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ON ISOTROPIC TENSORS

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1. Let M be an oriented Riemannian two-dimensional manifold of class C^∞ with the boundary ∂M. Let $\{U_\alpha\}$ be an open covering of M such that there is, on each U_α, a field of orthonormal frames $\{v_1, v_2\}$ with $v_1, v_2 \in T(M)$; let $\{\omega^1, \omega^2\}$ be the dual coframes. On U_α, the metric form of M is given by

$$g = (\omega^1)^2 + (\omega^2)^2 . \tag{1}$$

Let the 1-form ω_1^2 be defined by

$$\mathrm{d}\omega^1 = -\omega^2 \wedge \omega_1^2 , \quad \mathrm{d}\omega^2 = \omega^1 \wedge \omega_1^2 . \tag{2}$$

Then the Gauss curvature of M is given by

$$\mathrm{d}\omega_1^2 = -K\omega^1 \wedge \omega^2 . \tag{3}$$

On M, be given a quadratic form S; its expression in U_α be

$$S = S_{11}(\omega^1)^2 + 2S_{12}\omega^1\omega^2 + S_{22}(\omega^2)^2 . \tag{4}$$

The covariant derivatives $S_{ijk} = S_{jik}$ of the symmetric tensor S_{ij} (with respect to the coframe ω^1, ω^2) be defined by

$$\begin{aligned} \mathrm{d}S_{11} - 2S_{12}\omega_1^2 &= S_{111}\omega^1 + S_{112}\omega^2 , \\ \mathrm{d}S_{12} + (S_{11} - S_{22})\,\omega_1^2 &= S_{121}\omega^1 + S_{122}\omega^2 , \\ \mathrm{d}S_{22} + 2S_{12}\omega_1^2 &= S_{221}\omega^1 + S_{222}\omega^2 . \end{aligned} \tag{5}$$

Then

$$J(S) = S_{121}(S_{112} - S_{222}) + S_{122}(S_{221} - S_{111}) \tag{6}$$

is an invariant. We are going to prove this auxiliary result as well as the following

Theorem. *Let the data be as above. Further, suppose that:* (i) $K > 0$ *on* M; (ii) $J(S) \geqq 0$ *on* M; (iii) *there is a function* $\lambda : \partial M \to \mathcal{R}$ *such that* $S = \lambda g$ *on* ∂M. *Then there is a function* $\Lambda : M \to \mathcal{R}$ *such that* $S = \Lambda g$ *on* M.

I am going to prove this result by means of an integral formula which is a generalization of an integral formula introduced by A. Švec [1]. Švec constructed a certain 1-form τ on surfaces of E^3; he claims this form to be invariant without proving it. In what follows, I am going to prove this; in fact, I am going to show that τ is invariant on oriented surfaces of E^2. Because of that, I restrict myself to the case that M is a surface of E^3; it is easy to see that the general proof of our Theorem is exactly the same. From a general point of view, remark that I get an integral formula for a symmetric tensor without supposing this tensor to be of the Ricci type; see, p. ex., [2].

2. Let $M \subset E^3$ be a surface of class C^∞. Let us cover M by domains U_α such that there is, on each U_α, a field of orthonormal frames $\{m; v_1, v_2, v_3\}$ with $m \in U_\alpha \subset M$; $v_1, v_2 \in T_m(M)$. Then

(7) $$dm = \omega^1 v_1 + \omega^2 v_2 ,$$
$$dv_1 = \omega_1^2 v_2 + \omega_1^3 v_3 , \quad dv_3 = -\omega_1^2 v_1 + \omega_2^3 v_3 , \quad dv_3 = -\omega_1^3 v_1 - \omega_2^3 v_2 ;$$

(8) $$d\omega^i = \omega^j \wedge \omega_j^i , \quad d\omega_i^j = \omega_i^k \wedge \omega_k^j$$
$$\text{with} \quad \omega^3 = 0 , \quad \omega_i^j + \omega_j^i = 0 .$$

