Alois Švec
On the rigidity of certain surfaces in E^5

Persistent URL: http://dml.cz/dmlcz/101465

ON THE RIGIDITY OF CERTAIN SURFACES IN E^5

Alois Švec, Olomouc

E. Bompiani [1] has presented the classes of surfaces in E^n which may admit non-trivial higher order deformations (at least locally). In E^5, the general surfaces of this class are those possessing a conjugate net. In what follows, I just show the global infinitesimal rigidity of a subclass of surfaces with a conjugate net. Of course, Bompiani's results deserve a further study.

1. Let $G \subset \mathcal{R}^2$ be a bounded domain, ∂G its boundary and $M : \bar{G} \to E^5$, $\bar{G} = G \cup \partial G$, a surface in the 5-dimensional Euclidean space. To each point $m \in M \equiv M(\bar{G})$, associate an orthonormal frame $\{v_1, \ldots, v_5\}$ such that $v_1, v_2 \in T(M)$. Then

$$(1) \qquad \begin{aligned}
dm &= \omega^1 v_1 + \omega^2 v_2\,, \\
dv_1 &= \omega_1^2 v_2 + \omega_1^3 v_3 + \omega_1^4 v_4 + \omega_1^5 v_5\,, \\
dv_2 &= -\omega_1^2 v_1 + \omega_2^3 v_3 + \omega_2^4 v_4 + \omega_2^5 v_5\,, \\
dv_3 &= -\omega_1^3 v_1 - \omega_2^3 v_2 + \omega_3^4 v_4 + \omega_3^5 v_5\,, \\
dv_4 &= -\omega_1^4 v_1 - \omega_2^4 v_2 - \omega_3^4 v_3 + \omega_4^5 v_5\,, \\
dv_5 &= -\omega_1^5 v_1 - \omega_2^5 v_2 - \omega_3^5 v_3 - \omega_4^5 v_4
\end{aligned}$$

with the usual integrability conditions $d\omega^i = \omega^j \wedge \omega_j^i$, $d\omega_i^j = \omega_i^k \wedge \omega_k^j$. From

$$(2) \qquad \omega^3 = \omega^4 = \omega^5 = 0\,,$$

we get

$$(3) \qquad \omega^1 \wedge \omega_1^3 + \omega^3 \wedge \omega_2^3 = 0\,, \quad \omega^1 \wedge \omega_1^4 + \omega^2 \wedge \omega_2^4 = 0\,,$$

$$\omega^1 \wedge \omega_1^5 + \omega^2 \wedge \omega_2^5 = 0$$

and the existence of functions $a_1, \ldots, c_3$ such that

(4)
$$\omega_1^3 = a_1\omega^1 + a_2\omega^2 , \quad \omega_2^3 = a_2\omega^1 + a_3\omega^2 ,$$

$$\omega_1^4 = b_1\omega^1 + b_2\omega^2 , \quad \omega_2^4 = b_2\omega^1 + b_3\omega^2 ,$$

$$\omega_1^5 = c_1\omega^1 + c_2\omega^2 , \quad \omega_2^5 = c_2\omega^1 + c_3\omega^2 .$$

Let $T_m^2(M)$ denote the 2-osculating space of M at m, and suppose dim $T_m^2(M) = 4$ for each $m \in M$. The frames $\{v_i\}$ be chosen in such a way that $v_3, v_4 \in T^2(M)$, i.e.,

(5)
$$\omega_1^5 = \omega_2^5 = 0 .$$

Let $\{w_i\}$ be another field of moving frames, and let

(6)
$$v_1 = \varepsilon_1(\cos \alpha . w_1 - \sin \alpha . w_2) , \quad v_2 = \sin \alpha . w_1 + \cos \alpha . w_2 ,$$

$$v_3 = \varepsilon_2(\cos \beta . w_3 - \sin \beta . w_4) , \quad v_4 = \sin \beta . w_3 + \cos \beta . w_4 ,$$

$$v_5 = \varepsilon_3 w_5 ; \quad \varepsilon_1^2 = \varepsilon_2^2 = \varepsilon_3^2 = 1 .$$

