname{dom} \sup_{(F(A;u);\subseteq)} X = \bigcup\{\operatorname{dom} \varrho_{xy} \mid \varrho_{xy} \in X\} .$$

As X and dom ϱ_{xy} are finite sets for all $\varrho_{xy} \in X$, also dom $\sup_{(F(A;u);\subseteq)} X$ is finite. Since

$$\sup_{(F(A;u);\subseteq)} X \subseteq (\text{dom} \sup_{(F(A;u);\subseteq)} X)^2 ,$$

also $\sup_{(F(A;u);\subseteq)} X$ is finite. We suppose that σ is an infinite set, therefore $\sigma \subseteq \sup_{(F(A;u);\subseteq)} X$ does not hold. We have proved that from the covering $\{\varrho_{xy} \mid (x, y) \in \sigma\}$ no finite subcovering of the equivalence σ can be chosen. Hence σ is not a compact element in $(F(A; u); \subseteq)$.

Let, conversely, σ be a finite set. Let $X \subseteq F(A; u)$ and let $\sigma \subseteq \sup_{(F(A;u);\subseteq)} X$. Then, according to lemma 25, for every pair $(x, y) \in \sigma$ there exists a finite subset $X'(x, y) \subseteq X$, for which $(x, y) \in \sup_{(F(A;u);\subseteq)} X'(x, y)$. Then the set $X' =_{\text{Df}} \bigcup\{X'(x, y) \mid (x, y) \in \sigma\}$ is finite too. (By hypotheses, σ is finite.) Now, $X' \subseteq X$ and for every $(x, y) \in \sigma$

$$(x, y) \in \sup_{(F(A;u);\subseteq)} X'(x, y) \subseteq \sup_{(F(A;u);\subseteq)} X'$$

thus the inclusion $\sigma \subseteq \sup_{(F(A;u);\subseteq)} X'$ holds. This proves that σ is a compact element in $(F(A; u); \subseteq)$.

33. Theorem. *$(F(A; u); \subseteq)$ is an algebraic lattice.*

Proof. For $x, y \in A$ let us define $\varrho_{xy} =_{\text{Df}} \{x, y\}^2$ (see the beginning of the proof in section 32); then $\varrho_{xy} \in F(A; u)$, and, by section 32, ϱ_{xy} is a compact element in the complete lattice $(F(A; u); \subseteq)$. Let $\sigma \in F(A; u)$. Then the following holds:

$$\sigma = \bigcup\{\varrho_{xy} \mid (x, y) \in \sigma\} \subseteq \sup_{(F(A;u);\subseteq)} \{\varrho_{xy} \mid (x, y) \in \sigma\} .$$

For $(x, y) \in \sigma$ we have $\varrho_{xy} \subseteq \sigma$ and therefore the converse inclusion

$$\sup_{(F(A;u);\subseteq)} \{\varrho_{xy} \mid (x, y) \in \sigma\} \subseteq \sigma$$

also holds. Thus, every element $\sigma \in F(A; u)$ in the complete lattice $(F(A; u); \subseteq)$ can be expressed as a supremum of the set $\{\varrho_{xy} \mid (x, y) \in \sigma\}$ of compact elements.

34. Lemma. *Let $n \geqq 1$ be a natural number. Then the following statements hold:*

a) *A set A has n elements iff every maximal chain in $(G(A; u); \subseteq)$ has just n elements.*

b) *If a set A has n elements, then there exist at most $2^{n-1} - 1$ dual atoms in $(G(A; u); \subseteq)$ *).*

c) *A set A has n elements iff the set of all atoms in $(F(A; u); \subseteq)$ has just n elements*).*

---

*) The statements b), c) hold also for infinite cardinal numbers n.

Proof. Let us recall first the following characterization of the covering relation $-\!\!<_{(E(A);\subseteq)}$ in the complete lattice $(E(A); \subseteq)$ of all equivalences on A (see [9], page 163): For $\varrho, \sigma \in E(A)$ we have $\varrho -\!\!<_{(E(A);\subseteq)} \sigma$ iff there exist elements $X_0 \in A/\sigma$ and $Y_1, Y_2 \in A/\varrho$ with

$$Y_1 \neq Y_2\,, \quad X_0 = Y_1 \cup Y_2\,, \quad A/\sigma - \{X_0\} = A/\varrho - \{Y_1, Y_2\}\,.$$

a) From the above characterization of $-\!\!<_{(E(A);\subseteq)}$ we get, that the set A has n elements iff every maximal chain in $(E(A); \subseteq)$ has n elements. According to [5], section 29, every maximal chain in $(G(A; u); \subseteq)$ is maximal in $(E(A); \subseteq)$ too; and from this fact a) follows.

b) Following the above characterization of $-\!\!<_{(E(A);\subseteq)}$ there exist exactly $(1/2)$ . . card $(\exp A - \{\emptyset, A\}) = 2^{n-1} - 1$ dual atoms in the complete lattice $(E(A); \subseteq)$. According to [5], section 27, we have $-\!\!<_{(G(A;u);\subseteq)} \subseteq -\!\!<_{(E(A);\subseteq)}$ and so every dual atom in $(G(A; u); \subseteq)$ is a dual atom in $(E(A); \subseteq)$ too.

c) The atoms in $(F(A; u); \subseteq)$ are exactly the equivalences in A of the form $\{x\}^2$ for some $x \in A$. So A has the same cardinal number as the set of all atoms in $(F(A; u); \subseteq)$.

35. Corollary. *There exists an algebraic lattice $\mathscr{L}$, which is isomorphic neither to $(G(A; u); \subseteq)$ nor to $(F(A; u); \subseteq)$ for any poset $(A; u)$.*

Proof. By section 34 the lattice $\mathscr{L}$, the diagram of which is shown in fig. 2, has this property.

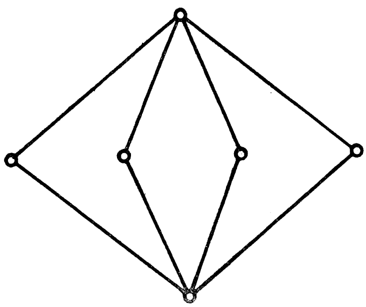

Fig. 2

36. Remarks. a) In section 45/c we shall exhibit another algebraic lattice, which is not isomorphic either to $(F(A; u); \subseteq)$ or to $(G(A; u); \subseteq)$.

b) At the end of this part we shall show a generalization of lemma 25; this generalization is proved (in contrast to section 25) by means of the axiom of choice.

37. Theorem. *Let $\mathscr{A}$ be a non-empty system of sets; let $\mathfrak{A} = (\mathscr{A}; \subseteq)$ be a complete lattice and let $\sup_{\mathfrak{A}} \mathscr{R} = \bigcup \mathscr{R}$ for every non-empty chain $\mathscr{R}$ in $\mathfrak{A}$. Let $\varphi : \mathscr{A} \to \mathscr{A}$ be an algebraic closure operator on A.* *) *Then the following holds:*

If $\mathscr{B} \subseteq \mathscr{A}$ and $b \in \varphi(\sup_{\mathfrak{A}} \mathscr{B})$, then there exists a finite subsystem $\mathscr{C}$ in $\mathscr{B}$ with $b \in \varphi(\sup_{\mathfrak{A}} \mathscr{C})$.

Proof. Let $\mathfrak{m}$ be the least cardinal number of a system $\mathscr{D}$, for which $\mathscr{D} \subseteq \mathscr{B}$ and $b \in (\sup_{\mathfrak{A}} \mathscr{D})$ Let us suppose, that $\aleph_0 \leqq \mathfrak{m}$; we shall derive a contradiction.

