Czechoslovak Mathematical Journal

Richard C. Brown; Allan M. Krall

On minimizing the sum of squares of $\mathcal{L}^2$ norms of differential operators under constraints

Czechoslovak Mathematical Journal, Vol. 27 (1977), No. 1, 132–143

Persistent URL: http://dml.cz/dmlcz/101451

Terms of use:

© Institute of Mathematics AS CR, 1977

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://dml.cz

Czechoslovak Mathematical Journal, 27 (102) 1977, Praha

ON MINIMIZING THE SUM OF SQUARES OF $\mathscr{L}^2$ NORMS OF DIFFERENTIAL OPERATORS UNDER CONSTRAINTS

RICHARD C. BROWN, University, and ALLAN M. KRALL, University Park

(Received April 8, 1975)

1. Introduction. This paper is concerned with minimizing the functional

$$J(y, \ldots, y^{(n-1)}) = \sum_{i=1}^{m} \int_0^1 |l_i y|^2 \, dt \,, \tag{1.1}$$

where $\{l_i y\}_{i=1}^m$ is a collection of differential expressions (degree $l_i(y) = d_i$, $i = 1, \ldots, m$) defined on $[0, 1]$, subject to the constraints

$$\lambda_j(y) = r_j \,, \tag{1.2}$$

$j = 1, \ldots, l$; $1 \leqq l < \infty$, where the λ_j, $j = 1, \ldots, l$, are arbitrary complex valued continuous functionals on the space $C^{n-1}[0, 1]$ with norm

$$\|x\| = \sum_{i=0}^{n-1} \sup_{[0,1]} |x^{(i)}(t)| \,.$$

Under these assumptions it is well known (see [5; p. 344]) that

$$\lambda_i(y) = \sum_{j=1}^{n} \int_0^1 d\nu_{ij} y^{(n-j)} \,, \tag{1.3}$$

where the ν_{ij} are uniquely defined measures of bounded variation.

Versions of this variational problem have been discussed recently by GOLOMB and JEROME [7] and SCHUMAKER [10]. Related problems have been frequently applied in engineering literature to the theory of electrical filters (e.g., BERKOVITZ and POLLARD [1]).

In [4] BROWN has studied the case where $m = 1$, the coefficients of $l_1(y)$ are suitably differentiable, and the measures ν_{ij} are singular with respect to Lebesgue measure. The solution of the resulting minimization problem is a "spline" which includes polynomial, L and Lg-splines as special cases, and has similar structural properties.

For the moment let us briefly review [4], since the same point of view will be employed in this paper: The single differential expression $l_1(y)$, together with the homogeneous side conditions

$$\lambda_j(y) = 0 ,$$

determine a normally solvable operator L with range in $\mathscr{L}_2[0, 1]$. Since the measures v_{ij} are singular, the domain of L is dense, and there exists a well defined $\mathscr{L}^2$-adjoint operator L^*, whose structure is known. Because of the Fredholm alternative (the mutual orthogonality between the range of L and the null space of L^*), it follows that the minimization problem has at least one solution y determined by the equation

$$(1,4) \qquad L^* L_r y = 0 ,$$

where L_r stands for the "r-translate" of L (i.e, the operator defined on a translate of the domain of L so that the constraints of (1.2) are satisfied). The constructive properties of the minimizing spline can be found by examining the equation (1.4) further.

The same procedure will be pursued here: We associate the homogeneous version of the system (1.1), (1.2) with an operator $\hat{L}$, whose adjoint can be found (Theorem 3.1) using recently developed techniques of KRALL and BROWN [7]. If $\hat{L}$ satisfies certain conditions (Theorems 4.1 – 4.3) which give it closed range, an equation formally similar to (1.4) is found to characterize the solution to the variational problem.

We also point out that the hypotheses of [4] will be considerably weakened. The differential expressions $l_i y$ will be more general than those previously considered, and the measures v_{ij} will not necessarily be singular. Also the domain of $\hat{L}$ may not be dense.

2. Notation and Preliminaries. Let L be a linear operator or linear relation on a Hilbert space. Then $D(L)$, $R(L)$, $N(L)$ and L^* denote its domain, range, null space and adjoint, respectively and $\mathbb{C}^m$ is the m-dimensional Euclidean space over the complex field.

