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FREE SUSLIN ALGEBRAS

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1. INTRODUCTION

Let ω = the non-negative integers, and for each ordinal $\varkappa$, $\varkappa^{<\omega} = \bigcup_{n<\omega}\varkappa^n$ = the set of all finite sequences of elements of $\varkappa$, and $\varkappa^\omega$ = the set of all functions mapping ω into $\varkappa$. Let n range over ω, f range over ω^ω, and $f \mid n$ denote the finite sequence consisting of the first n values of f. Consider a Boolean algebra B whose fundamental operations are written in additive notation. B is an *S-algebra* if it is a σ-algebra, and for each $a : \omega^{<\omega} \to B$, contains $\sum_f \prod_n a_{f|n}$. The latter element of B is denoted by $\mathscr{A}a$. An S-algebra is also closed under the *dual* $\mathscr{A}^*a = \prod_f \sum_n a_{f|n}$. These algebras were introduced by RIEGER (cf. [14]), who intended that they serve as the correct structure in which we can model $\prod_1^1$ analysis. An *S-field of sets* is a σ-field of sets closed under the operation $\mathscr{A}$, and an *S-homomorphism* is a σ-homomorphism between S-algebras which preserves $\mathscr{A}$. An S-algebra is *S-representable* if it is an S-homomorphic image of an S-field of sets. Rieger's [14] is an exhaustive study of the necessary and sufficient conditions for an algebra to be S-representable. His methods are basically algebraic and stem from his earlier work (cf. [13]) on characterizing various kinds of free algebras. [14] is written in Russian (which we can't read) and was only brought to our attention by CAMPBELL who generously provided us with a xerox of his translation. Campbell's Ph. D. thesis [1] (written under the direction of NERODE) introduces a special kind of S-algebra and several conjectures about them which we discuss in the last section of this paper. Our own work is quite different and has its origins in the metamathematics of HENKIN (cf. [5]) and his student KARP (cf. [7]). Additional material on S-algebras can be found in [2], [3], [11], and [12].

2. SR-ALGEBRAS

Let u range over $\omega^{<\omega}$. By $u \wedge n$ we mean that sequence obtained from u by adding n to it as a last element. Ω = the set of all countable ordinals and α, λ range over Ω with λ a limit ordinal. If B is an S-algebra and $a : \omega^{<\omega} \to B$ define

$$(1) \qquad a_u^0 = a_u\,, \quad a_u^{\alpha+1} = a_u^\alpha \cdot \sum_n a_{u\wedge n}^\alpha\,, \quad a_u^\lambda = \prod_{\alpha<\lambda} a_u^\alpha$$

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by ordinal recursion. Λ is the empty sequence. Using the same a as above let

$$(2) \qquad a^\alpha = a^\alpha_\Lambda\,, \quad a^*_\alpha = -(-a)^\alpha$$

where $-a$ has the value $-a_u$ at u. Then B is an *SR-algebra* if

$$(3) \qquad \mathscr{A}a = \prod\nolimits_\alpha a^\alpha$$

for every $a : \omega^{<\omega} \to B$. This definition was inspired by Rieger's algebraic analysis of Sierpinski's "constituent" method (cf. [17]).

Now let B' also be an S-algebra. B' is an S-subalgebra of B if it is a σ-subalgebra of B and the $\mathscr{A}$ operation applied to any $a : \omega^{<\omega} \to B'$ has the same value in B' as it has in B. A set $G \subseteq B$ *S-generates* B if the smallest S-subalgebra of B containing G is B itself. Clearly an S-subalgebra of an SR-algebra is an SR-algebra, but of course (3) is not necessarily preserved by S-homomorphisms. G *freely SR-generates* B if $G \subseteq B$ and any map from G into any SR-algebra B' can be extended to a unique S-homomorphism from B into B'. We intend showing that for any cardinal $\varkappa$ there exists an S-field of sets which is freely SR-generated by $\varkappa$ elements, i.e., every SR-algebra is S-representable. We do this by introducing a formal language L and an associated SR-logic which makes the Lindenbaum algebra L SR-free. We then prove a completeness theorem for SR-logic from which it is easy to show that the Lindenbaum algebra of L is isomorphic to an S-field of sets.

L is a proportional language containing a variable p_ξ for each $\xi < \varkappa$. It also contains connective symbols $\sim, \bigwedge, \bigvee, \mathscr{A}, \mathscr{A}^*$. Let P be the smallest set containing each p_ξ such that if $\varphi \in P$, $\sigma \subseteq P$ is at most countable, and $\mu : \omega^{<\omega} \to P$ then the concatenations $\sim\varphi$, $\bigwedge\sigma$, $\bigvee\sigma$, $\mathscr{A}\mu$, $\mathscr{A}^*\mu$ are all elements of P. Let φ, ψ range over P, σ range over at most countable subsets of P, and μ range over functions mapping $\omega^{<\omega}$ into P. Let $\to(\varphi, \psi) = \bigvee\{\sim\varphi, \psi\}$. Each φ is a *sentence* of L. There is an equivalent definition of P which is more useful for our purposes. Define

$$P_0 = \{p_\xi \mid \xi < \varkappa\}\,,$$

$$P_{\alpha+1} = P_\alpha \cup \{\sim\varphi \mid \varphi \in P_\alpha\} \cup$$

$$\cup \{\bigwedge\sigma \mid \sigma \subseteq P_\alpha\} \cup \{\bigvee\sigma \mid \sigma \subseteq P_\alpha\} \cup \{\mathscr{A}\mu \mid$$

$$\text{range } (\mu) \subseteq P_\alpha\} \cup \{\mathscr{A}^*\mu \mid \text{range } (\mu) \subseteq P_\alpha\}\,,$$

$$P_\lambda = \bigcup\nolimits_{\alpha<\lambda} P_\alpha\,.$$

Then $P = \bigcup_{\alpha<\Omega} P_\alpha$. Moreover this allows us to assign an ordinal rank to members of P. We let *rank* (φ) = the least α such that $\varphi \in P_\alpha$. We will not explicitly work with this function, but unless otherwise stated our recursions and inductions will be made with respect to *rank*.

Subsentences are defined by the recursion

(4)
$$\begin{aligned}
\operatorname{sub} p_\xi &= \{p_\xi\}\,,\\
\operatorname{sub} \sim \varphi &= \{\sim \varphi\} \cup \operatorname{sub} \varphi\,,\\
\operatorname{sub} \bigwedge\sigma &= \{\bigwedge\sigma\} \cup \textstyle\bigcup_{\varphi\in\sigma}\operatorname{sub} \varphi\,,\\
\operatorname{sub} \bigvee\sigma &= \{\bigvee\sigma\} \cup \textstyle\bigcup_{\varphi\in\sigma}\operatorname{sub} \varphi\,,\\
\operatorname{sub} \mathscr{A}\mu &= \{\mathscr{A}\mu\} \cup \textstyle\bigcup_{u}\operatorname{sub} \mu_u\,,\\
\operatorname{sub} \mathscr{A}^*\mu &= \{\mathscr{A}^*\mu\} \cup \textstyle\bigcup_{u}\operatorname{sub} \mu_u\,.
\end{aligned}$$

We then prove by induction that sub φ is at most countable for $\varphi \in P$. Let $\sim \sigma = \{\sim \varphi \mid \varphi \in \sigma\}$ and $\sim \mu$ take u into $\sim \mu_u$. Then we *move a quantifier inside* by

(5)
$$\begin{aligned}
(p_\xi) \sim &= \sim p_\xi\,,\\
(\sim \varphi) \sim &= \varphi\,,\\
(\bigwedge\sigma) \sim &= \bigvee \sim \sigma\,,\\
(\bigvee\sigma) \sim &= \bigwedge \sim \sigma\,,\\
(\mathscr{A}\mu) \sim &= \mathscr{A}^* \sim \mu\,,\\
(\mathscr{A}^*\mu) \sim &= \mathscr{A} \sim \mu\,.
\end{aligned}$$

Constituents are now defined as in (1)–(2). For $\mu : \omega^{<\omega} \to P$ let

(6)
$$\mu_u^0 = \mu_u\,, \quad \mu_u^{\alpha+1} = \mu_u^\alpha \wedge \textstyle\bigvee_n \mu^\alpha_{u\wedge n}\,, \quad \mu_u^\lambda = \textstyle\bigwedge_{\alpha<\lambda}\mu_u^\alpha\,,$$

(where we have abused Polish notation) and

(7)
$$\mu^\alpha = \mu^\alpha_\Lambda\,.$$

For our *SR-axioms* we take

(8) every instance of a tautology from finitary propositional logic,

(9)
$$(\varphi \sim) \leftrightarrow \sim \varphi\,,$$

(10)
$$\textstyle\bigwedge\sigma \to \varphi \quad \text{for any} \quad \varphi \in \sigma\,,$$

(11)
$$\textstyle\bigwedge_n \mu_{f|n} \to \mathscr{A}\mu \quad \text{for any} \quad f\,,$$

(12)
$$\mathscr{A}\mu \to \mu^\alpha \quad \text{for any} \quad \alpha\,.$$

For our *SR-rules of inference* we take

(13) $$\text{modul ponens}\,,$$

(14) $$\frac{\{\psi \to \varphi \mid \varphi \in \sigma\}}{\psi \to \bigwedge\sigma}\,,$$

(15) $$\frac{\{\bigwedge_n \mu_{f|n} \to \varphi \mid f \in \omega^\omega\}}{\mathscr{A}\mu \to \varphi}\,,$$

(16) $$\frac{\{\varphi \to \mu^\alpha \mid \alpha < \Omega\}}{\varphi \to \mathscr{A}\mu}\,.$$

Define *SR-proof* and *SR-theorem* in the usual way, where we put no restriction on the length of a proof. Write $SR \vdash \varphi$ if φ is an SR-theorem and omit the SR if there is no ambiguity. With the exception of (11), (12), (15) and (16) these axioms appear in [8]. (11) and (15) appear in [1]. (12) and (16) are new.