The equations (1) being now $dm = \Omega^i w_i$, $dw_i = \Omega_i^j w_j$, we get

(7)
$$\omega^1 = \varepsilon_1(\cos \alpha . \Omega^1 - \sin \alpha . \Omega^2) , \quad \omega^2 = \sin \alpha . \Omega^1 + \cos \alpha . \Omega^2 ,$$

$$\omega_1^3 = \varepsilon_1\varepsilon_2(\cos \alpha \cos \beta . \Omega_1^3 - \sin \alpha \cos \beta . \Omega_2^3 - \cos \alpha \sin \beta . \Omega_1^4 +$$

$$+ \sin \alpha \sin \beta . \Omega_2^4) ,$$

$$\omega_1^4 = \varepsilon_1(\cos \alpha \sin \beta . \Omega_1^3 - \sin \alpha \sin \beta . \Omega_2^3 + \cos \alpha \cos \beta . \Omega_1^4 -$$

$$- \sin \alpha \cos \beta . \Omega_2^4) ,$$

$$\omega_2^3 = \varepsilon_2(\sin \alpha \cos \beta . \Omega_1^3 + \cos \alpha \cos \beta . \Omega_2^3 - \sin \alpha \sin \beta . \Omega_1^4 -$$

$$- \cos \alpha \sin \beta . \Omega_2^4) ,$$

$$\omega_2^4 = \sin \alpha \sin \beta . \Omega_1^3 + \cos \alpha \sin \beta . \Omega_2^3 + \sin \alpha \cos \beta . \Omega_1^4 +$$

$$+ \cos \alpha \cos \beta . \Omega_2^4 .$$

Consider the functions

(8)
$$K = a_1 a_3 - a_2^2 + b_1 b_3 - b_2^2 , \quad k = a_1 b_2 - a_2 b_1 + a_2 b_3 - a_3 b_2$$

defined by

(9) $\omega_1^3 \wedge \omega_2^3 + \omega_1^4 \wedge \omega_2^4 = K\omega^1 \wedge \omega^2 , \quad \omega_1^3 \wedge \omega_1^4 + \omega_2^3 \wedge \omega_2^4 = k\omega^1 \wedge \omega^2$

resp. Similarly, K^* and k^* be defined by

(10) $\Omega_1^3 \wedge \Omega_2^3 + \Omega_1^4 \wedge \Omega_2^4 = K^*\Omega^1 \wedge \Omega^2 , \quad \Omega_1^3 \wedge \Omega_1^4 + \Omega_2^3 \wedge \Omega_2^4 = k^*\Omega^1 \wedge \Omega^2$

resp. From (7), we get

$$(11) \qquad\qquad K^* = K , \quad k^* = \varepsilon_1 \varepsilon_2 k .$$

Thus K is an invariant; it is easy to see that it is the Gauss curvature of M. The expression k depends just on the orientation of the moving frames. Let us look for the existence of a tangent vector field $V = \xi v_1 + \eta v_2$ such that there is a non-trivial tangent vector field $W = \xi' v_1 + \eta' v_2$ satisfying $WV \subset T(M)$. Because of

$$dV = dx \cdot v_1 + dy \cdot v_2 + x(\omega_1^2 v_2 + \omega_1^3 v_3 + \omega_1^4 v_4) + $$
$$+ y(-\omega_1^2 v_2 + \omega_2^3 v_3 + \omega_2^4 v_4) ,$$

we get

$$WV \equiv (a_1 \xi\xi' + a_2 \xi\eta' + a_2 \eta\xi' + a_3 \eta\eta') v_3 + $$
$$+ (b_1 \xi\xi' + b_2 \xi\eta' + b_2 \eta\xi' + b_3 \eta\eta') v_4 \pmod{T(M)} ;$$

because of $WV \equiv 0$ and $\xi'^2 + \eta'^2 \neq 0$, we get

$$(12) \qquad (a_1 b_2 - a_2 b_1) \xi^2 + (a_1 b_3 - a_3 b_1) \xi\eta + (a_2 b_3 - a_3 b_2) \eta^2 = 0 .$$

The equation (12) is the equation of the so-called conjugate directions $\{\xi v_1 + \eta v_2\}$ of M.

