Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 27 (1977), No. 1, 43–53

Persistent URL: `http://dml.cz/dmlcz/101444`

Czechoslovak Mathematical Journal, 27 (102) 1977, Praha

ON NORMAL SEMIGROUPS

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(Received December 28, 1974)

1. INTRODUCTION

A semigroup S is called *normal* if $xS = Sx$ for all elements x of S (S. Schwarz [15]). A subsemigroup A of a semigroup S is called a *bi-ideal* of S if $ASA \subseteq A$. The purpose of this paper is to give some properties which characterize normal semigroups and normal regular semigroups in terms of bi-ideals.

Let $\mathfrak{P}(S)$ be the set of all non-empty subsets of a semigroup S. Then $\mathfrak{P}(S)$ is a semigroup under the multiplication of subsets. In [3] S. Lajos has proved that if a semigroup S is normal then $\mathfrak{P}(S)$ is also normal. Let $\mathfrak{B}(S)$ be the set of all bi-ideals of a semigroup S. Then $\mathfrak{B}(S)$ is a subsemigroup of $\mathfrak{P}(S)$. In § 5 we shall prove that S is normal if and only if $\mathfrak{B}(S)$ is normal. A semigroup S is called *regular* if, for any element a of S, there exists an element x in S such that $a = axa$. J. LUH [9] has proved that a semigroup S is regular if and only if $\mathfrak{B}(S)$ is regular. It is clear that if $\mathfrak{B}(S)$ is idempotent then it is regular. S. Lajos has given an example of a semigroup S such that $\mathfrak{B}(S)$ is regular but is not idempotent and proved that, for an intra-regular semigroup S, $\mathfrak{B}(S)$ is idempotent if and only if $\mathfrak{B}(S)$ is regular ([7]). We shall prove in § 6 that, for a completely regular semigroup S or a normal semigroup S, $\mathfrak{B}(S)$ is idempotent if and only if $\mathfrak{B}(S)$ is regular.

2. DEFINITIONS

A subsemigroup A of a semigroup S is called *normal* if $xA = Ax$ for all elements x of S. A semigroup S is called *left* (*right*) *regular* if, for any element a of S, there exists an element x in S such that $a = xa^2$ ($a = a^2x$). A semigroup S is called *intra-regular* if, for any element a of S, there exist elements x and y in S such that $a = xa^2y$. A semigroup S is called *completely regular* if, for any element a of S, there exists an element x in S such that $a = axa$ and $ax = xa$.

We denote by $[x]_L$, $[x]_R$ and $[x]_B$ the principal left ideal, right ideal and bi-ideal of a semigroup S generated by x in S;

$$[x]_L = x \cup Sx\,, \quad [x]_R = x \cup xS\,, \quad [x]_B = x \cup x^2 \cup xSx\,.$$

For the terminology not defined here we refer to the book by A. H. CLIFFORD and G. B. PRESTON [1].

3. NORMAL IDEALS

Lemma 3.1. *Let A be any ideal of a semigroup S. Then*

(1) $A[x]_B = A[x]_L = Ax$ *for all* $x \in S$,

(2) $[x]_B A = [x]_R A = xA$ *for all* $x \in S$.

Proof. Let x be any element of S. Then

$$A[x]_L = A(x \cup Sx) = Ax \cup A(Sx) = Ax \cup (AS)\,x \subseteq Ax \subseteq A[x]_L$$

and

$$A[x]_B = A(x \cup x^2 \cup xSx) = Ax \cup Ax^2 \cup A(xSx) =$$
$$= Ax \cup (Ax)\,x \cup (AxS)\,x \subseteq Ax \subseteq A[x]_B\,.$$

Therefore

$$A[x]_B = A[x]_L = Ax$$

for all $x \in S$. Similarly we can prove that

$$[x]_B A = [x]_R A = xA$$

for all $x \in S$.