$\mathscr{L}^2_m[0, 1]$ denotes the Hilbert space of m-dimensional vector valued functions y over $\mathbb{C}$, defined on $[0, 1]$, under the norm

$$\|y\|_m = \left[\int_0^1 y^* y \; dt\right]^{1/2} = \left[\int_0^1 \sum_{i=1}^m |y_i|^2 \; dt\right]^{1/2} .$$

Likewise, $AC^n_m[0, 1]$ [1]) denotes the space of m-dimensional vector valued functions y for which $y^{(n-1)}$ exists and is absolutely continuous.

[1]) If the indices m and n are equal to 1, they will be omitted.

Concerning the expressions

$$l_i y = \sum_{k=0}^{d_i} a_{ik} y^{(d_i - k)} ,$$

$i = 1, \ldots, m$, we assume that the coefficients a_{ik} are measurable and essentially bounded on $[0, 1]$. If a_{i0}^{-1} is also essentially bounded $l_i y$ is said to be regular in the sense of Caratheodory (C-regular). If, in addition, $a_{ik} \in C^{d_i - k}[0, 1]$, and $a_{i0} > 0$ on $[0, 1]$, then $l_i y$ is said to be regular. If $l_i y$ is not at least C-regular, then it is singular.

As a notational convenience we will assume that

$$0 \leqq d_1 \leqq d_2 \leqq \ldots \leqq d_m = n .$$

As has been shown in [8] if the measures ν_{ij} have reasonably well behaved absolutely continuous parts, the functionals (1.3) can be written via integration by parts in the vector form

$$\begin{aligned} Uy &= \sum_{j=1}^{n} A_j y^{(n-j)}(0) + \sum_{j=1}^{n} B_j y^{(n-j)}(1) + \sum_{j=1}^{n} \int_0^1 \mathrm{d}K_j y^{(n-j)} , \\ Vy &= \sum_{j=1}^{n} \int_0^1 \mathrm{d}L_j y^{(n-j)} , \end{aligned}$$

where $A_j, B_j \in \mathbb{C}^r$, $0 \leqq r \leqq n$, and K_j, L_j are respectively $r \times 1$, $s \times 1$ matrix valued measures of bounded variation ($s < \infty$) with support in $(0, 1)$ such that K_j, L_j, $j = 1, \ldots, n - 1$ are singular.

It will be convenient here to write (2.1) in the more compact form

$$\begin{pmatrix} U \\ V \end{pmatrix} y = \int_0^1 \mathrm{d}\bar{w}\, \hat{y} ,$$

where

$$\bar{w} = \begin{pmatrix} K_n, \ldots, K_1 \\ L_n, \ldots, L_1 \end{pmatrix} + \begin{pmatrix} A_n\mu(0) + B_n\mu(1), \ldots, A_1\mu(0) + B_1\mu(1) \\ 0, \qquad \ldots, \qquad 0 \end{pmatrix}$$

($\mu(0)$ and $\mu(1)$ stand for point mass measures 1 concentrated at 0 and 1) and $\hat{y} = (y, \ldots, y^{(n-1)})^t$.

3. The Operator $\hat{L}$. Let D' denote the set of elements y in $AC^n[0, 1]$ satisfying $l_j y \in \mathscr{L}^2_m[0, 1]$, $j = 1, \ldots, m$.

Let D denote the set of elements y in D' satisfying $Uy = 0$, $Vy = 0$, and let

$$\hat{l} y = (l_1 y, \ldots, l_m y)^t = \sum_{k=0}^{n} a_k y^{(n-k)} ,$$

where a_k is the $m \times 1$ matrix $(a'_{k1}, \ldots. a'_{km})^t$, and

$$a'_{ki} = \begin{cases} a_{ki} & \text{if } d_i \geqq n - k \\ 0 & \text{otherwise} . \end{cases}$$

We now define the operator $\hat{L}: \mathscr{L}^2[0, 1] \to \mathscr{L}^2_m[0, 1]$ as the restriction of $\hat{l}$ to the domain D ($\hat{L}y = \hat{l}y$ for all $y \in D$). In the case $m = 1$, we write $\hat{L}$ as L and $\hat{l}$ as l. We note that L is the scalar version of the operator studied in [2], [3] and [8].