2. Let Φ be an infinitesimal second order deformation of our surface M; Φ is obviously given by (1) where we have to replace $\omega_3^4, \omega_3^5, \omega_4^5$ by $\omega_3^4 + t\varphi_4^3 + \dots$, $\omega_3^5 + t\varphi_3^5 + \dots$, $\omega_4^5 + t\varphi_4^5 + \dots$ resp. Comparing the terms at t in the integrability conditions, we get

$$(13) \qquad \omega_1^4 \wedge \varphi_3^4 + \omega_1^5 \wedge \varphi_3^5 = 0 , \quad \omega_1^3 \wedge \varphi_3^4 - \omega_1^5 \wedge \varphi_4^5 = 0 ,$$
$$\omega_2^4 \wedge \varphi_3^4 + \omega_2^5 \wedge \varphi_3^5 = 0 ,$$
$$\omega_2^3 \wedge \varphi_3^4 - \omega_2^5 \wedge \varphi_4^5 = 0 , \quad \omega_1^3 \wedge \varphi_3^5 + \omega_1^4 \wedge \varphi_4^5 = 0 ,$$
$$\omega_2^3 \wedge \varphi_3^5 + \omega_2^4 \wedge \varphi_4^5 = 0 ;$$

$$(14) \qquad\qquad d\varphi_3^4 = -\omega_3^5 \wedge \varphi_4^5 - \varphi_3^5 \wedge \omega_4^5 ,$$
$$d\varphi_3^5 = \omega_3^4 \wedge \varphi_4^5 + \varphi_3^4 \wedge \omega_4^5 , \quad d\varphi_4^5 = -\omega_3^4 \wedge \varphi_3^5 - \varphi_3^4 \wedge \omega_3^5 .$$

Because of (4) and (13), we get the existence of functions $A_1, \dots, C_4$ such that

$$(15) \qquad b_1\varphi_3^4 + c_1\varphi_3^5 = A_1\omega^1 + A_2\omega^2 , \quad a_1\varphi_3^4 - c_1\varphi_4^5 = B_1\omega^1 + B_2\omega^2 ,$$
$$b_2\varphi_3^4 + c_2\varphi_3^5 = A_2\omega^1 + A_3\omega^2 , \quad a_2\varphi_3^4 - c_2\varphi_4^5 = B_2\omega^1 + B_3\omega^3 ,$$
$$b_3\varphi_3^4 + c_3\varphi_3^5 = A_3\omega^1 + A_4\omega^2 , \quad a_3\varphi_3^4 - c_3\varphi_4^5 = B_3\omega^1 + B_4\omega^2 ,$$
$$a_1\varphi_3^5 + b_1\varphi_4^5 = C_1\omega^1 + C_2\omega^2 ,$$
$$a_2\varphi_{3}^5 + b_2\varphi_4^5 = C_2\omega^1 + C_3\omega^2 ,$$
$$a_3\varphi_3^5 + b_3\varphi_4^5 = C_3\omega^1 + C_4\omega^2 .$$

Write

(16) $\varphi_3^4 = x_1\omega^1 + x_2\omega^2$, $\varphi_3^5 = y_1\omega^1 + y_2\omega^2$, $\varphi_4^5 = z_1\omega^1 + z_2\omega^2$;

from (15), we obtain

(17)
$$b_2x_1 - b_1x_2 + c_2y_1 - c_1y_2 \qquad\qquad = 0,$$
$$b_3x_1 - b_2x_2 + c_3y_1 - c_2y_2 \qquad\qquad = 0,$$
$$a_2x_1 - a_1x_2 \qquad\qquad - c_2z_1 + c_1z_2 = 0,$$
$$a_3x_1 - a_2x_2 \qquad\qquad - c_3z_1 + c_2z_2 = 0,$$
$$a_2y_1 - a_1y_2 + b_2z_1 - b_1z_2 = 0,$$
$$a_3y_1 - a_2y_2 + b_3z_1 - b_2z_2 = 0.$$

We are now in the position to prove the following local

Theorem 1. *Let* $M \equiv M(G) \subset E^5$ *be a surface such that* $\dim T_m^2(M) = 5$ *for each* $m \in M$. *Then each infinitesimal second order deformation* Φ *of* M *is trivial.*