For $\varphi, \psi \in P$ let $\varphi \equiv \psi$ if $\vdash \varphi \leftrightarrow \psi$. It follows from (8) that $\equiv$ is an equivalence relation. Let $[\varphi]$ be the equivalence class containing φ and let $A_\varkappa = \{[\varphi] \mid \varphi \in P\}$. We then show

(17) $$\varphi \equiv \varphi' \text{ and } \psi \equiv \psi' \text{ implies } \sim\varphi \equiv \sim\varphi'\,,$$
$$\varphi \vee \psi \equiv \varphi' \vee \psi'\,, \text{ and } \varphi \wedge \psi \equiv \varphi' \wedge \psi'\,.$$

(18) $$\text{If } h : \sigma \to \sigma' \text{ is one-one and onto such that } \varphi' \equiv h(\varphi)$$
$$\text{for all } \varphi \in \sigma \text{ then } \bigwedge\sigma \equiv \bigwedge\sigma' \text{ and } \bigvee\sigma \equiv \bigvee\sigma'\,.$$

(19) $$\text{if } \mu_u \equiv \mu'_u \text{ for all } u \text{ then } \mathscr{A}\mu \equiv \mathscr{A}\mu' \text{ and } \mathscr{A}^*\mu \equiv \mathscr{A}^*\mu'\,.$$

We shall not prove (17)–(18); they are fairly well known. For (19), our hypotheses give

$\vdash \mu_{f|k} \to \mu'_{f|k}$ for all $k < \omega$, for all f,

$\vdash \bigwedge_n \mu_{f|n} \to \mu'_{f|k}$ for all k, for all f by (10),

$\vdash \bigwedge_n \mu_{f|n} \to \bigwedge_n \mu'_{f|n}$ for all f by (14),

$\vdash \bigwedge_n \mu_{f|n} \to \mathscr{A}\mu'$ for all f by (11), and finally

$\vdash \mathscr{A}\mu \to \mathscr{A}\mu'$ by (15). The reverse arrow is obtained in the same way. We now define algebraic operations on $A_\varkappa$ by

(20) $$-[\varphi] = [\sim\varphi]\,, \quad [\varphi] + [\psi] = [\varphi \vee \psi]\,, \quad \text{and} \quad [\varphi]\cdot[\psi] = [\varphi \wedge \psi]\,,$$

(21) $$\prod\{[\varphi] \mid \varphi \in \sigma\} = [\bigwedge\sigma]\,, \quad \text{and} \quad \sum\{[\varphi] \mid \varphi \in \sigma\} = [\bigvee\sigma]\,,$$

(22) $$\mathscr{A}(\lambda u[\mu_u]) = [\mathscr{A}\mu]\,, \quad \text{and} \quad \mathscr{A}^*(\lambda u[\mu_u]) = [\mathscr{A}^*\mu]$$

where λ is Church's functional symbol. These operations are well defined by (17)–(19). Thus we will regard $A_\varkappa$ as an algebraic system whose operations have the same names as those of an S-algebra. Our first result is

Theorem 1. *$A_\varkappa$ is an SR-algebra which is freely SR-generated by $\varkappa$ of its elements.*

Proof. It is well known that $A_\varkappa$ is a Boolean σ-algebra where $0 = [p_0 \wedge \sim p_0]$, and $1 = [p_0 \vee \sim p_0]$. Since $\vdash (\varphi \wedge \psi) \leftrightarrow \varphi$ if and only if $\vdash \varphi \to \psi$ we have established that $[\varphi] \leqq [\psi]$ if and only if $\vdash \varphi \to \psi$. Let $\nu = \lambda u[\mu_u]$. We must show that $\sup_f \inf_n \nu_{f|n} = \mathscr{A}\nu$. This will imply that $\mathscr{A}$ as defined in (22) is the same as that of an S-algebra. First $\vdash \bigwedge_n \mu_{f|n} \to \mu_{f|k}$ so $\prod_n \nu_{f|n} = [\bigwedge_n \mu_{f|n}] \leqq [\mu_{f|k}] = \nu_{f|k}$. Next if $[\varphi] \leqq \nu_{f|k}$ for all k then $\vdash \varphi \to \mu_{f|k}$ for all k and hence $\vdash \varphi \to \bigwedge_n \mu_{f|n}$, i.e., $[\varphi] \leqq$ $\leqq [\bigwedge_n \mu_{f|n}] = \prod_n \nu_{f|n}$. Thus we have shown that $\inf_n \nu_{f|n} = \prod_n \nu_{f|n}$. Now $\vdash \bigwedge_n \mu_{f|n} \to$ $\to \mathscr{A}\mu$ for any f so $\prod_n \nu_{f|n} = [\bigwedge_n \mu_{f|n}] \leqq [\mathscr{A}\mu] = \mathscr{A}\nu$. If $\prod_n \nu_{f|n} \leqq [\varphi]$ any f then $\vdash \bigwedge_n \mu_{f|n} \to \varphi$ for any f and hence $\vdash \mathscr{A}\mu \to \varphi$, i.e., $\mathscr{A}\nu = [\mathscr{A}\mu] \leqq [\varphi]$. Thus $\mathscr{A}\nu =$ $= \sup_f \prod_n \nu_{f|n} = \sup_f \inf_n \nu_{f|n}$. Therefore $A_\varkappa$ is an S-algebra for which the operations of (20)–(22) are properly defined. Next we omit an easy induction on α that $\nu^\alpha =$ $= [\mu^\alpha]$. Since $\vdash \mathscr{A}\mu \to \mu^\alpha$ for all α it follows that $\mathscr{A}\nu = [\mathscr{A}\mu] \leqq [\mu^\alpha] = [\mu]^\alpha = \nu^\alpha$ for all α, and if $[\varphi] \leqq \nu^\alpha$ for all α then $\vdash \varphi \to \mu^\alpha$ for all α and hence $\vdash \varphi \to \mathscr{A}\mu$, i.e., $[\varphi] \leqq [\mathscr{A}\mu] = \mathscr{A}\nu$. Thus $\mathscr{A}\nu = \inf_\alpha \nu^\alpha$ concluding our proof that $A_\varkappa$ is an SR-algebra. Now let B be any SR-algebra and let h be a function mapping the variables of P into B. We extend h to all of P by the recursion

(23) $$h(\sim \varphi) = -h(\varphi)\,, \quad h(\varphi \vee \psi) = h(\varphi) + h(\psi)\,,$$
$$\text{and} \quad h(\varphi \wedge \psi) = h(\varphi) \cdot h(\psi)\,,$$

(24) $$\text{if} \quad h(\sigma) = \{h(\varphi) \mid \varphi \in \sigma\} \quad \text{then} \quad h(\bigwedge\sigma) = \prod h(\sigma) \quad \text{and} \quad h(\bigvee\sigma) = \sum h(\sigma)\,,$$

(25) $$\text{if} \quad \nu = \lambda u h(\mu_u) \quad \text{then} \quad h(\mathscr{A}\mu) = \mathscr{A}\nu \quad \text{and} \quad h(\mathscr{A}^*\mu) = \mathscr{A}^*\nu\,.$$

We shall now prove

(26) $$\vdash \varphi \quad \text{implies that} \quad h(\varphi) = 1\,.$$

This is done by first showing that h maps each axiom into 1 and then showing that if each premise of a rule of inference gets mapped to 1 then so does its conclusion. It is clear that axioms of group (8) are mapped to 1. By our definition of $\to$ we have $h(\varphi \to \psi) = 1$ if and only if $h(\varphi) \leqq h(\psi)$. This easily shows that axioms of groups (10)–(12) get mapped into 1. For (12) we must show that $h(\mu^\alpha) = \nu^\alpha$ where ν is given as in (25). This is proved by induction on α. For (9) we use DeMorgan's laws. A proof that the rules of inference preserve the property being mapped into 1 is standard. We do (16) as an example. If $h(\varphi \to \mu^\alpha) = 1$ for all α then $h(\varphi) \leqq h(\mu^\alpha) = \nu^\alpha$ so $h(\varphi) \leqq \mathscr{A}\nu$ where ν is given as in (25), because B is an SR-algebra. From $h(\varphi) \leqq$ $\leqq h(\mathscr{A}\mu)$ we conclude that $h(\varphi \to \mathscr{A}\mu) = 1$. This completes our proof of (26).