As was noted in Krall and Brown [8] the results concerning adjoints of operators determined by Stieltjes boundary value systems, where the coefficients a_i are $m \times m$ matrices, carry over with slight notational changes to general $k \times m$ "rectangular" case. We state the appropriate generalization pertaining to $\hat{L}$.

3.1. Theorem. *The adjoint of $\hat{L}$, denoted by $\hat{L}^*$, is a closed linear relation in $\mathscr{L}^2_m[0, 1] \times \mathscr{L}^2[0, 1]$ with graph*

$$G(\hat{L}^*) = \{(z, \hat{l}_n^+ z);\ z \in D^*\}\,.$$

The domain D^* consists of those elements

$$z = (z_1, \ldots, z_m)^t$$

in $\mathscr{L}^2_m[0, 1]$ satisfying

1. $\hat{l}_j^+ z + K_{j+1}^* \phi + L_{j+1}^* \psi \in AC[0, 1]$,

$j = 0, 1, \ldots, n - 1$;

2. $\hat{l}_0^+ z$ *exists a.e. and is in* $\mathscr{L}^2[0, 1]$;

3. $\hat{l}_j^+ z(0+) = -A_{j+1}^* \phi$, $\hat{l}_j^+ z(1-) = B_{j+1}^* \phi$,

$j = 0, 1, \ldots, n - 1$; *where*

$$\begin{aligned}
\hat{l}_0^+ z &= \quad a_0^* z\,, \\
\hat{l}_1^+ z &= -[l_0^+ z + K_1^* \phi + L_1^* \psi]' + a_1^* z\,, \\
&\ldots \\
\hat{l}_n^+ z &= -[l_{n-1}^+ z + K_n^* \phi + L_n^* \psi]' + a_n^* z\,,
\end{aligned}$$

ϕ is a parameter in C^r; and ψ is a parameter in C^s.

Note that since K_j, L_j, $j = 1, \ldots, n - 1$ are singular,

$$\begin{aligned}
\hat{l}_j^+ z &= -(\hat{l}_{j-1}^+ z)' + a_j^* z\,, \quad \text{a.e.}\,, \\
&= \sum_{i=0}^{j} (-1)^{j-i} (a_i^* z)^{(j-i)}\,, \quad \text{a.e.}\,,
\end{aligned}$$

$j = 1, \ldots, n - 1$, and

$$l_n^+ z = \sum_{j=0}^{n} (-1)^{n-i} (a_i^* z)^{(n-i)} - K_n^{*\prime} \phi - L_n^{*\prime} \psi\,, \quad \text{a.e.}$$

The derivatives $K_n^{*\prime}$ and $L_n^{*\prime}$ represent only the derivatives of the absolutely continuous parts of K_n^* and L_n^*.

There is another characterization of D^* available if the coefficients of $\hat{l}y$ are sufficiently smooth. Suppose the components of $a_i \in C^{n-i}[0, 1]$, $i = 0, \ldots, n$. Then the partial adjoints $\hat{l}_j^+ z$ can be written, via Leibnitz's rule, as

$$\hat{l}_j^+ z = \sum_{r=0}^{n} \alpha_{jr} z^{(r)}, \quad \text{a.e.},$$

$j = 0, \ldots, n - 1$, and

$$\hat{l}_n^+ z = \sum_{r=0}^{n} \alpha_{nr} z^{(r)} - K_n^{*\prime} \phi - L_n^{*\prime} \psi, \quad \text{a.e.},$$

where

$$\alpha_{jr} = \sum_{i=0}^{j-r} (-1)^{j-i} \binom{j-i}{r} a_i^{*(j-i-r)}.$$

If the $n \times nm$ matrix B is defined by

$$B = \begin{bmatrix} \alpha_{00} & 0 & 0 \\ \alpha_{10} & \alpha_{11} & 0 \\ \vdots & \vdots & \vdots \\ \alpha_{n-1,0} & \alpha_{n-1,1} & \alpha_{n-1,n-1} \end{bmatrix},$$

then the following description of D^* in terms of B is immediate.