Proof. From (1), we obtain

(18) $v_1m = v_1,$ $v_2m = v_2,$

(19) $v_1v_1m = (\cdot)\,v_2 + a_1v_3 + b_1v_4 + c_1v_5,$

$\qquad\qquad\qquad v_2v_1m = (\cdot)\,v_2 + a_2v_3 + b_2v_4 + c_2v_5,$

$\qquad\qquad\qquad v_2v_2m = (\cdot)\,v_1 + a_3v_3 + b_3v_4 + c_3v_5.$

The vectors (19) are linearly independent because of the condition $\dim T_m^2(M) = 5$, and we may choose the frames in such a way that

(20) $b_1 = c_1 = c_2 = 0,$ $a_1b_2c_3 \neq 0.$

The system (17) is then reduced to

(21) $b_2x_1 = 0,$ $b_3x_1 - b_2x_2 + c_3y_1 = 0,$ $a_2x_1 - a_1x_2 = 0,$

$\qquad\qquad\qquad a_3x_1 - a_2x_2 - c_3z_1 = 0,$

$\qquad a_2y_1 - a_1y_2 + b_2z_1 = 0,$ $a_3y_1 - a_2y_2 + b_3z_1 - b_2z_2 = 0,$

and we get $x_1 = x_2 = y_1 = y_2 = z_1 = z_2 = 0$, i.e.,

(22) $\varphi_3^4 = \varphi_3^5 = \varphi_4^5 = 0.$

Thus Φ is trivial. QED.

3. Now, let us prove a global result.

Theorem 2. *Let $G \subset \mathcal{R}^2$ be a bounded domain and let $M : G \cup \partial G \to E^5$ be a surface such that:* (i) *$\dim T_m^2(M) = 4$ for each $m \in M$;* (ii) *there is $K \neq k$ or $K \neq -k$ on M;* (iii) *there are no real conjugate directions on M. Let Φ be an infinitesimal second order deformation of M which is trivial on the boundary of M. Then Φ is trivial on M.*

Proof. Choose moving frames $\{v_i\}$ of M and suppose $K \neq k$ on M; of course, we have (5). From (17) and (i), $x_1 = x_2 = 0$, i.e.,

$$(23) \qquad \qquad \varphi_3^4 = 0 .$$

From (13), (14) and (17),

$$(24) \qquad \omega_1^3 \wedge \varphi_3^5 + \omega_1^4 \wedge \varphi_4^5 = 0 , \quad \omega_2^3 \wedge \varphi_3^5 + \omega_2^4 \wedge \varphi_4^5 = 0 ,$$

$$(25) \qquad d\varphi_3^5 = \omega_3^4 \wedge \varphi_4^5 , \quad d\varphi_4^5 = -\omega_3^4 \wedge \varphi_3^5 ,$$

$$(26) \quad a_2 y_1 - a_1 y_2 + b_2 z_1 - b_1 z_2 = 0 , \quad a_3 y_1 - a_2 y_2 + b_3 z_1 - b_2 z_2 = 0 .$$

From $(16_{2,3})$ and (25), we get the existence of functions $S_1, \ldots, S_6$ such that

$$(27) \qquad dy_1 - y_2 \omega_1^2 - z_1 \omega_3^4 = S_1 \omega^1 + S_2 \omega^2 ,$$

$$dy_2 + y_1 \omega_1^2 - z_2 \omega_3^4 = S_2 \omega^1 + S_3 \omega^2 ,$$

$$dz_1 - z_2 \omega_1^2 + y_1 \omega_3^4 = S_4 \omega^1 + S_5 \omega^2 ,$$

$$dz_2 + z_1 \omega_1^2 + y_2 \omega_3^4 = S_5 \omega^1 + S_6 \omega^2 .$$

From $(4_{1-4}) + (5)$, we get the existence of functions $\alpha_1, \ldots, \beta_4$ such that

$$(28) \qquad da_1 - 2a_2 \omega_1^2 - b_1 \omega_3^4 = \alpha_1 \omega^1 + \alpha_2 \omega^2 ,$$