From $\omega^3 = 0$, we get the existence of functions a, b, c such that

(9) $$\omega_1^3 = a\omega^1 + b\omega^2 , \quad \omega_2^3 = b\omega^1 + c\omega^2 ;$$

further,

(10) $$da - 2b\omega_1^2 = \alpha\omega^1 + \beta\omega^2 ,$$
$$db + (a - c)\,\omega_1^2 = \beta\omega^1 + \gamma\omega^2 ,$$
$$dc + 2b\omega_1^2 = \gamma\omega^1 + \delta\omega^2 .$$

As always, let

(11) $$2H = a + c , \quad K = ac - b^2$$

define the mean and Gauss curvature resp.

Let $\{m; w_1, w_2, w_3\}$ be another field of orthonormal frames on U_α, i.e.,

(12) $$v_1 = \varepsilon_1 \cos\varphi \,.\, w_1 - \varepsilon_1 \sin\varphi \,.\, w_2 , \quad v_2 = \sin\varphi \,.\, w_1 + \cos\varphi \,.\, w_2 ,$$
$$v_3 = \varepsilon_2 w_3 ; \quad \varepsilon_1^2 = \varepsilon_2^2 = 1$$

and φ a function on U_α. Further, let

$$(13)\quad \mathrm{d}m = \tau^1 w_1 + \tau^2 w_2 ,$$

$$\mathrm{d}w_1 = \tau_1^2 w_2 + \tau_1^3 w_3 , \quad \mathrm{d}w_2 = -\tau_1^2 w_1 + \tau_2^3 w_3 , \quad \mathrm{d}w_3 = -\tau_1^3 w_1 - \tau_2^3 w_2 ;$$

$$(14)\quad \tau_1^3 = a^* \tau^1 + b^* \tau^2 , \quad \tau_2^3 = b^* \tau^1 + c^* \tau^2 ;$$

$$\mathrm{d}a^* - 2b^* \tau_1^2 = \alpha^* \tau^1 + \beta^* \tau^2 , \quad \text{etc.}$$

From (7), (13) and (12), we get

$$(15)\quad \tau^1 = \varepsilon_1 \cos \varphi \,.\, \omega^1 + \sin \varphi \,.\, \omega^2 , \qquad \tau^2 = -\varepsilon_1 \sin \varphi \,.\, \omega^1 + \cos \varphi \,.\, \omega^2 ,$$

i.e.,

$$(16)\quad \omega^1 = \varepsilon_1 \cos \varphi \,.\, \tau^1 - \varepsilon_1 \sin \varphi \,.\, \tau^2 , \quad \omega^2 = \sin \varphi \,.\, \tau^1 + \cos \varphi \,.\, \tau^2 .$$

Further, it is easy to see that

$$(17)\quad \tau_1^2 - \mathrm{d}\varphi = \varepsilon_1 \omega_1^2 ;$$

$$(18)\quad \tau_1^3 = \varepsilon_1 \varepsilon_2 \cos \varphi \,.\, \omega_1^3 + \varepsilon_2 \sin \varphi \,.\, \omega_2^3 ,$$

$$\tau_2^3 = -\varepsilon_1 \varepsilon_2 \sin \varphi \,.\, \omega_1^3 + \varepsilon_2 \cos \varphi \,.\, \omega_2^3 ;$$

$$(19)\quad \omega_1^3 = \varepsilon_1 \varepsilon_2 \cos \varphi \,.\, \tau_1^3 - \varepsilon_1 \varepsilon_2 \sin \varphi \,.\, \tau_2^3 ,$$

$$\omega_2^3 = \varepsilon_2 \sin \varphi \,.\, \tau_1^3 + \varepsilon_2 \cos \varphi \,.\, \tau_2^3$$