3.2. Theorem. *D^* consists of all elements z in $\mathscr{L}_m^2[0, 1]$ satisfying*

1. $B\hat{z} + \tilde{K}^*\phi + \tilde{L}^*\psi \in AC_{nm}[0, 1]$,
2. $B\hat{z}(0+) = -\tilde{A}^*\phi$, $B\hat{z}(1-) = \tilde{B}^*\phi$,
3. $\hat{l}_n^+ z \in \mathscr{L}^2[0, 1]$,

where

$$\hat{z} = (z, z', \ldots, z^{(n-1)})^t, \quad \tilde{K}^* = (K_1^*, \ldots, K_n^*)^t, \quad \tilde{L}^* = (L_1^*, \ldots, L_n^*)^t,$$
$$\tilde{A}^* = (A_1^*, \ldots, A_n^*)^t, \quad \tilde{B}^* = (B_1^*, \ldots, B_n^*)^t.$$

3.3. Remark. Taking $\phi = 0$, $\psi = 0$ we see that D^* is dense since for example it contains all m dimensional C^∞ functions with support in $(0, 1)$.

In the case $m = 1$ and ly is C-regular we have the result:

3.4. Theorem. *L is closed and has closed range. In other words, L is "normally solvable".*

Proof. See [3; Theorem 4.3].

4. The Minimization Problem. We begin by retracing the reasoning of [4] when $m = 1$. In order for the minimization problem to make sense, there must exist an element $\bar{y} \in AC^n[0, 1]$ such that $l\bar{y} \in \mathscr{L}^2[0, 1]$ and the constraints (1.2) are satisfied.

Define L_r by setting $L_r y = ly$ on $D_r = D + \bar{y} = \{y : y = x + \bar{y},\ x \in D\}$. Since L has closed range, its "translate" L_r has also. Since $R(L_r)$ is a closed flat in Hilbert space, there is a unique element f of minimum norm in $R(L_r)$, and, because of its minimal nature, f is orthogonal to $R(L)$ (see diagram).[2])

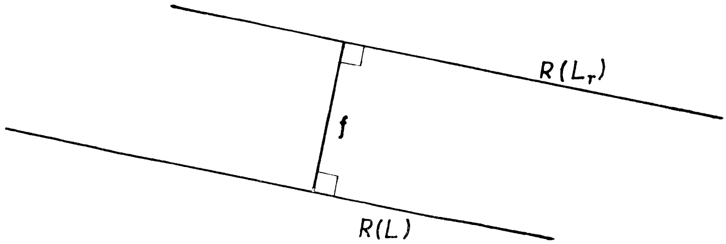

Fig. 1.

Since $R(L)^\perp = N(L^*)$, $f \in N(L^*)$. Hence y is a solution to the minimization problem if and only if $\lambda_i(y) = r_i$ and $L_r y = f \in N(L^*)$ or, equivalently, if and only if y satisfies (1.4). Moreover y is unique if and only if $N(L) = 0$.

For $m > 1$ we proceed in the same fashion, using the operator $\hat{L}$ (§ 3) and setting $\hat{L}_r y = \hat{l} y$ on D_r (defined as above). The only difficulty is in determining when $\hat{L}$ has closed range. Although this question has not been completely solved, we give various sufficient conditions implying closed range, which are adequate in most circumstances.

We require two preliminary lemmas.

4.1. Lemma. *Let T be a closed linear operator with domain and range in Banach spaces U, V. Then T has closed range if and only if*

$$\gamma(T) = \inf_{x \in D(T)} \frac{\|Tx\|_V}{d(x, N(T))} > 0 .$$

($0/0$ *is defined to be* ∞), *where* $d(x, N(T))$ [3]) *is the distance in the U metric between x and $N(T)$.*

Proof. See Goldberg [6], p. 98.

4.2. Lemma. *$\hat{L}$ is closed if one of the following conditions are satisfied:*

1. *The expressions $l_i y$, $i = 1, \ldots, m$, are C-regular.*
2. *The coefficients $a_i \in C^{n-i}[0, 1]$, $i = 1, \ldots, n$ and l_m is regular.*

Proof. Suppose $\{y_k\}$ is a sequence in D such that $y_k \to y$ and $\hat{L} y_k \to z$ where $z = (z_1, \ldots, z_m)^t$. Let L_m be the operator generated by l_m on D. Under either hypo-

---

[2]) This is a version the classical projection theorem for Hilbert spaces. For a proof consult Luenberger [9], p. 64.