Now $h(\varphi \leftrightarrow \psi) = 1$ if and only if $h(\varphi) = h(\psi)$. Since h was arbitrary on the variables of P we could have required that $h(p_\xi) \neq h(p_\eta)$, where $\xi, \eta < \varkappa$ and $\xi \neq \eta$. Thus $p_\xi \leftrightarrow p_\eta$ is not an SR-theorem and consequently $[p_\xi] \neq [p_\eta]$. Let $G = \{[p_\xi] \mid \xi < \varkappa\}$. We shall show that

$$\text{(27)} \qquad G \text{ freely SR-generates } A_\varkappa .$$

Let $H : G \to B$ where B is an SR-algebra. Define $h(p_\xi) = H([p_\xi])$ for $\xi < \varkappa$. Extend h to all of P by (23)–(25). Since $[\varphi] = [\psi]$ implies $\vdash \varphi \leftrightarrow \psi$ implies $h(\varphi) = h(\psi)$ the map $H([\varphi]) = h(\varphi)$ is well defined on $A_\varkappa$ and extends H on G. To show that H is an S-homomorphism we must show that it preserves the Boolean, the σ, and the $\mathscr{A}$ operations. We do the last. Others are done in the same way. $H(\mathscr{A}\lambda u[\mu_u]) = H([\mathscr{A}\mu]) = h(\mathscr{A}\mu) = \mathscr{A}\lambda u h(\mu_u) = \mathscr{A}\lambda u H([\mu_u])$. Finally we must show that this extension is unique. First note that G S-generates $A_\varkappa$. Let H' be an S-homomorphism mapping $A_\varkappa$ into B agreeing with H on G and let $D = \{x \in A_\varkappa \mid H(x) = H'(x)\}$. We show that D is an S-subalgebra of $A_\varkappa$ and so equals $A_\varkappa$. Thus we must show that D is closed under the Boolean, the σ, and the $\mathscr{A}$ operations We do the last. Suppose that μ maps into D. Then $H(\mathscr{A}\mu) = \mathscr{A}\lambda u H(\mu_u) = \mathscr{A}\lambda u H'(\mu_u) = H'(\mathscr{A}\mu)$. Thus $\mathscr{A}\mu \in D$. We finally remark that up to isomorphism $A_\varkappa$ is the unique SR-algebra which is freely SR-generated by $\varkappa$ of its elements. This is proved by the usual category argument. qed.

Our next task is to show that $A_\varkappa$ is isomorphic to an S-field of sets. We shall do even more, we actually give a description of what S-field of sets $A_\varkappa$ is isomorphic to Note that an S-field of sets is automatically an SR-algebra. This was proved by SIERPINSKI (cf. [17]) and will be discussed in detail later in our paper. Consider the complete Boolean algebra $\{0, 1\}$. Since $\{0, 1\}$ is isomorphic to an S-field of sets it is an SR-algebra. Hence by Theorem 1 any function $H : G \to \{0, 1\}$ can be extended to an S-homomorphism mapping $A_\varkappa$ into $\{0, 1\}$. We would like to arrange things so that if $x \in A_\varkappa$ and $x \neq 0$ then there is such an H with $H(x) = 1$. We do this by using a modified notion of consistency property (cf. [10] and [8]). Define

$$\text{(28)} \qquad \mu_\alpha^* = \sim (\sim \mu)^\alpha$$

where $\sim \mu = \lambda u(\sim \mu_u)$. A set $\mathscr{C}$ of countable sets of sentences of L is an *SR-consistency property* (cf. [2]) if for each $c \in \mathscr{C}$,

(29) $\qquad \varphi \notin c$ or $\sim \varphi \notin c$ for any $\varphi \in L$,

(30) $\qquad$ if $\sim \varphi \in c$ then $c \cup \{(\varphi) \sim\} \in \mathscr{C}$,

(31) $\qquad$ if $\bigwedge\sigma \in c$ then $c \cup \{\varphi\} \in \mathscr{C}$ for all $\varphi \in \sigma$,

(32) $\qquad$ if $\bigvee\sigma \in c$ then $c \cup \{\varphi\} \in \mathscr{C}$ for some $\varphi \in \sigma$,

(33) $\qquad$ if $\mathscr{A}^*\mu \in c$ then $c \cup \{\mu_\alpha^*\} \in \mathscr{C}$ for some α,

(34) $\qquad$ if $\mathscr{A}\mu \in c$ then $c \cup \{\bigwedge_n \mu_{f|n}\} \in \mathscr{C}$ for some f.

By an *assignment* we mean any function mapping the variables in P into $\{0, 1\}$. Let h range over assignments. By (23)–(25) we extend h to all of P. h *satisfies* φ if $h(\varphi) = 1$ (we use the same symbol for the assignement and its extension).

Lemma 1. *If $\mathscr{C}$ is an SR-consistency property and $c_0 \in \mathscr{C}$ then there exists an assignment satisfying every $\varphi \in c_0$.*

Proof. W.l.g. we may assume that every subset of an element in $\mathscr{C}$ is itself in $\mathscr{C}$. For each $c \in \mathscr{C}$ let $\gamma(c)$ be an enumeration of the least set S such that $c \subseteq S$, if $\varphi \in S$ then sub $\varphi \subseteq S$, and if $\sim \varphi \in S$ then $(\varphi) \sim \in S$. We then construct an increasing sequence $\{c_n\}_n \subseteq \mathscr{C}$ and a sequence of sequences of sentences $\{D_n\}_n$ by the following induction c_0 is the initially given element of $\mathscr{C}$ and $D_0 = \gamma(c_0)$. Assume that at stage n of our construction we have $c_m \in \mathscr{C}$ and D_m for $m \leqq n$. Finitely many sentences in $\bigcup_{m \leqq n}$ range (D_m) will be known as *treated*, the rest as *untreated* (initially no sentence of D_0 was treated). c_{n+1} will be defined as c_{nn} where c_{nm} is defined by the following subinduction. Assume that for some $m < n$ we have c_{nm}. Let φ be the first untreated sentence in D_{m+1}. If $c_{nm} \cup \{\varphi\} \notin \mathscr{C}$ let $c_{n(m+1)} = c_{nm}$. Otherwise let $c_{n(m+1)}$ be the least element in $\mathscr{C}$ such that

$$c_{nm} \cup \{\varphi\} \subseteq c_{n(m+1)} \tag{35}$$

$$\text{if } \varphi \text{ is of the form } \bigvee\sigma \text{ then } \psi \in c_{n(m+1)} \text{ for some } \psi \in \sigma\,, \tag{36}$$

$$\text{if } \varphi \text{ is of the form } \mathscr{A}\mu \text{ then } \bigwedge_n \mu_{f|n} \in c_{n(m+1)} \text{ for some } f\,, \tag{37}$$

$$\text{if } \varphi \text{ is of the form } \mathscr{A}^*\mu \text{ then } \mu^*_\alpha \in c_{n(m+1)} \text{ for some } \alpha\,. \tag{38}$$

Let $c_{n+1} = c_{nn}$, $D_{n+1} = \gamma(c_{n+1})$ and declare φ as treated. Set $c_\omega = \bigcup_n c_n$. Our assignment h is defined by $h(p_\xi) = 1$ if and only if $p_\xi \in c_\omega$.

We claim that $h(\varphi) = 1$ for each $\varphi \in c_\omega$. This is proved by induction. However we must do this induction using a notion of rank which assigns a higher rank to $\mathscr{A}^*\mu$ than it does to any μ^*_α. Such functions are not difficult to devise and we omit the details. By definition $h(p_\xi) = 1$ for each $p_\xi \in c_\omega$. Suppose that $\varphi \in c_\omega$. Then there is an n such that φ was treated at stage n and $\varphi \in c_{n+1}$. If $\varphi = \bigvee\sigma$ then $\psi \in c_{n+1}$ for some $\psi \in \sigma$ by (36), $h(\psi) = 1$ by induction hypothesis, and $h(\varphi) = 1$ by (24). If $\varphi = \bigwedge\sigma$ and $\psi \in \sigma$ then $\psi \in \text{range}(D_{n+1})$. If $\psi \notin c_{n+1}$ there will be a stage k at which ψ will be treated. But $c_k \cup \{\psi\} \in \mathscr{C}$ because subsets of $c \in \mathscr{C}$ are in $\mathscr{C}$. Thus $\psi \in c_{k+1}$ by (35) and we conclude that $\sigma \subseteq c_\omega$. We have shown that $h(\psi) = 1$ for all $\psi \in \sigma$ and hence h$(\varphi) = 1$ by (24). If $\varphi = \mathscr{A}\mu$ then $\bigwedge_n \mu_{f|n} \in c_{n+1}$ for some f by (37). Then $\mu_{f|n} \in c_\omega$ for each n by the previous case. $h(\mu_{f|n}) = 1$ by induction hypothesis and $h(\varphi) = 1$ by (25). If $\varphi = \mathscr{A}^*\mu$ then $\mu^*_\alpha \in c_{n+1}$ for some α. Then $h(\mu^*_\alpha) = 1$ by induction hypothesis. Let v be given as in (25). Then just as in the proof of Theorem 1 we show that $h(\mu^*_\alpha) = v^*_\alpha$. To complete this case we must show that $v^*_\alpha \leqq \mathscr{A}^*v$. But $\{0, 1\}$ is an SR-algebra and hence $\mathscr{A}(-v) \leqq (-v)^\alpha$ and $v^*_\alpha = -(-v)^\alpha \leqq$ $\leqq -\mathscr{A}(-v) = \mathscr{A}^*v$ by De Morgan's laws. Thus $1 = h(\mu^*_\alpha) = v^*_\alpha \leqq \mathscr{A}^*v = h(\varphi)$. If $\varphi =$