There exists ordinal number α with $\mathfrak{m} = \aleph_\alpha$ and there exists a system $\mathscr{B}_0$, for which

$$\mathscr{B}_0 \subseteq \mathscr{B}\,, \quad b \in \varphi\big(\sup_{\mathfrak{A}} \mathscr{B}_0\big)\,, \quad \operatorname{card} \mathscr{B}_0 = \aleph_\alpha\,.$$

We shall order the elements of $\mathscr{B}_0$ into a sequence $(X_\xi)_{\xi<\omega_\alpha}$, where ω_α is the least ordinal number of power $\aleph_\alpha$. Let us define $Y_\xi =_{\mathrm{Df}} \sup_{\mathfrak{A}} \{X_\zeta =_{\mathrm{Df}} \sup \{X_\zeta \mid \zeta < \xi\}$ for all $\xi < \omega_\alpha$. Then $Y_\xi \in \mathscr{A}$ for every $\xi < \omega_\alpha$ and $Y_\zeta \subseteq Y_\xi$ for $\zeta \leqq \xi < \omega_\alpha$. Especially $\{Y_\xi \mid \xi < \omega_\alpha\}$ is a non-empty chain in $\mathfrak{A}$ and therefore

$$Z =_{\mathrm{Df}} \sup_{\mathfrak{A}} \{Y_\xi \mid \xi < \omega_\alpha\} = \bigcup_{\xi<\omega_\alpha} Y_\xi\,.$$

Then $\xi + 1 < \omega_\alpha$ for every $\xi < \omega_\alpha$ and so, according to the definition of $Y_{\xi+1}$, we get that $X_\xi \subseteq Y_{\xi+1}$. There follows: $X_\xi \subseteq Z$ for every $\xi < \omega_\alpha$. Thus $\sup_{\mathfrak{A}} \mathscr{B}_0 \subseteq Z$; we shall derive the converse inclusion. The element $\sup \mathscr{B}_0$ in $\mathfrak{A}$ is an upper bound of the system $\{X_\zeta \mid \zeta < \xi\}$ for all $\xi < \omega_\alpha$ and therefore $Y_\xi \subseteq \sup_{\mathfrak{A}} \mathscr{B}_0$ for all $\xi < \omega_\alpha$. Thus $Z = \sup_{\mathfrak{A}} \{Y_\xi \mid \xi < \omega_\alpha\} \subseteq \sup_{\mathfrak{A}} \mathscr{B}_0$ and the equality

$$\sup_{\mathfrak{A}} \mathscr{B}_0 = \bigcup_{\xi<\omega_\alpha} Y_\xi$$

is derived.

Since $(Y_\xi)_{\xi<\omega_\alpha}$ is a non-dicreasing sequence of elements of $\mathscr{A}$, and since $\varphi : \mathscr{A} \to \mathscr{A}$ is an algebraic closure operator on $\mathfrak{A}$, there follows

$$\varphi(\sup_{\mathfrak{A}} \mathscr{B}_0) = \varphi\big(\bigcup_{\xi<\omega_\alpha} Y_\xi\big) = \bigcup_{\xi<\omega_\alpha} \varphi(Y_\xi)\,.$$

---

*) I.e., $\varphi(\mathscr{A})$ is the algebraic system of closed elements in the complete lattice $\mathfrak{A}$:

(i) If $\mathscr{X} \subseteq \varphi(\mathscr{A})$, then $\inf_{\mathfrak{A}} \mathscr{X} = \inf_{(\varphi(\mathscr{A});\subseteq)} \mathscr{X}$.

(ii) If $\mathscr{X} \subseteq \varphi(A)$ and if $(\mathscr{X}; \subseteq)$ is a non-empty chain, then $\sup_{\mathfrak{A}} \mathscr{X} = \sup_{\varphi(\mathscr{A});\subseteq)} \mathscr{X}$.

(The proof of the second equality: The inclusion $\bigcup_{\xi<\omega_\alpha} \varphi(Y_\xi) \subseteq \varphi(\bigcup_{\xi<\omega_\alpha} Y_\xi)$ follows from the fact that the closure operator is isotonic. We shall derive the converse inclusion. We have $\bigcup_{\xi<\omega_\alpha} \varphi(Y_\xi) \supseteq \bigcup_{\xi<\omega_\alpha} Y_\xi$, and therefore $\varphi(\bigcup_{\xi<\omega_\alpha} \varphi(Y_\xi)) \supseteq \varphi(\bigcup_{\xi<\omega_\alpha} Y_\xi)$. So we get that

$$\bigcup_{\xi<\omega_\alpha} \varphi(Y_\xi) = \sup_{\mathfrak{A}} \{\varphi(Y_\xi) \mid \xi < \omega_\alpha\} = \sup_{(\varphi(\mathscr{A});\subseteq)} \{\varphi(Y_\xi) \mid \xi < \omega_\alpha\} \in \varphi(\mathscr{A}),$$

because $(\{\varphi(Y_\xi) \mid \xi < \omega_\alpha\}; \subseteq)$ is a chain too. Hence $\varphi(\bigcup_{\xi<\omega_\alpha} \varphi(Y_\xi)) = \bigcup_{\xi<\omega_\alpha} \varphi(Y_\xi)$ and the converse inclusion

$$\varphi(\bigcup_{\xi<\omega_\alpha} Y_\xi) \subseteq \bigcup_{\xi<\omega_\alpha} \varphi(Y_\xi)$$

is derived.) Since $b \in (\sup_{\mathfrak{A}} \mathscr{B}_0)$, there exists an index $\nu < \omega_\alpha$ such that $b \in \varphi(Y_\nu)$. By the definition of Y_ν we get that

$$b \in \varphi(Y_\nu) = \varphi(\sup_{\mathfrak{A}} \{X_\xi \mid \xi < \nu\}),$$

and $\{X_\xi \mid \xi < \nu\} \subseteq \mathscr{B}$; card $\{X_\xi \mid \xi < \nu\} \leqq$ card $\nu < \aleph_\alpha$. But this is in a contradiction to the definition of cardinal number $\mathfrak{m} = \aleph_\alpha$. Therefore $\mathfrak{m} < \aleph_0$ and the proof is concluded.

INTERVALS IN $(G(A; u); \subseteq)$

38. Remark. In this final part we consider intervals in $(G(A; u); \subseteq)$. Therefore, given $\varrho, \sigma \in G(A; u)$, we shall write $\rangle, \sigma\rangle$ or $\langle\varrho, \sigma\rangle$ etc. instead of $\rangle, \sigma\rangle_{(G(A;u);\subseteq)}$ or $\langle\varrho, \sigma\rangle_{(G(A;u);\subseteq)}$ (see also section 1).

39. Theorem. (The local characterization of the elements of $F(A; u)$.) *Let $\alpha \in D(A)$, $\beta \in F(A; u)$ and let $\alpha \subseteq \beta$. Then $\alpha \in F(A; u)$ iff the following condition holds*:

$$(18) \qquad (\alpha \cap X^2) \in F(X; u) \quad \text{for all} \quad X \in A/\beta .$$

Proof. For $\alpha = \emptyset$ we have $u_\alpha = \emptyset = (u_\alpha)^{-1}$ and therefore $\alpha_u = u_\alpha \cap (u_\alpha)^{-1} = \emptyset$ and, according to the definition of the system $F(A; u)$ in section 1 (see also [5] section 18) we have $\emptyset \in F(A; u)$. At the same time, for $\alpha = \emptyset$ the condition (18) is satisfied. Therefore we can further suppose that $\alpha \neq \emptyset$; then $\beta \neq \emptyset$ and $A \neq \emptyset$.