[3]) Since T is closed, it is trivial that $N(T)$ is closed and therefore $d(x, N(T))$ exists.

thesis L_m is closed (Theorem 3.4). Hence $y \in D$ and $L_m y = z_m$. Suppose the first condition is satisfied. From standard theory (see [6] p. 145) the operators

$$l_i : AC^{d_i}[0, 1] \to L^2[0, 1], \quad i = 1, \ldots, m$$

are closed. Since $D \subset \bigcap AC^{d_i}[0, 1]$, and $l_i y_k \to z_i$ we conclude that $l_i y = z_i$, in other words $\hat{L}y = z$. Thus $\hat{L}$ is closed. Turning to the second condition, Green's relation ([7] Theorem 4.5) implies

$$\begin{array}{ccc} \langle \hat{l} y_i, z \rangle & = & \langle y_i, l^+ z \rangle \\ \downarrow & & \downarrow \\ \langle \psi, z \rangle & = & \langle y, l^+ z \rangle . \end{array} \tag{4,1}$$

Also

$$\langle \hat{l} y, z \rangle = \langle y, l^+ z \rangle \tag{4,2}$$

for all z in D^*. Subtracting (4.2) from (4.1) we have

$$\langle \psi - \hat{l} y, z \rangle = 0$$

for all z in D^*. Since D^* is dense. (Remark 3.3), $\hat{l} y = \psi$. Thus in this case as well $\hat{L}$ is closed.

4.3. Theorem. *Suppose $l_1 y = \alpha y$ where $\alpha \neq 0$ is a scalar and that one of the hypotheses of Lemma 4.2 hold for the operator $\hat{L}'$: $\mathscr{L}^2[0, 1] \to \mathscr{L}^2_{m-1}[0, 1]$ determined by $y \to (l_2 y, \ldots, l_m y)^t$ for y in D, then $R(\hat{L})$ is closed.*

Proof. By Lemma 4.2, $\hat{L}'$ is closed. It is trivial to verify that

$$\hat{L} y = \begin{pmatrix} \alpha y \\ \hat{L}' y \end{pmatrix} \tag{4,3}$$

is also closed. Now

$$\|\hat{L} y\|_m = \left[\int_0^1 \sum_{i=1}^m |l_i y|^2 \, dt \right]^{1/2} \geqq \|\alpha y\|$$

and since $N(\hat{L}) = \{0\}$

$$d(y, N(\hat{L} v)) = \|y\| .$$

Hence

$$\gamma(\hat{L}) \geqq \frac{\|\alpha y\|}{\|y\|} = |\alpha| > 0 .$$

Applying Lemma 4.1, $R(\hat{L})$ is closed.

4.4. Remark. It is well known (see [6], p. 56) that $\hat{L}$ is closed if and only if $\hat{L}^{**} = \hat{L}$. Consequently if $\hat{L}$ satisfies Lemma 4.2 or has the form (4.3) the adjoint of $\hat{L}^*$ is $\hat{L}$. Also, since $\hat{L}$ is an operator D^* is dense. Hence Remark 3.3 has been extended to cases where the a_i are not necessarily in $C^{n-i}[0, 1]$.

4.5. Theorem. *Suppose $N(L_m) = N(\hat{L})$ [4]) and one of the hypotheses of Lemma* 4.2 *hold for $\hat{L}$, then $R(\hat{L})$ is closed.*

Proof. We compute $\gamma(\hat{L})$. By Lemma 4.1 the normal solvability of L_m (Theorem 3.4) implies $\gamma(L_m) > 0$. As in the previous theorem, the definition of the norm in $\mathscr{L}_m[0, 1]$ easily gives the inequality

$$\|\hat{L}y\|_m \geqq \|l_m y\| . \tag{4,4}$$

By assumption $N(\hat{L}) = N(\hat{L}_m)$. Therefore

$$d(y, N(\hat{L})) = d(y, N(L_m)) . \tag{4,5}$$

Putting (4.4) and (4.5) together, we have

$$\gamma(\hat{L}) = \inf_{y \in D(\hat{L})} \frac{\|\hat{L}y\|_m}{d(y, N(\hat{L}))} \geqq \inf_{y \in D(L_m)} \frac{\|L_m y\|}{d(y, N(L_m))} = \gamma(L_m) > 0 .$$

Since $\gamma(\hat{L}) > 0$, it follows from Lemmas 4.1, 4.2 that $R(\hat{L})$ is closed.