and

$$(20)\quad a^* = \varepsilon_2 \cos^2 \varphi \,.\, a + 2\varepsilon_1 \varepsilon_2 \sin \varphi \cos \varphi \,.\, b + \varepsilon_2 \sin^2 \varphi \,.\, c ,$$

$$b^* = -\varepsilon_2 \sin \varphi \cos \varphi \,.\, a - \varepsilon_1 \varepsilon_2 \sin^2 \varphi \,.\, b + \varepsilon_1 \varepsilon_2 \cos^2 \varphi \,.\, b +$$

$$+ \varepsilon_2 \sin \varphi \cos \varphi \,.\, c ,$$

$$c^* = \varepsilon_2 \sin^2 \alpha \,.\, a - 2\varepsilon_1 \varepsilon_2 \sin \varphi \cos \varphi \,.\, b + \varepsilon_2 \cos^2 \varphi \,.\, c .$$

Thus

$$(21)\quad H^* = \varepsilon_2 H , \quad K^* = K ,$$

the well known results. By a little more complicated calculation, we obtain

$$(22)\quad \alpha^* = \varepsilon_1 \varepsilon_2 \cos^3 \varphi \,.\, \alpha + 3\varepsilon_2 \sin \varphi \cos^2 \varphi \,.\, \beta + 3\varepsilon_1 \varepsilon_2 \sin^2 \varphi \cos \varphi \,.\, \gamma +$$

$$+ \varepsilon_2 \sin^3 \varphi \,.\, \delta ,$$

$$\beta^* = -\varepsilon_1 \varepsilon_2 \sin \varphi \cos^2 \varphi \,.\, \alpha + \left(\varepsilon_2 \cos^3 \varphi - 2\varepsilon_2 \sin^2 \varphi \cos \varphi\right) \beta +$$

$$+ \left(2\varepsilon_1 \varepsilon_2 \sin \varphi \cos^2 \varphi - \varepsilon_1 \varepsilon_2 \sin^3 \varphi\right) \gamma + \varepsilon_2 \sin^2 \varphi \cos \varphi \,.\, \delta ,$$

$$\gamma^* = \varepsilon_1\varepsilon_2 \sin^2\varphi \cos\varphi \,.\, \alpha + (\varepsilon_2 \sin^3\varphi - 2\varepsilon_2 \sin\varphi \cos^2\varphi)\,\beta - $$
$$- (2\varepsilon_1\varepsilon_2 \sin^2\varphi \cos\varphi - \varepsilon_1\varepsilon_2 \cos^3\varphi)\,\gamma + \varepsilon_2 \sin\varphi \cos^2\varphi \,.\, \delta\,,$$
$$\delta^* = -\varepsilon_1\varepsilon_2 \sin^3\varphi \,.\, \alpha + 3\varepsilon_2 \sin^2\varphi \cos\varphi \,.\, \beta - 3\varepsilon_1\varepsilon_2 \sin\varphi \cos^2\varphi \,.\, \gamma +$$
$$+ \varepsilon_2 \cos^3\varphi \,.\, \delta$$

using (10), (14), (17) and (20).

Now, introduce the 1-form

(23)
$$\Phi = R_1\omega^1 + R_2\omega^2\,,$$
$$R_1 := (a - c)\,\beta + b(\gamma - \alpha)\,, \quad R_2 := (a - c)\,\gamma + b(\delta - \beta)\,,$$

this being exactly the form τ introduced in [1]. From (14) and (16), we get

(24)
$$R_1^* = \cos\varphi \,.\, R_1 + \varepsilon_1 \sin\varphi \,.\, R_2\,, \quad R_2^* = -\sin\varphi \,.\, R_1 + \varepsilon_1 \cos\varphi \,.\, R_2\,,$$

i.e.,

(25)
$$\Phi^* = \varepsilon_1 \Phi\,.$$

Thus we have proved that Φ *is an invariant form on oriented surfaces.*

3. Now, consider the quadratic form (10) on our surface M. From

(26)
$$S = S_{11}(\omega^1)^2 + 2S_{12}\omega^1\omega^2 + S_{22}(\omega^2)^2 = S_{11}^*(\tau^1)^2 + 2S_{12}^*\tau'\tau^2 + S_{22}^*(\tau^2)^2$$