Let the hypotheses of the theorem and the condition (18) be satisfied. Let $\mathscr{A}$ be the system of those equivalences τ, which satisfy the following:

$$(19) \qquad \tau \in F(A; u), \quad \alpha \subseteq \tau \subseteq \beta \quad \text{and, for all} \quad X \in A/\beta,$$

$$X^2 \cap \tau = X^2 \quad \text{or} \quad X^2 \cap \tau = X^2 \cap \alpha .$$

We have $\beta \in \mathscr{A}$, and so $\mathscr{A} \neq \emptyset$. According to [5] section 20 there exists

$$\gamma =_{\mathrm{Df}} \inf_{(F(A;u);\subseteq)} \mathscr{A} = \bigcap \mathscr{A} \in F(A; u) .$$

Then $\alpha \subseteq \tau$ for all $\tau \in \mathscr{A}$, and therefore $\alpha \subseteq \gamma$ too. We shall derive the converse inclusion thus proving that $\alpha \in F(A; u)$.

Let us show at first, that $\gamma \in \mathscr{A}$. We have derived that $\alpha \subseteq \gamma \in F(A; u)$. From the relation $\beta \in \mathscr{A}$ and from the definition of γ it follows that $\gamma \subseteq \beta$. Let $X \in A/\beta$. If $X^2 \cap \tau = X^2$ for all $\tau \in \mathscr{A}$, then also $X^2 \cap \gamma = X^2$. If there exists such $\tau_0 \in \mathscr{A}$, that $X^2 \cap \tau_0 = X^2 \cap \alpha$, there

$$X^2 \cap \gamma = X^2 \cap (\bigcap\{\tau \mid \tau \in \mathscr{A}\}) = \bigcap\{X^2 \cap \tau \mid \tau \in \mathscr{A}\} = X^2 \cap \alpha ,$$

because $\tau \cap X^2 = X^2$ or $\tau \cap X^2 = \alpha \cap X^2$ for all $\tau \in \mathscr{A}$ and $\tau_0 \cap X^2 = \alpha \cap X^2$. So the equivalence γ satisfies the condition (19), i.e. $\gamma \in \mathscr{A}$.

Let $\gamma - \alpha \neq \emptyset$. Then there exists $(a, b) \in \gamma - \alpha$; since $\gamma \subseteq \beta$, there exists $X_0 \in A/\beta$ with $a, b \in X_0$. By (19) we get that $X_0^2 \cap \alpha \neq X_0^2 \cap \tau = X_0^2$ for all $\tau \in \mathscr{A}$ and therefore, according to the definition of γ, $X_0^2 \subseteq \gamma$. From all this we get that $X_0 \in A/\gamma$. If we define

$$\delta =_{\mathrm{Df}} \gamma \cap (A - X_0)^2 , \quad \varepsilon =_{\mathrm{Df}} X_0^2 \cap \alpha , \quad \varphi =_{\mathrm{Df}} \delta \cup \varepsilon ,$$

then, by (18), $\varepsilon \in F(X_0; u)$ and according to [5] section 23, also $\varphi \in F(A; u)$. If $X \in A/\beta - \{X_0\}$, then $X^2 \cap \varphi = X^2 \cap \gamma$ and $X_0^2 \cap \varphi = X_0^2 \cap \varepsilon = X_0^2 \cap \alpha$. The element γ satisfies (19) and hence so does the element φ (the validity of the inclusion $\alpha \subseteq \varphi \subseteq \beta$ is evident). Thus $\varphi \in \mathscr{A}$. Also $\varphi \cap X_0^2 = \alpha \cap X_0^2 \subset X_0^2 \subseteq \gamma$, which is in a contradiction to the fact, that $\gamma = \inf_{(F(A;u);\subseteq)} \mathscr{A}$. The hypothesis $\gamma - \alpha \neq \emptyset$ leads to a contradiction, hence the inclusion $\gamma \subseteq \alpha$ holds. So $\alpha = \gamma \in F(A; u)$ and we have proved that the hypothesis of the theorem and the condition (18) imply $\alpha \in F(A; u)$.

We shall derive the converse implication. Let the hypothesis of the theorem be satisfied. Let $\alpha \in F(A; u)$ and $X \in A/\beta$. If $X^2 \cap \alpha = \emptyset$ then $X^2 \cap \alpha \in F(X; u)$ (see the first section of this proof) and therefore the condition (18) is for X satisfied. Let $X^2 \cap \alpha \neq \emptyset$. Let $n \geqq 1$ be a natural number and let

$$X_j \in X/\alpha , \quad (X_i, X_{i+1}) \in \dot{u} , \quad (X_n, X_0) \in \dot{u}$$

for every $i = 0, \ldots, n - 1$ and for every $j = 0, \ldots, n$. Then $X_j \in A/\alpha$ (because $X/\alpha \subseteq A/\alpha$ – see section 1, page 259) and according to lemma 2, from the hypothesis $\alpha \in F(A; u)$ there follows $X_0 = \ldots = X_n$. So, according to lemma 2, $(\alpha \cap X^2) \in F(X; u)$, because $X/\alpha = X/(\alpha \cap X^2)$. We have derived that $\alpha \in F(A; u)$ implies the condition (18).

40. Corollary. (The local characterization of the elements of $G(A; u)$.) *Let* $\alpha \in E(A)$, $\beta \in G(A; u)$ *and let* $\alpha \subseteq \beta$. *Then* $\alpha \in G(A; u)$ *iff the following holds*:

$$(18') \qquad (\alpha \cap X^2) \in G(X; u) \quad \textit{for all} \quad X \in A/\beta .$$

Proof. We have $G(A; u) = E(A) \cap F(A; u)$ (see [5] section 18) and $(\alpha \cap Y^2) \in E(Y)$ for $\alpha \in E(A)$ and $Y \subseteq A$. The statement follows directly from section 39.

41. Notation. Let $\sigma \in G(A; u)$. Then Δ_σ denotes the set of all elements, covered by σ in $(G(A; u); \subseteq)$, that means that Δ_σ is the set of all dual atoms in the complete lattice $(\rangle, \sigma\rangle; \subseteq)$. If $\varrho \in \rangle, \sigma\rangle$, then we define

$$(20) \qquad d_\sigma(\varrho) =_{\mathrm{Df}} \{\tau \mid \tau \in \Delta_\sigma,\ \varrho \subseteq \tau\} .$$

It is $d_\sigma(\varrho) \neq \emptyset$ for $\varrho \subset \sigma$ (see [5], section 28). Evidently, $d_\sigma : \rangle, \sigma\rangle \to \exp \Delta_\sigma$. If $X \in \exp \Delta_\sigma$ and $X \neq \emptyset$, then we define

$$(21) \qquad \psi_\sigma(X) =_{\mathrm{Df}} \inf_{(G(A;u);\subseteq)} X , \quad \psi_\sigma(\emptyset) =_{\mathrm{Df}} \sigma .$$

According to [5], section 20 we have $\psi_\sigma(X) = \bigcap X$ and evidently $\psi_\sigma(X) \in \rangle, \sigma\rangle$. Moreover

$$(21') \qquad \psi_\sigma(Y) = \inf_{(\rangle,\sigma\rangle;\subseteq)} Y$$

holds for all $Y \in \exp \Delta_\sigma$ and therefore $\psi_\sigma : \exp \Delta_\sigma \to \rangle, \sigma\rangle$.

42. Lemma. *Let* $\varrho, \sigma, \tau \in G(A; u)$ *and let* $\varrho \subset \tau \subseteq \sigma$. *Then* $d_\sigma(\tau) \subset d_\sigma(\varrho)$.