$= \sim \psi$ then $(\psi) \sim \in \text{range}(D_{n+1})$. If $(\psi) \sim \notin c_{n+1}$ there will be a stage k at which $(\psi) \sim$ will be treated. But $c_k \cup \{(\psi) \sim\} \in \mathscr{C}$ because subsets of $c \in \mathscr{C}$ are in $\mathscr{C}$. Thus $(\psi) \sim \in c_{k+1}$. If $\psi = p_\xi$ then $h((\psi) \sim) = 1$ by (29) and if $\psi = \sim \theta$ then $h((\psi) \sim) = h(\theta) = 1$ by induction. Otherwise $h((\psi) \sim) = 1$ by one of the previous cases. We know that $\sim \psi \leftrightarrow (\psi) \sim$ is an SR-axiom and just as in the proof of Theorem 1, using (26), we get $h(\sim \psi) = h((\psi) \sim)$. Hence $h(\varphi) = 1$. qed.

Lemma 2. *If $[\varphi] \in A_\varkappa$ and $[\varphi] \neq 0$ then φ is satisfied by some assignment h.*

Proof. Let $\mathscr{C}$ be the set of all finite subsets $c \subseteq P$ such that $\sim \bigwedge c$ is not an SR-theorem. We claim that $\mathscr{C}$ is an SR-consistency property. For (29), if $\varphi, \sim \varphi \in c \in \mathscr{C}$ then by (8) we have $\vdash \varphi \vee \sim \varphi$ and hence $\vdash \sim \bigwedge c$. For (30), if $\sim \varphi \in c$ and $c \cup \{(\varphi) \sim\} \notin \mathscr{C}$ then $\vdash \bigwedge c \to \sim (\varphi) \sim$, but $\vdash \varphi \leftrightarrow \sim (\varphi) \sim$ by (9) and hence $\vdash \sim \bigwedge c$ by (8). For (31), if $\bigwedge \sigma \in c$ but $c \cup \{\varphi\} \notin \mathscr{C}$ for some $\varphi \in \sigma$ then $\vdash \bigwedge c \to \sim \varphi$, but $\vdash \bigwedge c \to \varphi$ so $\vdash \sim \bigwedge c$ by (8). For (32), if $\bigvee \sigma \in c$ but $c \cup \{\varphi\} \notin \mathscr{C}$ for every $\varphi \in \sigma$ then $\vdash \bigwedge c \to \sim \varphi$ for every $\varphi \in \sigma$ so $\vdash \bigwedge c \to \bigwedge \sim \sigma$ by (14) where $\sim \sigma = \{\sim \varphi \mid \varphi \in \sigma\}$. But $\vdash \bigwedge \sim \sigma \to \sim \bigvee \sigma$ by (9) and hence $\vdash \sim \bigwedge c$. For (33), if $\mathscr{A}^* \mu \in c$ but $c \cup \{\mu_\alpha^*\} \notin \mathscr{C}$ for every α then $\vdash \bigwedge c \to \sim \mu_\alpha^*$ for every α. But $\sim \mu_\alpha^* = \sim \sim (\sim \mu)^\alpha$ by (28) and hence $\vdash \bigwedge c \to \mathscr{A} \sim \mu$ by (16). Thus $\vdash \bigwedge c \to \sim \mathscr{A}^* \mu$ by (9) giving $\vdash \sim \bigwedge c$. For (34), if $\mathscr{A} \mu \in c$ but $c \cup \{\bigwedge_n \mu_{f|n}\} \notin \mathscr{C}$ for every f then $\vdash \bigwedge_n \mu_{f|n} \to \sim \bigwedge c$ for every f. We get $\vdash \mathscr{A} \mu \to \sim \bigwedge c$ by (15) and hence $\vdash \sim \bigwedge c$. By contradiction then, $\mathscr{C}$ is an SR-consistency property.

Now to prove the lemma itself. Since $[\varphi] \neq 0 = [p_0 \wedge \sim p_0]$ and $\vdash \sim (p_0 \wedge \sim p_0)$ we know by (8) that $\sim \varphi$ is not an SR-theorem. Thus $\{\varphi\} \in \mathscr{C}$ and hence by Lemma 1 there is an assignment h such that $h(\varphi) = 1$. qed.

Give $\{0, 1\}$ the discrete topology and $X = \{0, 1\}^\varkappa$ the product topology. For $\xi < \varkappa$ let $q_\xi = \{g \in X \mid g(\xi) = 1\}$ and $Q = \{q_\xi \mid \xi < \varkappa\}$. Each q_ξ is a clopen (closed and open) subset of X. $B_\varkappa$ will be the S-field of subsets of X S-generated by Q.

Theorem 2. *$A_\varkappa$ is isomorphic to $B_\varkappa$.*

Proof. Recall that $G = \{[p_\xi] \mid \xi < \varkappa\}$ is a set of free SR-generators of $A_\varkappa$. Define H on G by $H([p_\xi]) = q_\xi$. Since $B_\varkappa$ is an S-field of sets it is an SR-algebra. Freeness allows us to extend H to an S-homomorphism mapping $A_\varkappa$ into $B_\varkappa$. The extension will also be called H. It maps onto $B_\varkappa$ because Q S-generates $B_\varkappa$.It only remains to show that H is one-one.

Every assignement h can be identified with exactly one $g \in X$ by setting $g(\xi) = h(p_\xi)$ for $\xi < \varkappa$. We use the symbol h to denote this function g. We claim that $H([\varphi]) = \{h \in X \mid h(\varphi) = 1\}$. This is proved by induction. That it is true for p_ξ follows immediately from $H([p_\xi]) = q_\xi$. If $\varphi = \mathscr{A} \mu$ then $H([\mathscr{A} \mu]) = H(\mathscr{A} \lambda u [\mu_u]) = \mathscr{A} \lambda u H([\mu_u]) = \bigcap_f \bigcup_n H([\mu_{f|n}])$ by (22) and the fact that H is an S-homomorphism. Also $h(\mathscr{A} \mu) = 1$ iff $(\exists f)(\forall n)\, h(\mu_{f|n}) = 1$ by (25). Thus $h \in H([\mathscr{A} u])$ iff $(\exists f)(\forall n)\, h \in H([\mu_{f|n}])$ iff $(\exists f)(\forall n)\, h(\mu_{f|n}) = 1$ iff $h(\mathscr{A} \mu) = 1$. We omit the remaining cases which are treated in exactly the same way. This proves our claim. Now assume that $[\varphi] \in A_\varkappa$ and $[\varphi] \neq 0$. Then $h(\varphi) = 1$ for some h and $h \in H([\varphi]) \neq \emptyset$. qed.

Corollary 1. *If B is an SR-algebra which is S-generated by $\varkappa$ of its elements then B is an S-homomorphic image of $B_\varkappa$.*

3. SS AND SE-ALGEBRAS

Before discussing these varieties of algebras we take an excursion thru the modern theory of complete Boolean algebras. Why we do so will soon become apparent. Until we say otherwise we suspend the notational conventions of the last two sections. Let M be the universe of sets and let $B \in M$ be a complete Boolean algebra. We are going to construct a B-valued model M^B of sets in the style of SCOTT (cf. [16]). Since [16] is not published, and probably never will be, we will outline the material we need in some detail. For information about Boolean valued models see [15] or [6]. Define

$$(39) \qquad M_0 = \emptyset\,, \quad M_{\alpha+1} = M_\alpha \cup \{u \mid u : M_\alpha \to B\}\,,$$
$$M_\lambda = \bigcup\nolimits_{\xi<\lambda} M_\xi\,, \quad \text{for limit } \lambda\,,$$

and let $M^B = \bigcup_\xi M_\xi$ where the union here is over all ordinals. Our object language will be that of ZFC (= Zermelo-Fraenkel + choice). It will contain variables $x, y, z, \ldots$ ranging over M^B, and constants $u, v, w, \ldots$ naming elements of M^B. We may as well let elements of M^B be their own names. Let $\varphi, \xi, \ldots$ (with constants) range over this language, and let δu denote the domain of the function u. We seek to assign a Boolean value $[\![\varphi]\!] \in B$ to each φ. By recursion on the least α such that $u, v \in M_\alpha$ define

$$(40) \qquad [\![u \in v]\!] = \sum\nolimits_{x\in\delta v} [\![u = x]\!] \cdot v(x)\,,$$
$$[\![u = v]\!] = \left(\prod\nolimits_{x\in\delta u} u(x) \to [\![x \in v]\!]\right) \cdot \left(\prod\nolimits_{x\in\delta v} v(x) \to [\![x \in u]\!]\right),$$

where $b_0 \to b_1 = (-b_0) + b_1$ for $b_0, b_1 \in B$. Then by recursion on sentence length define

$$(41) \qquad [\![\sim \varphi]\!] = -[\![\varphi]\!]\,,$$
$$[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cdot [\![\psi]\!]\,,$$
$$[\![\varphi \vee \psi]\!] = [\![\varphi]\!] + [\![\psi]\!]\,,$$
$$[\![\forall x\, \varphi(x)]\!] = \prod\nolimits_u [\![\varphi(u)]\!]\,,$$
$$[\![\exists x\, \varphi(x)]\!] = \sum\nolimits_u [\![\varphi(u)]\!]\,,$$

where $\prod$ and $\sum$ are taken over all of M^B. As usual $[\![\varphi \to \psi]\!] = 1$ if and only if $[\![\varphi]\!] \leq [\![\psi]\!]$.