4.6. Remark. To summarize Theorems 4.3 and 4.5, consider the statements:

(i) $l_i y$, $i = 1, \ldots, m$ are C-regular expressions.

(ii) $a_i \in C^{n-i}[0, 1]$, $i = 1, \ldots, n$.

(iii) $l_m y$ is regular.

(iv) $\hat{L}y$ has the form (4.3).

(v) $N(\hat{L}) = N(L_m)$.

(vi) $R(\hat{L})$ is closed.

Then the following has been shown: (i), (iv) $\Rightarrow$ (vi), (ii), (iii), (iv) $\Rightarrow$ (vi); (i), (v) $\Rightarrow$ $\Rightarrow$ (vi); (ii), (iii), (v) $\Rightarrow$ (vi).

Thus we have *four* sets of sufficient conditions for the closure of $R(\hat{L})$. A common feature of all of them is that an expression of highest degree $l_m y$ be regular or C-regular. Intuitively this seems the most essential condition. The behavior of $\hat{L}$ ought to be dominated by L_m. Probably however the last word on the closure of $R(\hat{L})$ has not been said. It seems a reasonable conjecture that conditions on $N(\hat{L})$ can be eliminated and that $R(\hat{L})$ *is closed if $l_m y$ is C-regular.*

Repeating the reasoning at the beginning of this section, we have:

4.7. Theorem. *If any of the sufficient conditions stated above hold for the closure of $R(\hat{L})$, then there is a solution s of the minimization problem* (1.1), (1.2). *This solution is completely characterized by the equation*

$$\hat{L}^* \hat{L}_r s = 0 , \tag{4,6}$$

[4]) This is true for example if L_m is 1-1.

where $\hat{L}$, $\hat{L}^*$ are the operators defined in § 3, *and $\hat{L}_r$ is defined by*

$$\hat{L}_r y = \hat{l} y$$

on the domain

$$D_r = \{y : y = x + \bar{y};\ x \in D,\ \bar{y} \text{ satisfies } (1.1), (1.2)\}\,.$$

4.8. Remark. Written out, (4.6) says that s satisfies

(1) $\lambda_j(s) = r_j$, $j = 1, \ldots, l$.

(2) $\hat{l}_n^+ \hat{l} s = 0$.

(3) $\hat{l}_j^+ \hat{l} s + K_{j+1}^* \phi + L_{j+1}^* \psi \in AC[0, 1]$, $j = 0, 1, \ldots, n - 1$.

(4) $\hat{l}_j^+ \hat{l} s(0+) = -A_{j+1}^* \phi$, $\hat{l}_j^+ \hat{l} s(1-) = B_{j+1} \phi$, $j = 0, 1, \ldots, n - 1$.

What can be said concerning the smoothness of s?

4.9. Theorem. *Let the expressions $l_i y$ be regular. Then*

1. $s \in AC^n[0, 1]$.

2. $s \in C^{n+j}[0, 1]$, $j = 0, 1, \ldots, n - 1$

if and only if $K_i^* \phi + L_i^* \psi$, $i = 1, \ldots, j + 1$ are continous.

3. *If O is an open set in the complement of the support of $\bar{w}$, then $s \in C^{2n}[O]$.*

4. *If $m = 1$, then ls vanishes on $[0, \alpha)$, $(\beta, 1]$, where $\alpha =$* inf. supp. $\bar{w}$, $\beta =$ sup. supp. $\bar{w}$.

Proof. The first statement is immediate from the definition of D_r.