Proof. From the inclusion $\varrho \subset \tau \subseteq \sigma$ and from (20) it follows that $d_\sigma(\tau) \subseteq d_\sigma(\varrho)$. If $\varrho \subset \tau$, then the following holds:

$$(22.\ \mathrm{a}) \qquad \forall Y \in A/\varrho\ \exists Z \in A/\tau \quad (Y \subseteq Z)$$

$$(22.\ \mathrm{b}) \qquad \exists Y_0 \in A/\varrho\ \exists Z_0 \in A/\tau \quad (Y_0 \subset Z_0)$$

It is

$$Z = \bigcup\{Y \mid Y \in A/\varrho,\ Y \subseteq Z\}$$

for all $Z \in A/\tau$ (ϱ, σ, τ are equivalences *on* A) and therefore according to (22. b)

$$(23) \qquad \mathrm{card}\,\{Y \mid Y \in A/\varrho,\ Y \subseteq Z_0\} \geqq 2 .$$

From the inclusion $\tau \subseteq \sigma$ it follows that there exists exactly one element $U_0 \in A/\sigma$ with $Z_0 \subseteq U_0$.

We shall construct an element ϱ_1 about which we shall show that $\varrho_1 \in d_\sigma(\varrho) - d_\sigma(\tau)$. We must distinguish two possibilities. If

(24) $\quad Y_0$ is not the $u_{A/\varrho}$-greatest element in $\{Y \mid Y \in A/\varrho,\ Y \subseteq Z_0\}$,

then we put

$$\varrho_1 =_{\mathrm{Df}} (\bigcup\{U^2 \mid U \in A/\sigma,\ U \neq U_0\}) \cup$$
$$\cup (\bigcup\{Y \mid Y \in A/\varrho,\ Y \subseteq U_0,\ (Y, Y_0) \in u_{A/\varrho}\})^2 \cup$$
$$\cup (\bigcup\{Y \mid Y \in A/\varrho,\ Y \subseteq U_0,\ (Y, Y_0) \notin u_{A/\varrho}\})^2\,;$$

(25) $\quad$ if Y_0 is the $u_{A/\varrho}$-greatest element in $\{Y \mid Y \in A/\varrho,\ Y \subseteq Z_0\}$

then we put

$$\varrho_1 =_{\mathrm{Df}} (\bigcup\{U^2 \mid U \in A/\varrho,\ U \neq U_0\}) \cup$$
$$\cup (\bigcup\{Y \mid Y \in A/\varrho,\ Y \subseteq U_0,\ (Y, Y_0) \in u_{A/\varrho} - \mathrm{id}_{A/\varrho}\})^2 \cup$$
$$\cup (\bigcup\{Y \mid Y \in A/\varrho,\ Y \subseteq U_0,\ (Y, Y_0) \notin u_{A,\varrho} - \mathrm{id}_{A/\varrho}\})^2\,.$$

We have $\varrho \in G(A; u)$ and therefore, according to section 2, $u_{A/\varrho}$ is an ordering on A/ϱ. From this, from the inclusions $\varrho \subset \tau \subseteq \sigma$ and from the definition of the relation ϱ_1 it follows, that ϱ_1 is an equivalence on A. According to (23) the system

$$\mathscr{A} =_{\mathrm{Df}} \{Y \mid Y \in A/\varrho_1,\ Y \subseteq U_0\}$$

has two elements and, by the definition of ϱ_1,

(26) $$A/\varrho_1 = (A/\sigma - \{U_0\}) \cup \mathscr{A}\,.$$

We get, that ϱ_1 is covered by σ in $(E(A); \subseteq)$ (see the characterization of the relation $-\!\!<_{(E(A);\subseteq)}$, page 281 in the first part of lemma 34). We denote the two-element system $\mathscr{A}$ by $\mathscr{A} = \{Y_1, Y_2\}$ and we choose the indices so that in case (24) $Y_0 \subseteq Y_1$ and in case (25) $Y_0 \subseteq Y_2$.

We shall show, that $(Y_2, Y_1) \notin \dot{u}$, by contradiction. Let us suppose, that $(Y_2, Y_1) \in \dot{u}$. Then there exist $y_2 \in Y_2$ and $y_1 \in Y_1$ with $(y_2, y_1) \in u$. Since $\varrho \in E(A)$, there exist $Y_1', Y_2' \in A/\varrho$, for which $y_1 \in Y_1'$ and $y_2 \in Y_2'$. Then

$$(Y_2', Y_1') \in \dot{u} \cap (A/\varrho)^2 \subseteq u_{A/\varrho}\,.$$

From inclusion $\varrho \subseteq \varrho_1$ it follows that $Y_1' \subseteq Y_1$, $Y_2' \subseteq Y_2$. According to the definition of ϱ_1, $(Y_1', Y_0) \in u_{A/\varrho}$ in case (24) because $Y_0 \subseteq Y_1$. Further, $(Y_2', Y_1') \in u_{A/\varrho}$ and therefore $(Y_2', Y_0) \in u_{A/\varrho}$. So in case (24) it is $Y_2' \subseteq Y_1$, but this is a contradiction, because also $Y_2' \subseteq Y_2$. This proves that, assuming (24), the relation $(Y_2, Y_1) \in \dot{u}$ is excluded. Let (25) hold. Then $Y_0 \subseteq Y_2$. We have $Y_1' \subseteq Y_1$ and therefore, according to the definition of ϱ_1, $(Y_1', Y_0) \in u_{A/\varrho} - \mathrm{id}_{A/\varrho}$. Also $(Y_2', Y_1') \in u_{A/\varrho}$ and we get $(Y_2', Y_0) \in u_{A/\varrho} - \mathrm{id}_{A/\varrho}$. According to the definition of ϱ_1, $Y_2' \subseteq Y_1$ and this is in a contradiction to the fact that $Y_2' \subseteq Y_2$.

We have verified that the relation $(Y_2, Y_1) \in \dot{u}$ does not hold in any case.
Thus the inclusion

$$\mathrm{id}_{\mathscr{A}} \subseteq \dot{u} \cap \mathscr{A}^2 \subseteq \{(Y_1, Y_1), (Y_2, Y_2), (Y_1, Y_2)\}$$

holds. From this inclusion it follows that $u_{\mathscr{A}} = \dot{u} \cap \mathscr{A}^2$ is an ordering on $\mathscr{A}$. Therefore, according to section 2, we have

$$\varrho_1 \cap U_0^2 = Y_1^2 \cup Y_2^2 \in G(U_0, u).$$

Also $\varrho_1 \cap U^2 = U^2$ for every $U \in (A/\varrho_1 - \{U_0\})$ and so $\varrho \cap U^2 \in G(U; u)$. According to section 40, we have $\varrho_1 \in G(A; u)$. We have shown before that $\varrho \subseteq \varrho_1 -\!\!\langle_{(E(A);\subseteq)}\sigma$ and therefore $\varrho_1 \in d_\sigma(\varrho)$.

Finally, we prove that $\varrho_1 \notin d_\sigma(\tau)$. From the definition of ϱ_1 and from (23) we get $Z_0 \cap Y_1 \neq \emptyset \neq Z_0 \cap Y_2$, because in case (24) we have

$$Y_1 = \bigcup\{Y \mid Y \in A/\varrho,\ Y \subseteq U_0,\ (Y, Y_0) \in u_{A/\varrho}\},$$

$$Y_2 = \bigcup\{Y \mid Y \in A/\varrho,\ Y \subseteq U_0,\ (Y, Y_0) \notin u_{A/\varrho}\},$$

and in case (25) we have

$$Y_1 = \bigcup\{Y \mid Y \in A/\varrho,\ Y \subseteq U_0,\ (Y, Y_0) \in u_{A/\varrho} - \mathrm{id}_{A/\varrho}\},$$

$$Y_2 = \bigcup\{Y \mid Y \in A/\varrho,\ Y \subseteq U_0,\ (Y, Y_0) \notin u_{A/\varrho} - \mathrm{id}_{A/\varrho}\},$$

If we choose $r \in Z_0 \cap Y_1$, $s \in Z_0 \cap Y_2$ then we get $(r, s) \in \tau - \varrho_1$. So the inclusion $\tau \subseteq \varrho_1$ does not hold and therefore $\varrho_1 \notin d_\sigma(\tau)$.