One of the first things proved in [16] is that equality behaves as expected. Namely,

$$
\begin{aligned}
(42) \qquad & [\![u = u]\!] = 1\,, \\
& [\![u = v]\!] = [\![v = u]\!]\,, \\
& [\![u = v]\!] \cdot [\![v = w]\!] \leqq [\![u = w]\!]\,, \\
& [\![u = v]\!] \cdot [\![v \in w]\!] \leqq [\![u \in w]\!]\,, \\
& [\![u \in v]\!] \cdot [\![v = w]\!] \leqq [\![u \in w]\!]\,.
\end{aligned}
$$

Say that φ is *valid* if $[\![\varphi]\!] = 1$, and that M^B is a *model* of a set of sentences Γ if each $\varphi \in \Gamma$ is valid. If φ is valied write $\models \varphi$. The second important thing proved in [16] is that

$$
(43) \qquad M^B \text{ is a } B\text{-valued model of ZFC}\,.
$$

The rest of [16] is concerned with showing that certain other φ are not valid, and thus providing independence proofs for those φ from ZFC. What we are interested in, however, is not independence proofs, but distributive laws about complete Boolean algebras. Thus we look at technical properties of the first order theory of M^B.

If $u \in M$ define $\tilde{u} \in M^B$ by the recursion $\delta\tilde{u} = \{\tilde{x} \mid x \in u\}$ and $\tilde{u}(x) = 1$ for $x \in \delta\tilde{u}$. From (40) we easily see that

$$
\begin{aligned}
(44) \qquad & [\![\tilde{u} \in \tilde{v}]\!] = 1 \quad \text{iff} \quad u \in v\,, \\
& [\![\tilde{u} = \tilde{v}]\!] = 1 \quad \text{iff} \quad u = v\,.
\end{aligned}
$$

Thus we can regard $\tilde{M} = \{\tilde{u} \mid u \in M\}$ as a substrncture of M^B which is isomorphic to M. From (40) and (42) we have $u(x) \leqq [\![x \in u]\!]$ and from (42) and an easy induction on sentence length we have

$$
(45) \qquad [\![\varphi(u)]\!] \cdot [\![u = v]\!] \leqq [\![\varphi(v)]\!]\,.
$$

We claim

$$
\begin{aligned}
(46) \qquad & [\![(\exists x \in u)\, \varphi(x)]\!] = \textstyle\sum_{x \in \delta u} u(x) \cdot [\![\varphi(x)]\!]\,, \\
& [\![(\forall x \in u)\, \varphi(x)]\!] = \textstyle\prod_{x \in \delta u} u(x) \to [\![\varphi(x)]\!]\,.
\end{aligned}
$$

By De Morgan's laws it suffices to prove the first of these equalities. $\sum_{x \in \delta u} u(x) \cdot \cdot [\![\varphi(x)]\!] \leqq \sum_{x \in \delta u} [\![x \in u]\!] \cdot [\![\varphi(x)]\!] \leqq \sum_x [\![x \in u]\!] \cdot [\![\varphi(x)]\!] = \sum_x \sum_{y \in \delta u} [\![x = y]\!] \cdot u(y) \cdot \cdot [\![\varphi(x)]\!] \leqq \sum_{y \in \delta u} u(y).\ [\![\varphi(y)]\!]$ by (45). Since the third term in this chain is $[\![(\exists x \in u) \cdot \cdot \varphi(x)]\!]$, we are done. A Δ_0 *sentence* of our object language is one built up from the (finitary) propositional connectives and the *restricted quantifiers* $(\exists x \in y)$, $(\exists x \in u)$, $(\forall x \in y)$, $(\forall x \in u)$. It follows from (44) and (46) that if φ is a Δ_0-sentence all of whose constants belong to $\tilde{M}$ (which we identify with M) then

$$
(47) \qquad [\![\varphi]\!] \doteq 1 \quad \text{iff } \varphi \text{ is true in } M\,.
$$

In classical set theory an *ordinal* is defined as a transitive connected set (cf. [6]). This definition can be cast as a Δ_0-sentence ord (u). Let ξ range over ordinals (in M). Then (46) implies

$$(48) \qquad [\![\mathrm{ord}\,(\tilde{\xi})]\!] = 1\,.$$

We prove by induction that

$$(49) \qquad u \in M_\xi \quad \text{implies that} \quad [\![\tilde{\xi} \in u]\!] = 0\,.$$

For otherwise $[\![\tilde{\xi} \in u]\!] = \sum_{x \in \delta u}[\![\tilde{\xi} = x]\!] \cdot u(x) > 0$ implies $[\![\tilde{\xi} = x]\!] > 0$ for some $x \in \delta u$. But $x \in M_\eta$ for some $\eta < \xi$ and $[\![\tilde{\eta} \in \tilde{\xi}]\!] = 1$ by (44). Then $[\![\tilde{\xi} = x]\!] = [\![\tilde{\eta} \in \tilde{\xi}]\!] \cdot [\![\tilde{\xi} = x]\!] \leqq [\![\tilde{\eta} \in x]\!]$ by (42) and the last term of this chain is 0 by induction hypothesis. Using (49) we prove

$$(50) \qquad [\![\mathrm{ord}\,(u)]\!] = \textstyle\sum_\xi [\![u = \tilde{\xi}]\!]\,.$$

From (44) and (48) we have $[\![u = \tilde{\xi}]\!] \leqq [\![u = \tilde{\xi}]\!] \cdot [\![\mathrm{ord}\,(\tilde{\xi})]\!] \leqq [\![\mathrm{ord}\,(u)]\!]$ hence $\sum_\xi [\![u = \tilde{\xi}]\!] \leqq [\![\mathrm{ord}\,(u)]\!]$. Now $[\![\mathrm{ord}\,(u)]\!] \leqq [\![u \in \tilde{\xi}]\!] + [\![u = \tilde{\xi}]\!] + [\![\tilde{\xi} \in u]\!]$ and use (49) to choose ξ so large that $[\![\tilde{\xi} \in u]\!] = 0$. Then $[\![\mathrm{ord}\,(u)]\!] \leqq \sum_{\eta \leqq \xi}[\![u = \tilde{\eta}]\!] \leqq \sum [\![u = \tilde{\xi}]\!]$ by (40) and (44). (50) easily gives $[\![(\exists x)\ \mathrm{ord}\,(x) \wedge \varphi(x)]\!] = \sum_\xi [\![\varphi(\tilde{\xi})]\!]$. We can express the fact an ordinal is finite by a Δ_0-sentence fin ord (u). Thus $[\![\mathrm{fin\ ord}\,(\tilde{n})]\!] = 1$ for each $n \in \omega$ and $[\![\mathrm{fin\ ord}\,(\tilde{\omega})]\!] = 0$. Thus we have

$$(51) \qquad [\![\omega = \tilde{\omega}]\!] = 1\,.$$

It is to be understood that the first ω in (51) is notational, and the whole expression $\omega = \tilde{\omega}$ merely says that $\tilde{\omega}$ is the first infinite ordinal. Then (46) gives $[\![(\exists x \in \omega)\ \varphi(x)]\!] = \sum_{n \in \omega}[\![\varphi(\tilde{n})]\!]$.