From the representation of D^* given by Theorem 3.3

$$B(\hat{l}s, \ldots, \hat{l}s^{(n-1)})^t + \tilde{K}^* \varphi + \tilde{L}^* \psi \in AC_{nm}[0, 1]\,.$$

Looking at the components, this expression says that $(-1)^j\, a_0^* a_0 s^{(n+j)} + T(s, \ldots$ $\ldots, s^{(n+j-1)}) + K_{j+1}^* \phi + L_{j+1}^* \psi$ is absolutely continuous, where $T(s, \ldots, s^{(n+j-1)})$ denotes a linear combination of terms involving lower order derivatives of s. By our characterization of s, (Remark 4.8–2) and the hypothesis, we know that the terms involving derivatives of order less than n are all continuous. Since the $l_i y$, $i = 1, \ldots, m$, are regular, $a_0^* a_0$ never vanishes, so that $s^{(n)}$ is discontinuous if and only if $K_1^* \phi + L_1^* \psi$ is discontinuous. Proceeding inductively, if $K_1^*, L_1^*, \ldots, K_j^*, L_j^*$ are continuous on $[0, 1]$, then $s^{(n+j)}$ is discontinuous if and only if $K_{j+1}^* \phi + L_{j+1}^* \psi$ is discontinuous, proving the second statement.

Since s is in the null space of $\hat{l}_n^+ \hat{l}$ which is an ordinary differential operator on O, we can write

$$(4{,}2) \qquad (-1)^n\, a_0^* a_0 s^{(2n)} + T(s, \ldots, s^{(2n-1)}) = 0\,.$$

From the second part $s \in C^{2n-1}$ on O. Our regularity assumptions and (4.2) force it to be in C^{2n}.

The last statement has been shown for singular measures in [4] Corollary 4.3. Since the proof is the same in this case we omit it.

4.10. Remarks. 1. The hypotheses of Theorem 4.6 can be weakened slightly. Since $a_0^* a_0$ involves only the coefficients of the expressions of maximum order, it is sufficient to require that only these coefficients be nonvanishing on $[0, 1]$.

2. Left open is the question of when s vanishes on $[0, \alpha)$, $(\beta, 1]$ in the case $m > 1$.

3. Also exactly what form Theorem 4.9 takes when the $l_i y$ are merely C-regular is unknown at this time.

4. The reader may verify that the sufficient conditions stated previously concerning the closure of the range of $\hat{L}$ are independent of the assumptions that K_j, L_j, $j = 1, \ldots, n-1$, are singular. Moreover, as was pointed out in [8], the characterization of $\hat{L}^*$ holds in the case where K_j, L_j, $j = 1, \ldots, n$ are *arbitrary* measures of bounded variation. Hence equation (4.6) remains valid. However, the technique of Theorem 4.9 fails, and the continuity properties of s are unknown.

4.8. A Simple Example. Consider minimizing

$$J(y, y', y'') = \int_0^2 (y')^2 \, dt + \int_0^2 (y'')^2 \, dt$$

under the constraints

$$y(0) = -1, \quad y'(0) = 1, \quad y'(1) + y(2) = 10. \tag{4,9}$$

Here

$$\hat{l}y = \begin{pmatrix} 1 \\ 0 \end{pmatrix} y'' + \begin{pmatrix} 0 \\ 1 \end{pmatrix} y'.$$

Writing the boundary conditions in vector form, we find

$$A_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \tag{4,10}$$

$$dk_2 = 0, \qquad dk_1 = \begin{pmatrix} 0 \\ u(1) \\ 0 \end{pmatrix},$$

hence

$$\psi = 0, \qquad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}.$$

Moreover

$$l_0^+ = (1,0)\, z\,, \quad \hat{l}_1^+ z = -(1,0)\, z' + (0,1)\, z\,, \quad \hat{l}^+ z = (1,0)\, z'' - (0,1)\, z'\,.$$

Therefore,

$$\hat{l}_0^+ \hat{l} y = y''\,, \quad \hat{l}_1^+ \hat{l} y = -y''' + y'\,, \quad \hat{l}^+ \hat{l} y = y^{\mathrm{IV}} - y''\,. \tag{4,11}$$