We have derived that $d_\sigma(\varrho) \neq d_\sigma(\tau)$. We have shown at the beginning of this proof that $d_\sigma(\tau) \subseteq d_\sigma(\varrho)$. Thus the proper inclusion $d_\sigma(\tau) \subset d_\sigma(\varrho)$ is proved and the proof is concluded.

43. Lemma. *Let $(X; \leqq)$ and $(Y; \preceq)$ be complete lettices and let mappings $\varphi : X \to Y$ and $\psi : Y \to X$ define a Galois' correspondence between $(X; \leqq)$ and $(Y; \preceq)$. Then the following statements hold.*

a) *$\psi\varphi : X \to X$ is a closure operator on $(X; \leqq)$ and $\varphi\psi : Y \to Y$ is a closure operator on $(Y; \preceq)$.*

b) *For $R \subseteq X$ and $S \subseteq Y$,*

$$\varphi(\sup_{(X;\leqq)} R) = \inf_{(Y;\preceq)} \varphi(R), \quad \psi(\sup_{(Y;\preceq)} S) = \inf_{(X;\leqq)} \psi(S).$$

c) *Let* $x < y$ *imply that* $\varphi(y) \prec \varphi(x)$ *for all* $x, y \in X$. *Then*:

(α) *Every element of* X *is* $\psi\varphi$*-closed.*

(β) *The mapping* $\varphi : X \to \varphi(X)$ *is an antitonic isomorphism from* $(X; \leqq)$ *onto* $(\varphi(X); \preceq)$. *In particular* $\varphi : X \to Y$ *is an injection. The partial mapping*

$$(\psi \mid \varphi(X)) : \varphi(X) \to X$$

is an isotonic isomorphism from $(\varphi(X); \succeq)$ *onto* $(X; \leqq)$ *and the mappings* φ, $\psi \mid \varphi(X)$ *are mutually inverse.*

Proofs of these statements can be found in the literature. E.g. the statement a) is proved [4], theorem 11.1.2 (page 241 of the Russian translation), the statement c/(α) is proved in [4], theorem 11.1.4 (page 242 of the Russian translation) and the statement c/(β) is proved in [3], section VI. 11.1 (page 290–291 of the Czech translation; by (α) it is $\psi\, \varphi(X) = X$). The statement b) is also well-known and it is given, in a special case e.g. in [2] (page 61 of the Russian translation).

44. Theorem. *Let* $\sigma \in G(A; u)$. *Then the mappings*

$$d_\sigma \colon \rangle, \sigma\rangle \to \exp \Delta_\sigma\,, \qquad \psi_\sigma \colon \exp \Delta_\sigma \to \rangle, \sigma\rangle$$

define a Galois' correspondence between the complete lattices $(\rangle, \sigma\rangle; \subseteq)$ *and* $(\exp \Delta_\sigma; \subseteq)$. *The following statements hold*:

a) *The mapping* $d_\sigma \colon \rangle, \sigma\rangle \to \exp \Delta_\sigma$ *is an injection.*

b) *Every element from* $\rangle, \sigma\rangle$ *is* $\psi_\sigma d_\sigma$ *– closed*).*

c) *Mappings* d_σ *and* $\psi_\sigma \mid d_\sigma(\rangle, \sigma\rangle)$ *are mutually inverse and the complete lattices* $(\rangle, \sigma\rangle; \subseteq)$, $(d_\sigma(\rangle, \sigma\rangle); \supseteq)$ *are isomorphic.*

d) *If* $\emptyset \neq X \subseteq \rangle, \sigma\rangle$ *and* $\sup_{(G(A;u);\subseteq)} X \neq \sigma$, *then*

$$\sup_{(G(A;u);\subseteq)} X = \bigcap \bigcap \{d_\sigma(\varrho) \mid \varrho \in X\}\,.$$

Proof. $(G(A; u); \subseteq)$ is a complete lattice and $\rangle, \sigma\rangle$ is a principal ideal in this lattice; therefore $(\rangle, \sigma\rangle; \subseteq)$ is a complete lattice. For $Y, Z \in \exp \Delta_\sigma$, $Y \subseteq Z$ holds

$$\psi_\sigma(Z) = \inf_{(\rangle,\sigma\rangle;\subseteq)} Z \subseteq \inf_{(\rangle,\sigma\rangle;\subseteq)} Y = \psi_\sigma(Y)$$

(see (21′)). From this and from section 42 it follows that the mappings d_σ, ψ_σ define a Galois' correspondence between the complete lattices $(\rangle, \sigma\rangle; \subseteq)$ and $(\exp \Delta_\sigma, \subseteq)$.

*) That means that for all $\varrho \in \rangle, \sigma\rangle$, $\varrho = \psi_\sigma\, d_\sigma(\varrho)$.

The statements a), b), c) are direct consequences of sections 42 and 43. We shall verify the statement d). According to section 43/b,

$$d_\sigma(\sup_{(\rangle,\sigma\rangle;\subseteq)} X) = \inf_{(\exp\Delta_\sigma;\subseteq)} d_\sigma(X) = \bigcap\{d_\sigma(\varrho) \mid \varrho \in X\} \tag{27}$$

as X is non-empty, we have $d_\sigma(X) \neq \emptyset$. From the hypothesis that $\sup_{(G(A;u);\subseteq)} X \neq \sigma$, $X \subseteq \rangle, \sigma\rangle$ it follows, that $\sup_{(G(A;u);\subseteq)} X \subset \sigma$ and therefore, according to a), c), $d_\sigma(\sup_{(\rangle,\sigma\rangle,\subseteq)} X) \neq \emptyset$ (by hypotheses $\sup_{(\rangle,\sigma\rangle;\subseteq)} X = \sup_{(G(A;u);\subseteq)} X$). As a consequence of (27) we get $\bigcap d_\sigma(X) \neq \emptyset$. According to b), (21) and [5], section 20, it follows from (27) that

$$\sup_{(G(A;u);\subseteq)} X = \psi_\sigma d_\sigma(\sup_{(\rangle,\sigma\rangle;\subseteq)} X) = \psi_\sigma(\bigcap d_\sigma(X)) = \inf_{(G(A;u);\subseteq)} (\bigcap d_\sigma(X)) = \bigcap \bigcap d_\sigma(X)\,.$$

45. Remarks. a) The statement 44/b is a basic generalization of lemma 28 in [5]. According to section 44/b and [5] section 20, the following statement hold:

If $\varrho, \sigma \in G(A; u)$ and if $\varrho \subset \sigma$, then $\varrho = \bigcap d_\sigma(\varrho)$.