and (15), (16), we get

(27)
$$S_{11}^* = \cos^2\varphi \,.\, S_{11} + 2\varepsilon_1 \sin\varphi \cos\varphi \,.\, S_{12} + \sin^2\varphi \,.\, S_{22}\,,$$
$$S_{12}^* = -\sin\varphi \cos\varphi \,.\, S_{11} - \varepsilon_1 \sin^2\varphi \,.\, S_{12} + \varepsilon_1 \cos^2\varphi \,.\, S_{12} +$$
$$+ \sin\varphi \cos\varphi \,.\, S_{22}\,,$$
$$S_{22}^* = \sin^2\varphi \,.\, S_{11} - 2\varepsilon_1 \sin\varphi \cos\varphi \,.\, S_{12} + \cos^2\varphi \,.\, S_{22}\,.$$

From (11) and similar equations

(28)
$$dS_{11}^* - 2S_{12}^*\tau_1^2 = S_{111}^*\tau^1 + S_{112}^*\tau^2\,,$$
$$dS_{12}^* + (S_{11}^* - S_{22}^*)\,\tau_1^2 = S_{121}^*\tau' + S_{122}^*\tau^2\,,$$
$$dS_{22}^* + 2S_{12}^*\tau_1^2 = S_{221}^*\tau^1 + S_{222}^*\tau^2$$

we obtain

$$\cos^2\varphi \,.\, S_{111} + 2\varepsilon_1 \sin\varphi \cos\varphi \,.\, S_{121} + \sin^2\varphi \,.\, S_{221} =$$
$$= \varepsilon_1 \cos\varphi \,.\, S_{111}^* - \varepsilon_1 \sin\varphi \,.\, S_{112}^*\,,$$

$$\cos^2\varphi\,.\,S_{112}+2\varepsilon_1\sin\varphi\cos\varphi\,.\,S_{122}+\sin^2\varphi\,.\,S_{222}=$$
$$=\sin\varphi\,.\,S^*_{111}+\cos\varphi\,.\,S^*_{112}\,,$$

$$-\sin\varphi\cos\varphi\,.\,S_{111}-\varepsilon_1\sin^2\varphi\,.\,S_{121}+\varepsilon_1\cos^2\varphi\,.\,S_{121}+\sin\varphi\cos\varphi\,.\,S_{221}=$$
$$=\varepsilon_1\cos\varphi\,.\,S^*_{121}-\varepsilon_1\sin\varphi\,.\,S^*_{122}\,,$$

$$-\sin\varphi\cos\varphi\,.\,S_{112}-\varepsilon_1\sin^2\varphi\,.\,S_{122}+\varepsilon_1\cos^2\varphi\,.\,S_{122}+\sin\varphi\cos\varphi\,.\,S_{222}=$$
$$=\sin\varphi\,.\,S^*_{121}+\cos\varphi\,.\,S^*_{122}\,,$$

$$\sin^2\varphi\,.\,S_{111}-2\varepsilon_1\sin\varphi\cos\varphi\,.\,S_{121}+\cos^2\varphi\,.\,S_{221}=$$
$$=\varepsilon_1\cos\varphi\,.\,S^*_{121}-\varepsilon_1\sin\varphi\,.\,S^*_{222}\,,$$

$$\sin^2\varphi\,.\,S_{112}-2\varepsilon_1\sin\varphi\cos\varphi\,.\,S_{122}+\cos^2\varphi\,.\,S_{222}=$$
$$=\sin\varphi\,.\,S^*_{221}+\cos\varphi\,.\,S^*_{222}\,,$$