By Theorem 4.9 the solution $s \in C^1[0, 2]$. Using this fact, as well as (4.10), (4.11) and (4.7), s must satisfy the eight equations

$$\begin{aligned} s''(0+) &= -\phi_3\,, \\ -s'''(0+) + s'(0+) &= -\phi_1\,, \\ s''(2-) &= 0\,, \\ -s'''(2-) + s'(2-) &= \phi_2\,, \\ s''(1+) - s''(1-) &= -\phi_2\,, \\ -s'''(1+) + s'(1+) - (-s'''(1-) + s'(1-)) &= 0\,, \\ s(1+) - s(1-) &= 0\,, \\ s'(1+) - s'(1-) &= 0\,, \end{aligned} \tag{4,12}$$

as well as the three interpolation conditions (4.9). Hence

$$s = \begin{cases} c_1 + c_2 t + c_3 e^t + c_4 e^{-t}\,, & 0 \leqq t < 1 \\ d_1 + d_2 t + d_3 e^t + d_4 e^{-t}\,, & 1 < t \leqq 2\,. \end{cases} \tag{4,13}$$

We may substitute (4.13) into the 11 equations (4.9), (4.12) obtaining the system

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & e & -e^{-1} & 1 & 2 & e^2 & e^{-2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^2 & e^{-2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -e & -e^{-1} & 0 & 0 & e & e^{-1} & 0 & 1 & 0 \\ 0 & 2 & e & -e^{-1} & 0 & 0 & -e & e^{-1} & 0 & 0 & 0 \\ -1 & -1 & -e & -e^{-1} & 1 & 1 & e & e^{-1} & 0 & 0 & 0 \\ 0 & -1 & -e & e^{-1} & 0 & 1 & e & -e^{-1} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ d_1 \\ d_2 \\ d_3 \\ d_4 \\ \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 10 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this we find

$$(4{,}14)\qquad s = \begin{cases} -4\cdot9393 + 3\cdot5682t + \cdot68556e^{t} + 3\cdot2537e^{-t}, & 0 \leqq t < 1 \\ -1\cdot3711 + 3\cdot5682t + \cdot029230e^{t} - 1\cdot5959e^{-t}, & 1 < t \leqq 2 \end{cases}$$

and

$$\phi_1 = -\phi_2 = c_2 = -3\cdot5682\,, \quad \phi_3 = -3\cdot9393\,.$$

The minimum value of J may easily be calculated from (4.14).

Variational problems with more complicated differential expressions and/or multipoint side conditions lead of course to much higher dimensional systems. However all such problems may be handled in principle the same way as the above example.

References

[1] *L. D. Berkovitz* and *H. Pollard:* A nonclassical variational problem arising from an optimal filter problem II, Arch. Ratl. Mech. Anal. *38* (1970), 161—172.

[2] *R. C. Brown:* Duality theory for *n*-th order differential operators under Stieltjes boundary conditions, S.I.A.M. J. Math. Anal., to appear.

[3] *R. C. Brown:* Duality theory for *n*-th order differential operators under Stieltjes boundary conditions, II: Nonsmooth coefficients and nonsingular measures, Ann. Mat. Pura. Appl., to appear.

[4] *R. C. Brown:* Adjoint domains and generalized splines, Czech. Math. J., *25* (1975), 134—147.

[5] *N. Dunford* and *J. T. Schwartz:* Linear operators, Part 1, Interscience, New York, 1975.

[6] *S. Goldberg:* Unbounded linear operators, McGraw-Hill, New York, 1966.

[7] *M. Golomb* and *J. Jerome:* Linear ordinary differential equations with boundary conditions on arbitrary point sets, Trans. Amer. Math. Soc., *153* (1971), 235—264.

[8] *A. M. Krall* and *R. C. Brown:* *n*-th order differential systems under Stieltjes boundary conditions, MRC Tech. Summ. Rept. #1581.

[9] *D. G. Lunberger:* Optimization by vector space methods, John Wiley, New York, 1969.

[10] *L. Schumaker:* Spline functions: Theory and applications, in press.

Authors' addresses: R. C. Brown, Department of Mathematics, the University of Alabama, University, Alabama 35486, U.S.A.; A. M. Krall, Department of Mathematics, the Pennsylvania State University, University Park, Pennsylvania 16802, U.S.A.