(That means that in the complete lattice $(\rangle, \sigma\rangle; \subseteq)$ there exist sufficiently many dual atoms, which are above ϱ.)

b) The statement 44/d exhibits one possible form of a supremum in $(G(A; u); \subseteq)$; this question has not played any important role in [5] (see also section 24). We have $A^2 \in G(A; u)$; if we choose $\sigma = A^2$ in section 44) d we get:

Let $X \subseteq G(A; u)$. If $X = \emptyset$ then $\sup_{(G(A;u);\subseteq)} X = \mathrm{id}_A$. If $X \neq \emptyset$ and if τ is an upper bound of X in $(G(A; u); \subseteq)$, $\tau \neq A^2$, then

$$\sup_{(G(A;u);\subseteq)} X = \bigcap\{\chi \mid \chi \text{ is a dual atom, which is an upper bound of } X \text{ in } (G(A; u); \subseteq)\}\,.$$

(Δ_{A^2} is the set of all dual atoms in $(G(A; u); \subseteq)$. Evidently,

$$\{\chi \mid \chi \in \Delta_{A^2},\ \forall \varrho \in X(\varrho \subseteq \chi)\} = \bigcap d_{A^2}(X)\,.$$

From this and from section 44/d the statement follows).

c) Let $\mathscr{L}$ be at least three-element finite chain. Then, e.g. by section 44/a, there exists no poset $(A; u)$, for which the lattices $\mathscr{L}$ and $(G(A; u); \subseteq)$ are isomorphic. If we consider that $G(A; u)$ is a principal filter in $(F(A; u); \subseteq)$, determined by id_A, and that the set of all atoms in $(F(A; u); \subseteq)$ has the same cardinal number as A (see section 34/c) evidently $\mathscr{L}$ is not isomorphic to $(F(A; u); \subseteq)$ either. Yet $\mathscr{L}$ is an algebraic lattice (see section 35, where another counterexample is exhibited).

46. Remark. Let us recall the following notation (see [5], section 54). $\alpha, \beta \in E(A)$ and $\alpha \subseteq \beta$, then we define

$$(X, Y) \in \beta/\alpha \Leftrightarrow_{\mathrm{Df}} \forall x \in X\ \forall y \in Y((x, y) \in \beta)$$

for $A \neq \emptyset$ and for all $X, Y \in A/\alpha$. According to section 3, $\beta/\alpha = \emptyset/\emptyset = \{(\emptyset, \emptyset)\}$ for $A = \emptyset$.

Let $\alpha, \beta, \gamma \in E(A)$. Then the following statements hold; the proof is left to reader.

a) If $\alpha \subseteq \beta$, then $\beta/\alpha \in E(A/\alpha)$.

b) If $\alpha \subseteq \beta$ and $\alpha \subseteq \gamma$ then $\beta/\alpha \subseteq \gamma/\alpha$ iff $\beta \subseteq \gamma$.

c) For $\delta \in E(A/\alpha)$ there exist exactly one equivalence $\delta' \in E(A)$ with $\alpha \subseteq \delta'$ and such that $\delta = \delta'/\alpha$. (If we define for any $x, y \in A$ the relation δ' by $(x, y) \in \delta'$ iff there exist $X, Y \in A/\alpha$ for which $x \in X$, $y \in Y$ and $(X, Y) \in \delta$, then $\alpha \subseteq \delta'$, $\delta' \in E(A)$ and $\delta'/\alpha = \delta$; the unicity of such δ' follows from proposition b).)

The mapping

$$\delta \mapsto \delta' \quad (\delta \in E(A/\alpha))$$

is an isomorphism from the complete lattice $(E(A/\alpha); \subseteq)$ onto the complete lattice $(\langle \alpha, \langle_{(E(A);\subseteq)}; \subseteq)$. (See [2] chap. II, section 3.)

47. Lemma. *Let $\varrho \in G(A; u)$ and $\tau \in E(A/\varrho)$. Then $\tau \in G(A/\varrho; u_{A/\varrho})$* *) *iff there exists $\tau' \in \langle \varrho, \langle$ with $\tau = \tau'/\varrho$; such τ' is unique* (for a given τ). *The mapping*

$$\tau \mapsto \tau' \quad (\tau \in G(A/\varrho; u_{A/\varrho}))$$

is an isomorphism from the complete lattice $(G(A/\varrho; u_{A/\varrho}); \subseteq)$ onto the complete lattice $(\langle \varrho, \langle; \subseteq)$.

Proof. For $A = \emptyset$ the statement trivial (we have $\varrho = \emptyset$, $A/\varrho = \{\emptyset\}$, $E(A/\varrho) = G(A/\varrho; u_{A/\varrho}) = \{\{(\emptyset, \emptyset)\}\}$, $\tau = \{(\emptyset, \emptyset)\}$, $\tau' = \emptyset$ and $\langle \varrho, \langle = \{\emptyset\}$). Suppose $A \neq \emptyset$.

Let $\tau \in G(A/\varrho; u_{A/\varrho})$. We have $G(A/\varrho; u_{A/\varrho}) \subseteq E(A/\varrho)$ and therefore, according to section 46/c, there exists a unique $\tau' \in \langle \varrho, \langle_{(E(A);\subseteq)}$ with $\tau = \tau'/\varrho$. We shall show that $\tau' \in G(A; u)$. The relation $X \in A/\tau'$ holds iff there exists $\mathscr{X} \in (A/\varrho)/\tau$, for which $X/\varrho = \mathscr{X}$. Let $n \geqq 1$ be a natural number and let

$$X_j \in A/\tau', \quad (X_i, X_{i+1}) \in \dot{u}, \quad (X_n, X_0) \in \dot{u},$$

hold for all $i = 0, \ldots, n-1$ and $j = 1, \ldots, n$. Then for all j there exist $x_j, x'_j \in X_j$ such that $(x_i, x'_{i+1}) \in u$, $(x_n, x'_0) \in u$ for all i. There also exist $Y_j, Y'_j \in X_j/\varrho$, for which the relations $x_j \in Y_j$, $x'_j \in Y'_j$ hold. Then $(Y_i, Y'_{i+1}) \in \dot{u}$, $(Y_n, Y'_0) \in \dot{u}$ for all i. Therefore,

$$(X_i/\varrho, X_{i+1}/\varrho) \in (u_{A/\varrho})^{\cdot}, \quad (X_n/\varrho, X_0/\varrho) \in (u_{A/\varrho})^{\cdot},$$

*) We have $\varrho \in G(A; u)$ and therefore $(A/\varrho; u_{A/\varrho})$ is a poset; so $G(A/\varrho; u_{A/\varrho})$ is defined. Let us recall that $\langle \varrho, \langle$ always denotes the interval in $(G(A; u); \subseteq)$.

because from $(Y_i, Y'_{i+1}) \in \dot{u}$ it follows that

$$(Y_i, Y'_{i+1}) \in \dot{u} \cap (A/\varrho)^2 \subseteq u_{A/\varrho}.$$

We have

$$\tau \in G(A/\varrho; u_{A/\varrho}), \quad (X_j/\varrho) \in ((A/\varrho)/\tau),$$

and therefore, according to section 2, $X_i/\varrho = X_{i+1}/\varrho$ for all i. So $X_0 = \ldots = X_n$ and therefore, according to section 2, $\tau' \in G(A; u)$. There follows $\tau' \in \langle \varrho, \langle$.