B satisfies the (ω, λ)-*DL* (*distributive law*) if for every function $b : \omega \times \lambda \to B$ we have

$$(52) \qquad \textstyle\prod_{n<\omega}\sum_{\alpha<\lambda} b_{n\alpha} = \sum_{g \in \lambda^\omega}\prod_{n<\omega} b_{ng(n)}\,.$$

Now assume that B is an (ω, ω)-DL complete Boolean algebra. We claim

$$(53) \qquad [\![\tilde{\omega}^{\tilde{\omega}} = (\omega^\omega)^\sim]\!] = 1\,.$$

If $f \in \omega^\omega$ then $[\![\tilde{f} : \tilde{\omega} \to \tilde{\omega}]\!] = 1$ by (47). Hence $[\![(\forall f \in (\omega^\omega)^\sim)\, f \in \tilde{\omega}^{\tilde{\omega}}]\!] = \prod_{f \in \omega^\omega}[\![\tilde{f} \in \tilde{\omega}^{\tilde{\omega}}]\!] = 1$ by (46). Conversely, if $f \in M^B$ then $[\![f : \tilde{\omega} \to \tilde{\omega}]\!] \leqq [\![(\forall x \in \tilde{\omega})\,(\exists y \in \tilde{\omega})\, f(x) = y]\!] = \prod_{x \in \omega}\sum_{y \in \omega}[\![f(\tilde{x}) = \tilde{y}]\!] = \sum_{g \in \omega^\omega}\prod_{x \in \omega}[\![f(\tilde{x}) = (g(x))^\sim]\!]$. Without difficulty $[\![(g(x))^\sim = \tilde{g}(\tilde{x})]\!] = 1$, so $[\![f : \tilde{\omega} \to \tilde{\omega}]\!] \leqq \sum_{g \in \omega^\omega}\prod_{x \in \omega}[\![f(\tilde{x}) = \tilde{g}(\tilde{x})]\!] \leqq \sum_{g \in \omega^\omega}[\![f = \tilde{g}]\!] \leqq [\![f \in (\omega^\omega)^\sim]\!]$. This completes our proof of (53). It is well known (cf. [19]) that an (ω, ω)-DL Boolean algebra is $(\omega, 2)$-DL, and if complete, is (ω, Ω)-DL. From these facts we obtain

$$(54) \qquad [\![\tilde{2}^{\tilde{\omega}} = (2^\omega)^\sim]\!] = 1\,,$$

$$(55) \qquad [\![\tilde{\Omega}^{\tilde{\omega}} = (\Omega^\omega)^\sim]\!] = 1\,,$$

in exactly the same way as we obtained (53). An ordinal ξ is *countable* if there is a function f mapping ω *onto* ξ (in symbols $f : \omega \twoheadrightarrow \xi$). When this happens, $[\![\tilde{f} : \tilde{\omega} \twoheadrightarrow \tilde{\xi}]\!] = 1$ by (47). Thus if ξ is countable then $[\![\tilde{\xi}$ is countable$]\!] = 1$. Since Ω is uncountable in M, range $(f) \neq \Omega$ for any $f : \omega \to \Omega$, and hence $[\![$range $(\tilde{f}) = \tilde{\Omega}]\!] = 0$. Thus $[\![(\forall f \in \tilde{\Omega}^{\tilde{\omega}})$ range $(f) = \tilde{\Omega}]\!] = [\![(\forall f \in (\Omega^{\omega})^{\sim})$ range $(f) = \tilde{\Omega}]\!] = \prod_{f \in \Omega^{\omega}} [\![$range $(\tilde{f}) = \tilde{\Omega}]\!] = 0$. We have shown that $[\![\tilde{\Omega}$ is countable$]\!] = 0$ and thus

$$[\![\Omega = \tilde{\Omega}]\!] = 1 \tag{56}$$

where as in (51) $\Omega = \tilde{\Omega}$ asserts that $\tilde{\Omega}$ is the first uncountable ordinal. This concludes our review of Boolean valued model theory. We return to the notational conventions that were introduced in the last section.

We shall examine a construction due to Shoenfield (cf. [18]) which is used for showing that Σ^1_2 predicates are absolute with respect to constructible sets. Let R be the real numbers and let $D : \omega^{<\omega} \to$ the power set of R. Consider the coanalytic set $S = \bigcap_f \bigcup_n D_{f|n}$. x denotes real numbers. We partially order $\omega^{<\omega}$ by $u \prec v$ if v is a proper initial segment of u and then define a set $E^x \subseteq \omega^{<\omega}$ by

$$u \in E^x \quad \text{iff} \quad (\forall v)(u \prec v \to x \notin D_v)\,. \tag{57}$$

It easily follows that

$$x \in S \ \text{iff} \ E^x \text{ is well founded under } \prec\,. \tag{58}$$

Now E^x is well founded if and only if there is an order preserving map taking E^x into Ω. We shall try to build such a map by finite approximations. Each $u \in \omega^{<\omega}$ can be coded by an integer $\#(u) \in \omega$. We do this in such a way that u $\prec v$ implies $\#(v) < \#(u)$. Moreover, we identify u with $\#(u)$. Let s range over $\Omega^{<\omega}$, g range over Ω^{ω}, and define a set $D^*_s \subseteq R$ by

$$x \in D^*_s \quad \text{iff} \quad (\forall u.\, v)(u \prec v \wedge s(u) \geqq s(v) \to u \notin E^x)\,. \tag{59}$$

It follows from (58) that $x \in S$ iff $(\exists g)(\forall n)\, x \in D^*_{g|n}$. Thus we have shown that

$$\textstyle\bigcap_f \bigcup_n D_{f|n} = \bigcup_g \bigcap_n D^*_{g|n}\,. \tag{60}$$

Now we Booleanize the whole argument. Let B be an (ω, ω)-DL complete Boolean algebra and let $d : \omega^{<\omega} \to B$. Choose, $x, D \in M^B$ such that $[\![x$ is a real$]\!] = 1$, $[\![(\forall u \in \tilde{\omega}^{<\tilde{\omega}})\, D_u$ is a set of reals$]\!] = 1$, and $[\![x \in D_{\tilde{u}}]\!] = d_u$ for all $u \in \omega^{<\omega}$. Now $[\![x \in \bigcap_f \bigcup_n D_{f|n}]\!] = \prod_f \sum_n d_{f|n}$ by (51) and (52), and $[\![x \in \bigcup_g \bigcap_n D^*_{g|n}]\!] = \sum_g \prod_n d^*_{g|n}$ by (51) and (55) where we have set $d^*_s = [\![x \in D^*_{\tilde{s}}]\!]$. Now (60) is a theorem of set theory in which we have used the fact that every countable well founded relation can be mapped into Ω in an order preserving way. By (56) we give a Boolean value of 1 to (60). Thus

$$\textstyle\prod_f \sum_n d_{f|n} = \sum_g \prod_n d^*_{g|n}\,. \tag{61}$$

It only remains to evaluate the function d^*. Let $e_u = [\![\tilde{u} \in E^x]\!]$. It follows from (57), (59) that

$$e_u = \prod\{-d_v \mid u \prec v\}\,, \tag{62}$$
$$d_s^* = \prod\{-e_u \mid (\exists v)\, u \prec v \wedge s(u) \geqq s(v)\}\,.$$

We say that an S-algebra is an *SS-algebra* if it satisfies (61) with d^* defined as in (62). In our previous notation we would write (61) in the form $\mathscr{A}^* d = \sum_g \prod_n d^*_{g|n}$. Thus we have shown

Lemma 3. *An* (ω, ω)*-DL complete Boolean algebra is an SS-algebra.*

There are probably more direct ways to prove this lemma. We have chosen the Boolean valued approach because it gives much insight into what is going on and provides an intuitive way to study other distributive laws. Let B be an S-algebra. $G \subseteq B$ freely *SS-generates* B if any map from G into any SS-algebra B' can be extended to a unique S-homomorphism from B into B'. We prove, just as in Theorem 1, that there exist free SS-algebras. P, the set of sentences of our formal language is the same as in Section 2. Our SS-axioms and SS-rules of inference are the same as those of SR-logic except (12) and (16) are replaced by (63) and (64) below. Let $\mu : \omega^{<\omega} \to P$. Define $v_u = \bigwedge\{\sim\mu_v \mid u \prec v\}$ and $\mu_s^* = \bigwedge\{\sim v_u \mid (\exists v)\, u \prec v \wedge s(u) \geqq$ $\geqq s(v)\}$.

$$\bigwedge\nolimits_n \mu^*_{g|n} \to \mathscr{A}^*\mu\,, \tag{63}$$

$$\frac{\{\bigwedge_n \mu^*_{g|n} \to \varphi \mid g \in \Omega^\omega\}}{\mathscr{A}^*\mu \to \varphi}\,. \tag{64}$$

The Lindenbaum algebra of SS-logic will be called $A'_\varkappa$. Then we claim

Theorem 3. *$A'_\varkappa$ is an SS-algebra which is freely SS-generated by $\varkappa$ of its elments.*

We do not prove this result (which is done in exactly the same way as Theorem 1). The method is quite general and will work for any reasonable kind of algebra. A little more care is necessary for completeness. *SS-consistency* property is defined in the same way as consistency property except that (33) is replaced by (65)

$$\text{if} \quad \mathscr{A}^*\mu \in c \quad \text{then} \quad c \cup \{\bigwedge\nolimits_n \mu^*_{g|n}\} \in \mathscr{C} \quad \text{for some } g\,. \tag{65}$$

We then prove a lemma for SS-consistency properties which is exactly like Lemma 1. We must note that in proving Lemma 1 we used the fact that the algebra $\{0, 1\}$ is an SR-algebra. In the present context we must know that $\{0, 1\}$ is an SS-algebra. But this is clear because $\{0, 1\}$ is a complete field of sets and hence is (ω, ω)-DL. Lemma 3 finishes the job. We end with