$$da_2 + (a_1 - a_3) \omega_1^2 - b_2 \omega_3^4 = \alpha_2 \omega^1 + \alpha_3 \omega^2 ,$$

$$da_3 + 2a_2 \omega_1^2 - b_3 \omega_3^4 = \alpha_3 \omega^1 + \alpha_4 \omega^2 ,$$

$$db_1 - 2b_2 \omega_1^2 + a_1 \omega_3^4 = \beta_1 \omega^1 + \beta_2 \omega^2 ,$$

$$db_2 + (b_1 - b_3) \omega_1^2 + a_2 \omega_3^4 = \beta_2 \omega^1 + \beta_3 \omega^2 ,$$

$$db_3 + 2b_2 \omega_1^2 + a_3 \omega_3^4 = \beta_3 \omega^1 = \beta_4 \omega^2 .$$

The differential consequences of (26) are then

$$(29) \quad a_2 S_1 - a_1 S_2 + b_2 S_4 - b_1 S_5 = -\alpha_2 y_1 + \alpha_1 y_2 - \beta_2 z_1 + \beta_1 z_2 ,$$

$$a_2 S_2 - a_1 S_3 + b_2 S_5 - b_1 S_6 = -\alpha_3 y_1 + \alpha_2 y_2 - \beta_3 z_1 + \beta_2 z_2 ,$$

$$a_3 S_1 - a_2 S_2 + b_3 S_4 - b_2 S_5 = -\alpha_3 y_1 + \alpha_2 y_2 - \beta_3 z_1 + \beta_2 z_2 ,$$

$$a_3 S_2 - a_2 S_3 + b_3 S_5 - b_2 S_6 = -\alpha_4 y_1 + \alpha_3 y_2 - \beta_4 z_1 + \beta_3 z_2 .$$

254

Consider the system

(30) $\qquad a_1 X + b_1 Y - a_2 Z - b_2 T = Q_1, \quad a_2 X + b_2 Y - a_3 Z - b_3 T = Q_2$

for X, Y, Z, T. It is easy to show that the convenable combinations of the equations (30) lead to

(31) $\qquad (K - k)\, X = \left(a_3 b_2 - a_2 b_3 + b_1 b_3 - b_2^2\right)(X + T) +$
$$+ \left(a_2 b_2 - a_3 b_1\right)(Y - Z) + \left(a_3 - b_2\right) Q_1 + \left(b_1 - a_2\right) Q_2,$$

$\qquad (K - k)\, Y = \left(a_2 b_2 - a_1 b_3\right)(X + T) + \left(a_1 a_3 - a_2^2 + a_3 b_2 - a_2 b_3\right)(Y - Z) +$
$$+ \left(a_2 + b_3\right) Q_1 - \left(a_1 + b_2\right) Q_2,$$

$\qquad (K - k)\, Z = \left(a_2 b_2 - a_1 b_3\right)(X + T) + \left(a_1 b_2 - a_2 b_1 - b_1 b_3 + b_2^2\right)(Y - Z) +$
$$+ \left(a_2 + b_3\right) Q_1 - \left(a_1 + b_2\right) Q_2,$$

$\qquad (K - k)\, T = \left(a_1 a_3 - a_2^2 - a_1 b_2 + a_2 b_1\right)(X + T) + \left(a_3 b_1 - a_2 b_2\right)(Y - Z) +$
$$+ \left(b_2 - a_3\right) Q_1 + \left(a_2 - b_1\right) Q_2.$$

Applying this auxiliary result to the systems (26), $(29_{1,3})$ and $(29_{2,4})$ resp., we get

(32) $\qquad (K - k)\, y_2 = \left(a_3 b_2 - a_2 b_3 + b_1 b_3 - b_2^2\right)(y_2 + z_1) +$
$$+ \left(a_2 b_2 - a_3 b_1\right)(z_2 - y_1),$$

$\qquad (K - k)\, z_2 = \left(a_2 b_2 - a_1 b_3\right)(y_2 + z_1) +$
$$+ \left(a_1 a_3 - a_2^2 + a_3 b_2 - a_2 b_3\right)(z_2 - y_1),$$