i.e.,

(29)
$$S^*_{111}=\varepsilon_1\cos^2\varphi\,.\,S_{111}+\sin\varphi\cos^2\varphi\,.\,S_{112}+2\sin\varphi\cos^2\varphi\,.\,S_{121}+$$
$$+2\varepsilon_1\sin^2\varphi\cos\varphi\,.\,S_{122}+\varepsilon_1\sin^2\varphi\cos\varphi\,.\,S_{221}+\sin^3\varphi\,.\,S_{222}\,,$$

$$S^*_{112}=-\varepsilon_1\sin\varphi\cos^2\varphi\,.\,S_{111}+\cos^3\varphi\,.\,S_{112}-2\sin^2\varphi\cos\varphi\,.\,S_{121}+$$
$$+2\varepsilon_1\sin\varphi\cos^2\varphi\,.\,S_{122}-\varepsilon_1\sin^3\varphi\,.\,S_{221}+\sin^2\varphi\cos\varphi\,.\,S_{222}\,,$$

$$S^*_{121}=-\varepsilon_1\sin\varphi\cos^2\varphi\,.\,S_{111}-\sin^2\varphi\cos\varphi\,.\,S_{112}-$$
$$-\sin^2\varphi\cos\varphi\,.\,S_{121}+\cos^3\varphi\,.\,S_{121}-\varepsilon_1\sin^3\varphi\,.\,S_{122}+$$
$$+\varepsilon_1\sin\varphi\cos^2\varphi\,.\,S_{122}+\varepsilon_1\sin\varphi\cos^2\varphi\,.\,S_{221}+$$
$$+\sin^2\varphi\cos\varphi\,.\,S_{222}\,,$$

$$S^*_{122}=\varepsilon_1\sin^2\varphi\cos\varphi\,.\,S_{111}-\sin\varphi\cos^2\varphi\,.\,S_{112}+\sin^3\varphi\,.\,S_{121}-$$
$$-\sin\varphi\cos^2\varphi\,.\,S_{121}-\varepsilon_1\sin^2\varphi\cos\varphi\,.\,S_{122}+\varepsilon_1\cos^3\varphi\,.\,S_{122}-$$
$$-\varepsilon_1\sin^2\varphi\cos\varphi\,.\,S_{221}+\sin\varphi\cos^2\varphi\,.\,S_{222}\,,$$

$$S^*_{221}=\varepsilon_1\sin^2\varphi\cos\varphi\,.\,S_{111}+\sin^3\varphi\,.\,S_{112}-2\sin\varphi\cos^2\varphi\,.\,S_{121}-$$
$$-2\varepsilon_1\sin^2\varphi\cos\varphi\,.\,S_{122}+\varepsilon_1\cos^3\varphi\,.\,S_{221}+\sin\varphi\cos^2\varphi\,.\,S_{222}\,,$$

$$S^*_{222}=-\varepsilon_1\sin^3\varphi\,.\,S_{111}+\sin^2\varphi\cos\,.\,S_{112}+2\sin^2\varphi\cos\varphi\,.\,S_{121}-$$
$$-2\varepsilon_1\sin\varphi\cos^2\varphi\,.\,S_{122}-\varepsilon_1\sin\varphi\cos^2\varphi\,.\,S_{221}+\cos^3\varphi\,.\,S_{222}\,.$$

From this

(30) $$S_{121}(S_{112} - S_{222}) + S_{122}(S_{221} - S_{111}) = \\ = S^*_{121}(S^*_{112} - S^*_{222}) + S^*_{122}(S^*_{221} - S^*_{111}),$$

i.e.,

(31) $$J(S) = J^*(S).$$

Consider the 1-form

(32) $$\Psi = T_1\omega^1 + T_2\omega^2,$$
$$T_1 := (S_{11} - S_{22}) S_{121} + S_{12}(S_{221} - S_{111}),$$
$$T_2 := (S_{11} - S_{22}) S_{122} + S_{12}(S_{222} - S_{112}).$$

From (27) and (30),

(33) $$T^*_1 = \cos\varphi . T_1 + \varepsilon_1 \sin\varphi . T_2, \quad T^*_2 = -\sin\varphi . T_1 + \varepsilon_1 \cos\varphi . T_2,$$

i.e.,

(34) $$\Psi^* = \varepsilon_1 \Psi,$$

and the form Ψ (32) is thus invariant on oriented surfaces.