Theorem 4. *$A'_\varkappa$ is isomorphic to $B_\varkappa$.*

A set $F \subseteq \omega^{<\omega}$ is called *full* if $(\forall f)(\exists n) f \mid n \in F$. Let F range over full sets and let $D : \omega^{<\omega} \to$ the power set of R. We claim that

$$\bigcup_f \bigcap_n D_{f|n} = \bigcap_F \bigcup_{u \in F} D_u \,. \tag{66}$$

Assume that x belongs to the left hand side of (66). Then there is an f such that $x \in D_{f|n}$ for all n. Let F be full. Then $f \mid n \in F$ for some n, i.e., $x \in \bigcup_{u \in F} D_u$. Conversely, assume that x does not belong to the left hand side of (66). Then there is no f such that $x \in D_{f|n}$ for all n. Let $F_0 = \{u \mid x \notin D_u\}$. It follows that F_0 is full and $x \notin \bigcup_{u \in F_0} D_u$, so we are done. For the moment let T be the power set of $\omega^{<\omega}$. Since $\omega^{<\omega}$ is essentially ω, via our coding $\#$, (51), and (54) give

$$[\![T = \tilde{T}]\!] = 1\,. \tag{67}$$

It then follows from (51) and (53) that

$$[\![\tilde{F} \text{ is full}]\!] = 1 \quad \text{iff } F \text{ is full}\,, \tag{68}$$

where for (67) and (68) we assume that B is an (ω, ω)-DL complete Boolean algebra. Let $d \in \omega^{<\omega} \to B$ and let x, D be defined as in the proof of Lemma 3 (just after (60)). We have already shown that $[\![x \in \bigcup_f \bigcap_n D_{f|n}]\!] = \sum_f \prod_n d_{f|n}$. From (67) and (68) we get $[\![x \in \bigcap_F \bigcup_{u\in F} D_u]\!] = \prod_F \sum_{u \in F} d_u$. Finally, (66) gets Boolean value 1 because it is a theorem of set theory. Thus

$$\sum_f \prod_n d_{f|n} = \prod_F \sum_{u \in F} d_u\,. \tag{69}$$

We say that an S-algebra is an *SE-algebra* if it satisfies (69). From the previous argument

Lemma 4. *An (ω, ω)-DL complete Boolean algebra is an SE-algebra.*

Our motivation for SE-algebras is from is from [2] where we obtained a completeness theorem for SE-logic. Define free SE-algebra canonically and prove one exists via SE-logic. This logic is the same as SR-logic except that we replace (12) and (16) by

$$\mathscr{A}\mu \to \bigvee_{u \in F} \mu_u\,, \tag{70}$$

$$\frac{\{\varphi \to \bigvee_{u\in F} \mu_u \mid F \text{ is full}\}}{\varphi \to \mathscr{A}\mu}\,. \tag{71}$$

Let $A''_\varkappa$ be the Lindenbaum algebra of SE-logic. We then prove exactly as before

Theorem 5. *$A''_\varkappa$ is an SE-algebra which is freely SE-generated by $\varkappa$ of its elements.*

Theorem 6. *$A''_\varkappa$ is isomorphic to $B_\varkappa$.*

By now it is clear that the Boolean methods of the last section give a heuristic way to obtain S-representable S-algebras. We invite the reader to try his own algebra using his own favorite distributive axiomatization. Up to now our distributive laws have been very powerful. We now turn to a much less direct situation. Let $j : \omega \times \times \omega \to \omega$ be a one-one onto mapping given by $j(x, y) = \frac{1}{2}(x + y)(x + y + 1) + x$. Its first, second inverse k, l respectively are defined by $j(k(x), l(x)) = x$. If $u, v \in \omega^{<\omega}$ are of the same length n define $j(u, v) = w$ to be that sequence of length n such that $j(u_i, v_i) = w_i$ for $i < n$. k and l are extended to $\omega^{<\omega}$ in a natural way. If $f, g \in \omega^\omega$ define $j(f, g) = h$ to be that function h such that $j(f(x), g(x)) = h(x)$ for $x < \omega$. k and l are then extended to ω^ω in a natural way, e.g., $(kf)(x) = k(fx)$. Finally define $u^{(x)} = \lambda y u_{j(x,y)}$ for $u \in \omega^{<\omega}$ and $f^{(x)} = \lambda y f j(x, y)$ for $f \in \omega^\omega$.

Now let B be an S-algebra and let $b : (\omega^{<\omega}) \times (\omega^{<\omega}) \to B$. B is closed under the $\mathscr{A}^2$ operator which is defined by $\mathscr{A}^2 b = \sum_f \prod_n \sum_g \prod_m b(f \mid n, g \mid m)$. With b we associate $\tilde{b} : \omega^{<\omega} \to B$ as follows. If $u \in \omega^{<\omega}$ has length x,

$$\tilde{b}_u = b(k(u) \mid k(x), l(u)^{(kx)} \mid l(x)) . \tag{72}$$

An S-algebra is an *SC-algebra* if it satisfies $\mathscr{A}^2 b = \mathscr{A}\tilde{b}$ for every $b : (\omega^{<\omega}) \times \times (\omega^{<\omega}) \to B$. These algebra were introduced by Campbell (cf. [1]) who hoped that this condition (which he called the $\mathscr{A}^2 = \mathscr{A}$ law) would be necessary and sufficient for S-representability. An error in Campbell's argument was discovered by Richard Platek (unpublished) which in turn casts doubt on one of the main results of [14]; namely that a weakly distributive free S-algebra satisfies the weak zero condition. Our result is more modest. We will prove that a complete SC-algebra is S-representable. We start with the following lemma from [1] (whose proof was not quite correct, which we hope is correct here).

Lemma 5. *If B is an SC-algebra then B satisfies the (ω, ω)-DL.*

Proof. Let $a : \omega \times \omega \to B$. if $u \in \omega^{<\omega}$ let $lh(u)$ be the length of u. Now define

$$b_{u,v} = a(lh(u), v(lh(u))) \quad \text{if} \quad lh(u) < l\,h(v),$$
$$b_{u,v} = 1 \quad \text{if} \quad lh(u) \geqq lh(v).$$

Now $\prod_m b_{f|n,g|m} = a_{ng(n)}$ and hence

$$\mathscr{A}^2 b = \sum_f \prod_n \sum_g \prod_m b_{f|n,g|m} = \prod_n \sum_g a_{ng(n)} =$$
$$= \prod_n \sum_m a_{nm}. \; \mathscr{A}\tilde{b} = \sum_f \prod_n \tilde{b}_{f|n} = \sum_f \prod_n b(k(f \mid n) \mid k(n), l(f \mid n)^{(kn)} \mid l(n)) =$$
$$= \sum_f \prod_n a(k(n), l(f \mid n)^{(kn)}(kn)) = \sum_f \prod_n a(n, f^{(n)}(n)) = \sum_f \prod_n a_{nf(n)} .$$

The (ω, ω)-DL follows. qed.

Lemma 6. (cf. [1]). *If B is a complete S-algebra satisfying the (ω, ω)-DL then B is an SC-algebra.*

Proof. Any (ω, ω)-DL complete Boolean algebra satisfies the (ω, ω^ω)-DL (cf. [19]). Let $b : (\omega^{<\omega}) \times (\omega^{<\omega}) \to B$. Then

$$\textstyle\sum_f \prod_n \sum_g \prod_m b(f \mid n,\ g \mid m) =$$
$$= \textstyle\sum_f \sum_g \prod_n \prod_m b(f \mid n,\ g^{(n)} \mid m) = \sum_f \prod_n \prod_m b(k(f) \mid n,\ l(f)^{(n)} \mid m) =$$
$$= \textstyle\sum_f \prod_n b(k(f) \mid k(n),\ l(f)^{(kn)} \mid l(n)) = \sum_f \prod_n b(k(f \mid n) \mid k(n),\ \ l(f \mid n)^{(kn)}\ l(n)) = \mathscr{A}\tilde{b}.$$

qed.

Thus we see that a complete Boolean algebra is an SC-algebra if and only if it satisfies the (ω, ω)-DL. We now continue with some ideas of Rieger (cf. [14]). Let $a : \omega^{<\omega} \to B$. Recall our notions a_u^α and a^α from (1) and (2) and then define

$$a_\alpha = a^\alpha \cdot \textstyle\prod_u (a_u^\alpha \to a_u^{\alpha+1})\,. \tag{73}$$

For the following lemmas we assume B is a complete (ω, ω)-DL Boolean algebra.

Lemma 7. $\prod_n a_{f|n} \leqq a_{f|m}^\alpha$ *for all m, α.*

Proof. By induction on α. Obvious for $\alpha = 0$ and if true for α then $\prod_n a_{f|n} \leqq$ $\leqq a_{f|m}^\alpha \cdot a_{f|(m+1)}^\alpha \leqq a_{f|m}^\alpha \cdot \sum_n a_{(f|m)\wedge n}^\alpha = a_{f|m}^{\alpha+1}$ and if $\prod_n a_{f|n} \leqq a_{f|m}^\alpha$ for all $\alpha <$ limit λ then $\prod_n a_{f|n} \leqq \prod_{\alpha<\lambda} a_{f|m}^\alpha = a_{f|m}^\lambda$. qed.

Lemma 8. $\mathscr{A}a \leqq \prod_\alpha a^\alpha$.

Proof. $\prod_n a_{f|n} \leqq a_{f|0}^\alpha = a^\alpha$. Hence $\mathscr{A}a \leqq a^\alpha$ for all α and $\mathscr{A}a \leqq \prod_\alpha a^\alpha$. qed.