$\qquad (K - k)\, y_1 = \left(a_2 b_2 - a_1 b_3\right)(y_2 + z_1) +$
$$+ \left(a_1 b_2 - a_2 b_1 - b_1 b_3 + b_2^2\right)(z_2 - y_1),$$

$\qquad (K - k)\, z_1 = \left(a_1 a_3 - a_2^2 - a_1 b_2 + a_2 b_1\right)(y_2 + z_1) +$
$$+ \left(a_3 b_1 - a_2 b_2\right)(z_2 - y_1);$$

(33) $\quad (K - k)\, S_2 = \left(a_3 b_2 - a_2 b_3 + b_1 b_3 - b_2^2\right)(S_2 + S_4) + \left(a_2 b_2 - a_3 b_1\right) \cdot$
$$\cdot (S_5 - S_1) + \left(a_3 - b_2\right) q_1 + \left(b_1 - a_2\right) q_2,$$

$\qquad (K - k)\, S_5 = \left(a_2 b_2 - a_1 b_3\right)(S_2 + S_4) + \left(a_1 a_3 - a_2^2 + a_3 b_2 - a_2 b_3\right) \cdot$
$$\cdot (S_5 - S_1) + \left(a_2 + b_3\right) q_1 - \left(a_1 + b_2\right) q_2,$$

$\qquad (K - k)\, S_1 = \left(a_2 b_2 - a_1 b_3\right)(S_2 + S_4) + \left(a_1 b_2 - a_2 b_1 - b_1 b_3 + b_2^2\right) \cdot$
$$\cdot (S_5 - S_1) + \left(a_2 + b_3\right) q_1 - \left(a_1 + b_2\right) q_2,$$

$\qquad (K - k)\, S_4 = \left(a_1 a_3 - a_2^2 - a_1 b_2 + a_2 b_1\right)(S_2 + S_4) + \left(a_3 b_1 - a_2 b_2\right) \cdot$
$$\cdot (S_5 - S_1) + \left(b_2 - a_3\right) q_1 + \left(a_2 - b_1\right) q_2$$

with

$$q_1 = \alpha_2 y_1 - \alpha_1 y_2 + \beta_2 z_1 - \beta_1 z_2 \,, \quad q_2 = \alpha_3 y_1 - \alpha_2 y_2 + \beta_3 z_1 - \beta_2 z_2 \,;$$

(34) $\quad (K - k) S_3 = (a_3 b_2 - a_2 b_3 + b_1 b_3 - b_2^2)(S_3 + S_5) + (a_2 b_2 - a_3 b_1) \,.$
$$\qquad\qquad \cdot (S_6 - S_2) + (a_3 - b_2) q_1' + (b_1 - a_2) q_2' \,,$$

$$(K - k) S_6 = (a_2 b_2 - a_1 b_3)(S_3 + S_5) + (a_1 a_3 - a_2^2 + a_3 b_2 - a_2 b_3) \,.$$
$$\qquad\qquad \cdot (S_6 - S_2) + (a_2 + b_3) q_1' - (a_1 + b_2) q_2' \,,$$

$$(K - k) S_2 = (a_2 b_2 - a_1 b_3)(S_3 + S_5) + (a_1 b_2 - a_2 b_1 - b_1 b_3 + b_2^2) \,.$$
$$\qquad\qquad \cdot (S_6 - S_2) + (a_2 + b_3) q_1' - (a_1 + b_2) q_2' \,,$$

$$(K - k) S_5 = (a_1 a_3 - a_2^2 - a_1 b_2 + a_2 b_1)(S_3 + S_5) + (a_3 b_1 - a_2 b_2) \,.$$
$$\qquad\qquad \cdot (S_6 - S_2) + (b_2 - a_3) q_1' + (a_2 - b_1) q_2'$$

with

$$q_1' = \alpha_3 y_1 - \alpha_2 y_2 + \beta_3 z_1 - \beta_2 z_2 \,, \quad q_2' = \alpha_4 y_1 - \alpha_3 y_2 + \beta_4 z_1 - \beta_3 z_2 \,.$$

It follows from (32) and $K \neq k$ that any linear combination of y_1, y_2, z_1, z_2 may be written as a combination of $y_2 + z_1$ and $y_1 - z_2$.