The equations (11) imply

(35) $$\{dS_{111} - (S_{112} + 2S_{121})\,\omega^2_1\} \wedge \omega^1 + \\ + \{dS_{112} + (S_{111} - 2S_{122})\,\omega^2_1\} \wedge \omega^2 = 2S_{12}K\omega^1 \wedge \omega^2,$$
$$\{dS_{121} + (S_{111} - S_{122} - S_{221})\,\omega^2_1\} \wedge \omega^1 + \\ + \{dS_{122} + (S_{112} + S_{121} - S_{222})\,\omega^2_1\} \wedge \omega^2 = (S_{22} - S_{11})\,K\omega^1 \wedge \omega^2,$$
$$\{dS_{221} + (2S_{121} - S_{222})\,\omega^2_1\} \wedge \omega^1 + \\ + \{dS_{222} + (2S_{122} + S_{221})\,\omega^2_1\} \wedge \omega^2 = -2S_{12}K\omega^1 \wedge \omega^2,$$

and we get the existence of functions $T_1, \ldots, T_9$ such that

(36) $$dS_{111} - (S_{112} + 2S_{121})\,\omega^2_1 = T_1\omega^1 + (T_2 - S_{12}K)\,\omega^2,$$
$$dS_{112} + (S_{111} - 2S_{122})\,\omega^2_1 = (T_2 + S_{12}K)\,\omega^1 + T_3\omega^2,$$
$$dS_{121} + (S_{111} - S_{122} - S_{221})\,\omega^2_1 = T_4\omega^1 + (T_5 + S_{11}K)\,\omega^2,$$
$$dS_{122} + (S_{112} + S_{121} - S_{222})\,\omega^2_1 = (T_5 + S_{22}K)\,\omega^1 + T_6\omega^2,$$
$$dS_{221} + (2S_{121} - S_{222})\,\omega^2_1 = T_7\omega^1 + (T_8 + S_{12}K)\,\omega^2,$$
$$dS_{222} + (2S_{122} + S_{221})\,\omega^2_1 = (T_8 - S_{12}K)\,\omega^1 + T_9\omega^2.$$

By means of these formulas, we get

$$(37) \qquad \mathrm{d}\Psi = -\{2S_{121}(S_{112} - S_{222}) + 2S_{122}(S_{221} - S_{111}) + \\ + ((S_{11} - S_{22})^2 + 4S_{12}^2)K\}\,\omega^1 \wedge \omega^2 .$$

The Stokes formula $\int_{\partial M} \Psi = \int_M \mathrm{d}\Psi$ reads now

$$(38) \qquad \int_{\partial M} \{(S_{11} - S_{22})(S_{121}\omega^1 + S_{122}\omega^2) + \\ + S_{12}(S_{221}\omega^1 - S_{111}\omega^1 + S_{222}\omega^2 - S_{112}\omega^2)\} = \\ = -\int_M \{2J(S) + ((S_{11} - S_{22})^2 + 4S_{12}^2)K\}\,\omega^1 \wedge \omega^2 .$$

The proof of our Theorem follows easily. On the boundary ∂M, we have $S_{11} = S_{22} = \lambda$, $S_{12} = 0$, and the left-hand side of (38) is thus equal to zero. Because of $K > 0$ and $J(S) \geqq 0$, we get $(S_{11} - S_{22})^2 + 4S_{12}^2 = 0$, i.e., $S_{11} - S_{22} = S_{12} = 0$. We are finished setting $S_{11} = S_{22} = \Lambda$.

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