Theorem 3.2. *For an ideal A of a semigroup S the following conditions are equivalent*:

(1) A *is normal,*

(2) $XA = AX$ *for all* $X \in \mathfrak{P}(S)$,

(3) $XA = AX$ *for all* $X \in \mathfrak{B}(S)$,

(4) $[x]_B A = A[x]_B$ *for all* $x \in S$,

(5) $[x]_B A = A[x]_L$ *for all* $x \in S$,

(6) $[x]_B A = Ax$ *for all* $x \in S$,

(7) $[x]_R A = A[x]_B$ *for all* $x \in S$,

(8) $[x]_R A = A[x]_L$ *for all* $x \in S$,

(9) $[x]_R A = Ax$ *for all* $x \in S$,

(10) $xA = A[x]_B$ *for all* $x \in S$,

(11) $xA = A[x]_L$ *for all* $x \in S$.

Proof. First we assume that A is normal. Let X be any non-empty subset of S and xa $(x \in X,\ a \in A)$ any element of XA. Then we have

$$xa \in xA = Ax \subseteq AX$$

and so

$$XA \subseteq AX\,.$$

Similarly we can see that the converse inclusion holds. Therefore

$$XA = AX$$

for all $X \in \mathfrak{P}(S)$ and (1) implies (2). It is clear that (2) implies (3) and (3) implies (4). It follows from Lemma 3.1 that (1) and (4) $\sim$ (11) are equivalent. This completes the proof of the theorem.

As is easily seen, the product AB of ideals A and B of a semigroup S is also an ideal of S. For normal ideals of a semigroup we have the following:

Theorem 3.3. *Let A and B be any normal ideals of a semigroup S. Then the products AB and BA are also normal ideals of S and*

$$AB = BA$$

holds.

Proof. It follows from (2) of Theorem 3.2 that $AB = BA$ holds. For any element x of S we have

$$x(AB) = (xA)\,B = (Ax)\,B = A(xB) = A(Bx) = (AB)\,x\,.$$

Thus AB is normal.

Theorem 3.4. *For an ideal A of a regular semigroup S the following conditions are equivalent:*

(1) *A is normal,*
(2) $eA = Ae$ *for all idempotents e of S,*
(3) $[e]_B A = A[e]_B$ *for all idempotents e of S,*
(4) $[e]_B A = A[e]_L$ *for all idempotents e of S,*
(5) $[e]_B A = Ae$ *for all idempotents e of S,*
(6) $[e]_R A = A[e]_B$ *for all idempotents e of S,*
(7) $[e]_R A = A[e]_L$ *for all idempotents e of S,*
(8) $[e]_R A = Ae$ *for all idempotents e of S,*
(9) $eA = A[e]_B$ *for all idempotents e of S,*
(10) $eA = A[e]_L$ *for all idempotents e of S.*

Proof. It is clear that (1) implies (2). That (2) $\sim$ (10) are equivalent can be proved in a similar way as in the proof that (1) and (4) $\sim$ (11) are equivalent in Theorem 3.2.

Assume that (2) holds. In order to prove that (1) holds, let x be any element of S. Then, since S is regular, there exists an element y in S such that $x = xyx$ and yx is idempotent. Then we have

$$xA = (xyx)\,A = x((yx)\,A) = x(A(yx)) = (xAy)\,x \subseteq Ax\,.$$

Similarly we can prove that the converse inclusion holds. Therefore we obtain that

$$xA = Ax$$

for all $x \in S$ and that (2) implies (1). This completes the proof of the theorem.

4. MINIMAL NORMAL IDEALS

Let A be a normal ideal of a semigroup S, and x any element of S. Then we have

$$(xA)\,S = x(AS) \subseteq xA$$

and

$$S(xA) = S(Ax) = (SA)\,x \subseteq Ax\,.$$

This means that xA is an ideal of S. Thus we have the following:

Lemma 4.1. *Let A be any normal ideal of a semigroup S. Then, for any element x of S, xA is an ideal of S.*

S. LAJOS has given the following:

Lemma 4.2. ([4] Theorem 8). *The product of a bi-ideal and of a non-empty subset of a semigroup S is also a bi-ideal of S.*

An element z of a semigroup S is called a *zero element* of S if $az = za = z$ for all $a \in S$. Then we have the following:

Theorem 4.3. *Any minimal ideal of a semigroup S is a zero element of $\mathfrak{B}(S)$.*

Proof. Let A be a minimal ideal of S. Then it is clear that $A \in \mathfrak{B}(S)$. Let X be any bi-ideal of S. Then we have

$$XA \subseteq SA \subseteq A\,.$$

Then it follows from Lemma 4.2 and the minimality of A that

$$XA = A\,.$$

Similarly we can prove that

$$AX = A$$

for all $X \in \mathfrak{B}(S)$. This means that A is a zero element of $\mathfrak{B}(S)$.