From (27), we get

(35) $\quad d(y_1 - z_2) - (y_2 + z_1)(\omega_1^2 + \omega_3^4) = (S_1 - S_5)\,\omega^1 + (S_2 - S_6)\,\omega^2 \,,$

$$d(y_2 + z_1) + (y_1 - z_2)(\omega_1^2 + \omega_3^4) = (S_2 + S_4)\,\omega^1 + (S_3 + S_5)\,\omega^2 \,.$$

Let G be covered by a system of isothermic coordinates (u, v); let

(36) $\qquad I = r^2(du^2 + dv^2), \quad r(u, v) > 0 \,, \quad \text{i.e.,} \quad \omega^1 = r\,du \,, \quad \omega^2 = r\,dv \,.$

Then

(37)
$$\frac{\partial(y_1 - z_2)}{\partial u} = (S_1 - S_5)\,r + \varrho_1(y_2 + z_1) \,,$$

$$\frac{\partial(y_1 - z_2)}{\partial v} = (S_2 - S_6)\,r + \varrho_2(y_2 + z_1) \,,$$

$$\frac{\partial(y_2 + z_1)}{\partial u} = (S_2 + S_4)\,r + \varrho_3(y_1 - z_2) \,,$$

$$\frac{\partial(y_2 + z_1)}{\partial v} = (S_3 + S_5)\,r + \varrho_4(y_1 - z_2) \,,$$

256

$\varrho_1, \ldots, \varrho_4$ being easy to calculate. From (37) and $(33_1) + (34_3)$ or $(33_2) + (34_4)$ resp., we get a system of the form $(i = 1, 2)$

$$(38) \qquad a_{i1}\frac{\partial(y_1 - z_2)}{\partial u} + a_{i2}\frac{\partial(y_1 - z_2)}{\partial v} + b_{i1}\frac{\partial(y_2 + z_1)}{\partial u} +$$

$$+ b_{i2}\frac{\partial(y_2 + z_1)}{\partial v} = c_{i1}(y_1 - z_2) + c_{i2}(y_2 + z_1)$$

with

$$(39) \quad a_{11} = a_2b_2 - a_3b_1, \qquad\qquad a_{12} = b_1b_3 - b_2^2 + a_2b_1 - a_1b_2,$$

$$a_{21} = a_1a_3 - a_2^2 + a_3b_2 - a_2b_3, \quad a_{22} = a_2b_2 - a_3b_1,$$

$$b_{11} = b_2^2 - b_1b_3 + a_2b_3 - a_3b_2, \quad b_{12} = a_2b_2 - a_1b_3,$$

$$b_{21} = a_1b_3 - a_2b_2. \qquad\qquad b_{22} = a_1a_3 - a_2^2 + a_2b_1 - a_1b_2.$$

Recall that the system (38) is called elliptic if the quadratic form

$$(40) \quad \varphi = (a_{12}b_{22} - a_{22}b_{12})\mu^2 - (a_{11}b_{22} - a_{21}b_{12} + a_{12}b_{21} - a_{22}b_{11})\mu\nu +$$

$$+ (a_{11}b_{21} - a_{21}b_{11})\nu^2$$

is definite. In our case,

$$(41) \quad \varphi = (k - K)\{(a_1b_2 - a_2b_1)\mu^2 + (a_1b_3 - a_3b_1)\mu\nu + (a_2b_3 - a_3b_2)\nu^2\},$$

and φ is definite because of the suppositions of our Theorem. Thus $y_1 - z_2 = y_2 + z_1 = 0$ in G and (32) implies $y_1 = y_2 = z_1 = z_2 = 0$ in G.

In the case $K \neq -k$, we have to use $y_1 + z_2, y_2 - z_1$ instead of $y_1 - z_2, y_2 + z_1$ resp. QED.

Let us remark that locally Φ does not need to be trivial.

Bibliography

[1] *E. Bompiani:* Geometrie riemanniane di specie superiore. R. Acc. Italia, Mem. Cl. di Sci. Fiz., Mat. e Nat., Vol. VI, 1935.

Author's address: 771 46 Olomouc, Leninova 26, ČSSR (Přírodovědecká fakulta University Palackého).