Suppose, conversely, that $\tau' \in \langle \varrho, \langle$. Then, according to section 46/a, $\tau = \tau'/\varrho$ is an equivalence on A/ϱ. Let $n \geqq 1$ be an integer and let the relations

$$\mathscr{X}_j \in (A/\varrho)/\tau, (\mathscr{X}_i, \mathscr{X}_{i+1}) \in (u_{A/\varrho})^{\cdot}, \quad (\mathscr{X}_n, \mathscr{X}_0) \in (u_{A/\varrho})^{\cdot}$$

hold for all $j = 0, \ldots, n$ and $i = 0, \ldots, n-1$. Then for all j there exist $X_j, X'_j \in \mathscr{X}_j$ such that for all i $(X_i, X'_{i+1}) \in u_{A/\varrho}$, $(X_n, X'_0) \in u_{A/\varrho}$. According to [5], section 17, is $\boldsymbol{u}_{A/\sigma} = u_{A/\varrho}$, and so relations

$$(x_i, x'_{i+1}) \in u_\varrho, \quad (x_n, x'_0) \in u_\varrho$$

hold for every $x_j \in X_j$ and $x'_j \in X'_j$ (see [5], section 16; or section 1 of the present paper, page 259). By the above, $(x_j, x'_j) \in \tau'$ and, via [5] section 13, it follows from the inclusion $\varrho \subseteq \tau'$ that $u_\varrho \subseteq u_{\tau'}$. Hence

$$(x_i, x'_{i+1}) \in u_{\tau'}, \quad (x'_{i+1}, x_{i+1}) \in \tau' \subseteq u_{\tau'}$$

and, from the fact that the quasiordering $u_{\tau'}$ is transitive on A, we get that $(x_0, x_j) \in$ $\in u_{\tau'}$ for all $x_0 \in X_0$ and for all $x_j \in X_j$. We have

$$x_0 \in \bigcup \mathscr{X}_0 \in A/\tau', \quad x_j \in \bigcup \mathscr{X}_j \in A/\tau',$$

and for all $x \in \bigcup \mathscr{X}_0$ and for all $y \in \bigcup \mathscr{X}_j$ we get $(x, y) \in u_{\tau'}$ (see [5] section 15). Therefore $(\bigcup \mathscr{X}_0, \bigcup \mathscr{X}_j) \in u_{A/\tau'}$. Analogously, we derive the relation $(\bigcup \mathscr{X}_j, \bigcup \mathscr{X}_0) \in u_{A/\tau'}$. Since, by hypotheses, $\tau' \in G(A; u)$, we get $\bigcup \mathscr{X}_0 = \bigcup \mathscr{X}_j$ for all indices j (see lemma 2). This implies $\mathscr{X}_0 = \mathscr{X}_1 = \ldots = \mathscr{X}_n$; therefore, by lemma 2, $\tau \in G(A/\varrho; u_{A/\varrho})$. *)

We have derived that the mapping

$$\tau \mapsto \tau' \quad (\tau \in G(A/\varrho; u_{A/\varrho}))$$

is a bijection from $G(A/\varrho; u_{A/\varrho})$ onto $\langle \varrho, \langle$. According to section 46/b, this mapping is an isomorphism from the complete lattice $(G(A/\varrho; u_{A/\varrho}); \subseteq)$ onto the complete lattice $(\langle \varrho, \langle; \subseteq)$.

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*) This part of the proof also shows, that the relational structure

$$((A/\varrho)/(\tau'/\varrho); (u_{A/\varrho})_{(A/\varrho)/(\tau'/\varrho)})$$

is a poset for $\varrho, \tau' \in G(A; u)$, $\varrho \subseteq \tau'$. This fact has been claimed in [5], section 55.

48. Corollary. *Let $\varrho, \sigma \in G(A; u)$, $\varrho \subseteq \sigma$ and let $X \in A/\sigma$. Then for $\tau_X \in G(X/\varrho, u_{X/\varrho})$ there exists exactly one element $\tau'_X \in (\langle \varrho \cap X^2 \langle_{(G(X;u);\subseteq)}$ with $\tau_X = \tau'_{X/(\varrho \cap X^2)}$. The mapping*

$$\tau_X \mapsto \tau'_X \quad (\tau_X \in G(X/\varrho; u_{X/\varrho}))$$

is an isomorphism from the complete lattice $(G(X/\varrho; u_{X/\varrho}); \subseteq)$ onto the complete lattice $(\langle \varrho \cap X^2, \langle_{(G(X;u)\subseteq)}; \subseteq)$.

Proof. We have $(\varrho \cap X^2) \in G(X; u)$ and $u_{X/(\varrho \cap X^2)} = (u \cap X^2)_{X/(\varrho \cap X^2)}$ (see theorem 6 and lemma 5). If in lemma 47 we substitute the set A for the set X, and the equivalence ϱ for the equivalence $\varrho \cap X^2$, then the proof follows.

49. Remark. Let I be a set and let $(X_i; u_i)$ be posets for all $i \in I$. Then we denote the cardinal product of the family $((X_i, u_i))_{i \in I}$ by $\prod_{i \in I} (X_i; u_i)$. Let us recall that the base set of the poset $\prod_{i \in I} (X_i; u_i)$ is usual cartesian product $\prod_{i \in I} X_i$ and the ordering of u on $\prod_{i \in I} X_i$ is defined as follows:

$$(x, y \in \prod_{i \in I} X_i) \Rightarrow ((x, y) \in u \Leftrightarrow_{\mathrm{Df}} \forall i \in I((x(i), y(i)) \in u_i)) .$$

The proof of the following statement follows directly from the definition of the cardinal product;

a) *Let $((X_i; u_i))_{i \in I}$ and $((Y_i; v_i))_{i \in I}$ be families of posets and let $\varphi_i : X_i \to Y_i$ be an isotonic isomorphism from $(X_i; u_i)$ onto $(Y_i; v_i)$ for every $i \in I$. Let us define a mapping $\varphi : \prod_{i \in I} X_i \to \prod_{i \in I} Y_i$ as follows:*

$$(x \in \prod_{i \in I} X_i) \Rightarrow (\varphi(x)(i) =_{\mathrm{Df}} \varphi_i(x(i))) .$$

Then φ: $\prod_{i \in I} X_i \to \prod_{i \in I} Y_i$ is an isotonic isomorphism from the cardinal product $\prod_{i \in I} (X_i; u_i)$ onto the cardinal product $\prod_{i \in I} (Y_i; v_i)$.

(The existence of an isomorphism follows also from the fact that the cardinal product is a product in the usual category Ord of posets, which is uniquely determined, up to isomorphism.) Let us mention that the cardinal product of a family of lattices is the complete direct product of this family.

50. Lemma. *Let $\varrho, \sigma \in G(A; u)$ and let $\varrho \subseteq \sigma$. Let us define a mapping*

$$\psi : \prod_{X \in A/\sigma} \langle \varrho \cap X^2, \langle_{(G(X;u);\subseteq)} \to \langle \varrho, \sigma \rangle$$

as follows

(28) $$(\tau \in \prod_{X \in A/\sigma} \langle \varrho \cap X^2, \langle_{(G(X;u);\subseteq)}) \Rightarrow (\psi(\tau) =_{\mathrm{Df}} \bigcup \{\tau(X) \mid X \in A/\sigma\}) .$$

Then ψ is an isotonic isomorphism from $\prod_{X \in A/\sigma} (\langle \varrho \cap X^2, \langle_{(G(X;u);\subseteq)}; \subseteq)$ onto $(\langle \varrho, \sigma \rangle; \subseteq)$.