Lemma 9. *If $\alpha \leqq \beta$ then $a_u^\beta \leqq a_u^\alpha$.*

Proof. By induction on β. Obvious for $\beta = \alpha$ and if true for β then $a_u^{\beta+1} =$ $= a_u^\beta \cdot \sum_n a_{u\wedge n}^\beta \leqq a_u^\beta \leqq a_u^\alpha$. If true for all $\beta <$ limit λ then $a_u^\lambda = \prod_{\beta<\lambda} a_u^\beta \leqq a_u^\alpha$. qed.

Lemma 10. $\sum_\alpha a_\alpha \leqq \mathscr{A}a$.

Proof. In Lemmas (7)–(9) we have not used the (ω, ω)-DL. It will be used here. $\alpha_\alpha = a^\alpha \cdot \prod_u (\alpha_u^\alpha \to a_u^{\alpha+1}) = \prod_u a^\alpha \cdot (\alpha_u^\alpha \to a_u^{\alpha+1}) = \prod_u \sum_n a^\alpha \cdot (a_u^\alpha \to a_{u\wedge n}^\alpha)$. Now let g range over functions mapping $\omega^{<\omega}$ into ω and use (ω, ω)-DL to get $a_\alpha = \sum_g \prod_u a^\alpha \cdot$ $\cdot (a_u^\alpha \to a_{u\wedge g(u)}^\alpha)$. Fix g and let $x = \prod_u a^\alpha \cdot (a_u^\alpha \to a_{u\wedge g(u)}^\alpha)$. Then $x \leqq a^\alpha = a_\Lambda^\alpha$. Assume that $x \leqq a_u^\alpha$. Since we also have $x \leqq a_u^\alpha \to a_{u\wedge g(u)}^\alpha$ we get $x \leqq a_{u\wedge g(u)}^\alpha$. Define a function $f : \omega \to \omega$ by the recursion $f(n) = g(f \mid n)$. Then $x \leqq \prod_n a_{f|n}^\alpha$. By Lemma 9 we have $x \leqq \prod_n a_{f|n}$ so $x \leqq \mathscr{A}a$. Now summing on g gives $a_\alpha \leqq \mathscr{A}a$, and hence $\sum_\alpha a_\alpha \leqq \mathscr{A}a$. qed.

We have shown that $\sum_\alpha a_\alpha \leqq \mathscr{A}a \leqq \prod_\alpha a^\alpha$ (remember that α ranges over Ω). Rieger's *weak zero condition* (*WZC*) asserts that if $b : \Omega \times \omega \to B$ such that $b_{\alpha n} \cdot b_{\beta n} = 0$ for $\alpha < \beta$ then $\prod_\alpha \sum_n b_{\alpha n} = 0$.

Lemma 11. *If B satisfies the WZC then* $\prod_\alpha \sum_u (a_u^\alpha - a_u^{\alpha+1}) = 0$.

Proof. If $\alpha < \beta$ then $(a_u^\alpha - a_u^{\alpha+1}) \cdot (a_u^\beta - a_u^{\beta+1}) \leqq a_u^\beta - a_u^{\alpha+1} = 0$ because $a_u^\beta \leqq a_u^{\alpha+1}$ by Lemma 9. Now use the WZC. qed.

Lemma 12. (Rieger [14]). *If B satisfies the WZC then* $\sum_\alpha a_\alpha = \mathscr{A} a = \prod_\alpha a^\alpha$.

Proof. We already know $\sum_\alpha a_\alpha \leqq \mathscr{A} a \leqq \prod_\alpha a^\alpha$.

$$\prod_\beta a^\beta - \sum_\alpha a_\alpha = \prod_\beta \prod_\alpha (a^\beta - a_\alpha) \leqq \prod_\alpha (a^\alpha - a_\alpha) =$$

$$= \prod_\alpha [a^\alpha - a^\alpha \cdot \prod_u (a_u^\alpha \to a_u^{\alpha+1})] = \prod_\alpha \sum_u a^\alpha \cdot (a_u^\alpha - a_u^{\alpha+1}) \leqq \prod_\alpha \sum_u (a_u^\alpha - a_u^{\alpha+1}) = 0$$

by Lemma 11. Hence $\sum_\alpha a_\alpha = \prod_\alpha a^\alpha = \mathscr{A} a$. qed.

Lemma 13. *A complete (ω, ω)-DL Boolean algebra satisfies the WZC.*

Proof. Let $b : \Omega \times \omega \to B$ satisfy $b_{\alpha n} \cdot b_{\beta n} = 0$ for $\alpha < \beta$ (α and β both range over Ω). Define $f \in M^B$ such that $[\![(\tilde{\alpha}, \tilde{n}) \in f]\!] = b_{\alpha n}$. Then $[\![f \text{ is a function}]\!] = \prod_n \prod_\beta \prod_{\alpha < \beta} - [\![(\tilde{\alpha}, \tilde{n}) \in f \wedge (\tilde{\beta}, \tilde{n}) \in f]\!] = \prod_n \prod_\beta \prod_{\alpha < \beta} - b_{\alpha n} \cdot b_{\beta n} = 1$. By (51) and (56) $[\![f \text{ is a function} \to (\exists \alpha < \tilde{\Omega})(\forall n < \tilde{\omega})\, \alpha \neq f(n)]\!] = 1$. Thus $1 = \sum_\alpha \prod_n - b_{\alpha n}$. Then by De Morgan's laws $\prod_\alpha \sum_n b_{\alpha n} = 0$. qed.

From Lemma 5, 12, and 13 we conclude that if B is a complete SC-algebra then B is an SR-algebra and hence by Theorem 2 is an S-homomorphic image of an S-field of sets. In fact if B is S-generated by $\varkappa$ of its elements then B is an S-homomorphic image of $B_\varkappa$ (cf. Theorem 2). We prefer however to give a proof that is independent of Theorem 2. An S-algebra B is an *SSR-algebra* if it satisfies

$$\sum_\alpha a_\alpha = \mathscr{A} a = \prod_\alpha a^\alpha \tag{74}$$

for every $a : \omega^{<\omega} \to B$.

Lemma 14. *If A, B are SSR-algebras and $H : A \to B$ is a σ-homomorphism then H is an S-homomorphism.*

Proof. Let $a : \omega^{<\omega} \to A$ and let $b = \lambda u H(a_u)$. We must show that $H(\mathscr{A} a) = \mathscr{A} b$. By induction on α we prove that $H(a^\alpha) = b^\alpha$ and $H(a_\alpha) = b_\alpha$. This readily follows from the fact that H is a σ-homomorphism. Then $b_\alpha = h(a_\alpha) \leqq h(\mathscr{A} a) \leqq h(a^\alpha) = b^\alpha$ and $\sum_\alpha b_\alpha = \mathscr{A} b = \prod_\alpha b^\alpha$ because A and B are both SSR-algebras. Thus $h(\mathscr{A} a) = \mathscr{A} b$. qed.

Theorem 7. *If A is a complete SC-algebra then A is an S-homomorphic image of an S-field of sets.*

Proof. Let X be the Stone space of A. For any set $a \subseteq X$ let a^- be the closure of a and a^0 the interior of a. By the Stone representation theorem A is isomorphic to the field of clopen subsets of A. We therefore identify A with this field. Since A is a complete Boolean algebra we know from [4] that X is a complete Boolean space (here *complete* means that the closure of an open set is open). If $a, b \subseteq X$ let $a \Delta b$ be the symmetric difference of a and b, and write $a \sim b$ if $a \Delta b$ is meager (of Category I). a has the *Baire property* if $a \sim b$ for some open b. Let B be the S-field of subsets of X S-generated by A. Since having the Baire property is preserved under the $\mathscr{A}$ operation (cf. [9]) it is not hard to show that every element of B has the Baire property. If $a \subseteq X$ is open then $a \sim a^-$ by general considerations and $a^- = a^{-0}$ is clopen by completeness. Moreover we easily show that if a, b are both clopen and $a \sim b$ then $a = b$. Since $\sim$ is an equivalence relation we can combine these facts and prove that for each $b \in B$ there is a unique $a \in A$ such that $a \sim b$. We then set $H(b) = a$. Then H maps B onto A. We claim that H is a σ-homomorphism. Since $a \Delta b = (-a) \Delta (-b)$ we easily show that $H(-b) = -H(b)$. If $\{b_n \mid n < \omega\}$ is a sequence of element of B then $H(\bigcup_n b_n) \Delta \bigcup_n H(b_n) \subseteq \bigcup_n b_n \Delta H(b_n)$. Since the latter set is meager we have $H(\bigcup_n b_n) \sim \bigcup_n H(b_n) \sim (\bigcup_n H(b_n))^- = \sum_n H(b_n)$ where the $\sum$ is performed in A. Since clopen sets in the $\sim$ relation are equal we get $H(\bigcup_n b_n) = \sum_n H(b_n)$. We already know that A is an SSR-algebra, and B is one also because B is an S-field of sets. Then by Lemma 14 H is an S-homomorphism. qed.

Since freeness and complete Boolean algebras are somewhat incompatible we shall not attempt to refine Theorem 7. We have shown in this paper that SR-algebras, SS-algebras, SE-algebras, and complete SC-algebras are all S-representable. Moreover we have given a general method in Section 3 for intuiting sufficient conditions for S-representability. It is still an open question as to whether an arbitrary SC-algebra is S-representable.

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