Proof. Let us denote $C =_{\mathrm{Df}} \prod_{X \in A/\sigma} \langle \varrho \cap X^2, \langle_{(G(X;u);\subseteq)}$. We shall show first, that (28) defines a mapping $\psi : C \to \langle \varrho, \sigma \rangle$ correctly. Let $\tau \in C$. Then $\tau(X) \in E(X)$ for every $X \in A/\sigma$. The system A/σ is disjoint and therefore $\psi(\tau) = \bigcup\{\tau(X) \mid X \in A/\sigma\}$ is an equivalence on A. We have $\varrho \cap X^2 \subseteq \tau(X) \subseteq X^2$ for all $X \in A/\sigma$ and hence

$$\varrho = \bigcup\{\varrho \cap X^2 \mid X \in A/\sigma\} \subseteq \bigcup\{\tau(X) \mid X \in A/\sigma\} =$$
$$= \psi(\tau) \subseteq \bigcup\{X^2 \mid X \in A/\sigma\} = \sigma .$$

So $\psi(\tau) \in \langle \varrho, \sigma \rangle_{(E(A);\subseteq)}$. Moreover $\tau(X) \in G(X; u)$ for all $X \in A/\sigma$, because $\tau(X) \in \in \langle \varrho \cap X^2, \langle_{(G(X;u);\subseteq)}$. Hence

$$\psi(\tau) \cap X^2 = \tau(X) \in G(X; u) .$$

According to section 40, $\psi(\tau) \in G(A; u)$, because $\psi(\tau) \subseteq \sigma \in G(A; u)$ and $\psi(\tau)$ satisfies (18′) in section 40.

We have derived that $\psi(\tau) \in \langle \varrho, \sigma \rangle_{(E(A);\subseteq)} \cap G(A; u) = \langle \varrho, \sigma \rangle$, for $\tau \in C$. Therefore $\psi : C \to \langle \varrho, \sigma \rangle$. We shall show that the mapping $\psi : C \to \langle \varrho, \sigma \rangle$ is a surjection. Let $\alpha \in \langle \varrho, \sigma \rangle$. Then $X^2 \cap \varrho \subseteq X^2 \cap \alpha \subseteq X^2 \cap \sigma$ for all $X \in A/\sigma$ and via theorem 6 we get

$$\alpha \cap X^2 \in \langle \varrho \cap X^2, \langle_{(G(X;u);\subseteq)} .$$

Let us define a mapping

$$\alpha^* : A/\sigma \to \bigcup\{\langle \varrho \cap X^2, \langle_{(G(X;u);\subseteq)} \mid X \in A/\sigma\}$$

as follows:

$$(X \in A/\sigma) \Rightarrow (\alpha^*(X) =_{\mathrm{Df}} \alpha \cap X^2) .$$

Evidently, $\alpha^* \in C$ and, by its definition,

$$\psi(\alpha^*) = \bigcup\{\alpha^*(X) \mid X \in A/\sigma\} = \bigcup\{\alpha \cap X^2 \mid X \in A/\sigma\} = \alpha .$$

We have derived, that $\psi : C \to \langle \varrho, \sigma \rangle$ is a surjection. Let us denote

$$(C; v) =_{\mathrm{Df}} \prod_{X \in A/\sigma} (\langle \varrho \cap X^2, \langle_{(G(X;u);\subseteq)}; \subseteq) .$$

We shall show, that $\psi : C \to \langle \varrho, \sigma \rangle$ is an isotonic isomorphism from the cardinal product (C, v) onto $(\langle \varrho, \sigma \rangle; \subseteq)$. We have

$$(\tau_1, \tau_2) \in v \Leftrightarrow \forall X \in A/\sigma(\tau_1(X) \subseteq \tau_2(X)) \Leftrightarrow \psi(\tau_1) =$$
$$= \bigcup\{\tau_1(X) \mid X \in A/\sigma\} \subseteq \bigcup\{\tau_2(X) \mid X \in A/\sigma\} = \psi(\tau_2)$$

for $\tau_1, \tau_2 \in C$. (The second equivalence is a consequence of the fact, that A/σ is a partition of A and that the unions $\{\tau_k(X) \mid X \in A/\sigma\}$ $k = 1, 2$, are disjoint; that means that $\tau_k(X) \cap \tau_k(Y) = \emptyset$ for $X, Y \in A/\sigma$ with $X \neq Y$.)

51. Theorem. *Let $\varrho, \sigma \in G(A; u)$ and let $\varrho \subseteq \sigma$. Then the complete lattice $(\langle \varrho, \sigma \rangle; \subseteq)$ is isomorphic to the cardinal product $\prod\limits_{X \in A/\sigma} (G(X/\varrho; u_{X/\varrho}); \subseteq)$.*

Proof. According to sections 48 and 49/a, the cardinal products

$$\prod_{X \in A/\sigma} (G(X/\varrho; u_{X/\varrho}); \subseteq), \quad \prod_{X \in A/\sigma} (\langle \varrho \cap X^2, \langle_{(G(X;u);\subseteq)}; \subseteq)$$

are isomorphic. Therefore, the proof follows directly from lemma 50.

52. Remark. Theorem 51 guarantees the existence of a certain isomorphism. With the help of sections 48, 49/a and 50 this isomorphism can be constructed.

53. Corollary. *Let $\sigma \in G(A; u)$. Then the complete lattice $(\rangle, \sigma\rangle; \subseteq)$ is isomorphic to the cardinal product $\prod\limits_{X \in A/\sigma} (G(X; u); \subseteq)$.*

Proof. Let $X \in A/\sigma$. Then

$$X/\mathrm{id}_A = X/(\mathrm{id}_A \cap X^2) = X/\mathrm{id}_X = \{\{x\} \mid x \in X\}$$

and, by the definition of $\dot{u}$ (see section 1) the following holds for all $x, y \in X$:

$$(x, y) \in u \Leftrightarrow (\{x\}, \{y\}) \in \dot{u} \cap (X/\mathrm{id}_X)^2 .$$

Therefore the relation $\dot{u} \cap (X/\mathrm{id}_X)^2$ is transitive and so

$$u_{X/\mathrm{id}_A} = u_{X/\mathrm{id}_X} = \dot{u} \cap (X/\mathrm{id}_X)^2 .$$

It follows that the mapping $x \mapsto \{x\}$ $(x \in X)$ is an isotonic isomorphism from $(X; u)$ onto $(X/\mathrm{id}_X; u_{X/\mathrm{id}_X})$. From section 20 we get that the lattices $(G(X; u); \subseteq)$ and $(G(X/\mathrm{id}_X; u_{X/\mathrm{id}_X}); \subseteq)$ are isomorphisms too. Since $\langle \mathrm{id}_A, \sigma \rangle = \rangle, \sigma\rangle$, we get, via theorem 51,

$$(\rangle, \sigma\rangle; \subseteq) \quad \text{and} \quad \prod_{X \in A/\sigma} (G(X/\mathrm{id}_A; u_{X/\mathrm{id}_A}); \subseteq)$$

are isomorphic and also

$$\prod_{X \in A/\sigma} (G(X/\mathrm{id}_A; u_{X/\mathrm{id}_A}; \subseteq) \quad \text{and} \quad \prod_{X \in A/\sigma} (G(X; u); \subseteq)$$

are isomorphic.

54. Remark. Concluding this section, we shall notice the algebraic character of intervals in the lattices $(F(A; u); \subseteq)$ and $(G(A; u); \subseteq)$. This question is solved by means of the following statement a: (we suppose, that this statement is already known)

a) *Let $\mathscr{L} = (L; \leqq)$ be an algebraic lattice. Then each interval $(\langle a, b \rangle_{\mathscr{L}}; \leqq)$ is the algebraic lattice, for $a, b \in L$, $a \leqq b$.*

Proof follows from the fact, that the set

$$K =_{\mathrm{Df}} \{a \vee k \mid k \text{ is a compact element in } \mathscr{L},\ k \leqq b\}$$

is evidently the set of compact elements in the complete lattice $(\langle a, b\rangle; \leqq)$ and for every $x \in \langle a, b\rangle$ there exists $K_x \subseteq K$ that $x = \sup_{(\langle a,b\rangle; \leqq} K_x$.

b) *Intervals in* $(F(A; u); \subseteq)$ *and* $(G(A; u); \subseteq)$ *are algebraic lattices.*

Proof follows from a) and from theorems 33 and 30.

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