Theorem 4.4. *Any minimal normal ideal of a semigroup is a group.*

Proof. Let A be a minimal normal ideal of a semigroup S. Let x be any element of S. Then we have

$$Ax = xA \subseteq SA \subseteq A\,.$$

Then it follows from Lemma 4.1 and the minimality of A that

$$Ax = xA = A\,.$$

This implies that

$$Ax = xA = A$$

for all $x \in A$. This means that A is a group.

5. NORMAL SEMIGROUPS

Now we give a characterization of a normal semigroup.

Theorem 5.1. *Following conditions concerning a semigroup S are equivalent to each other*:

(1) S *is normal*,
(2) $XS = SX$ *for all* $X \in \mathfrak{P}(S)$,
(3) $XS = SX$ *for all* $X \in \mathfrak{B}(S)$,
(4) $[x]_B S = S[x]_B$ *for all* $x \in S$,
(5) $[x]_B S = S[x]_L$ *for all* $x \in S$,
(6) $[x]_B S = Sx$ *for all* $x \in S$,
(7) $[x]_R S = S[x]_B$ *for all* $x \in S$,
(8) $[x]_R S = S[x]_L$ *for all* $x \in S$,
(9) $[x]_R S = Sx$ *for all* $x \in S$,
(10) $xS = S[x]_B$ *for all* $x \in S$,
(11) $xS = S[x]_L$ *for all* $x \in S$,
(12) $\mathfrak{B}(S)$ *is normal*,
(13) $[x]_B \mathfrak{B}(S) = \mathfrak{B}(S)[x]_B$ *for all* $x \in S$,
(14) $[x]_B \mathfrak{B}(S) = \mathfrak{B}(S)[x]_L$ *for all* $x \in S$,
(15) $[x]_B \mathfrak{B}(S) = \mathfrak{B}(S)\,x$ *for all* $x \in S$,
(16) $[x]_R \mathfrak{B}(S) = \mathfrak{B}(S)[x]_B$ *for all* $x \in S$,
(17) $[x]_R \mathfrak{B}(S) = \mathfrak{B}(S)[x]_L$ *for all* $x \in S$,
(18) $[x]_R \mathfrak{B}(S) = \mathfrak{B}(S)\,x$ *for all* $x \in S$,
(19) $x\,\mathfrak{B}(S) = \mathfrak{B}(S)[x]_B$ *for all* $x \in S$,

(20) $x\,\mathfrak{B}(S) = \mathfrak{B}(S)\,[x]_L$ *for all* $x \in S$,
(21) $x\,\mathfrak{B}(S) = \mathfrak{B}(S)\,x$ *for all* $x \in S$.

Proof. Since the semigroup S itself is an ideal of S, it follows from Theorem 3.2 that (1) $\sim$ (11) are equivalent.

Assume that (1) holds. Let A and X be any bi-ideals of S and a any element of A. Then we have

$$aX \subseteq aS = Sa \subseteq SA \subseteq \mathfrak{B}(S)\,A$$

and so

$$A\,\mathfrak{B}(S) \subseteq \mathfrak{B}(S)\,A\,.$$

Similarly we can prove that the converse inclusion holds. Thus we obtain that

$$A\,\mathfrak{B}(S) = \mathfrak{B}(S)\,A$$

for all $A \in \mathfrak{B}(S)$, and that $\mathfrak{B}(S)$ is normal. Therefore (1) implies (12). It is clear that (12) implies (13). Assume that (13) holds. In order to prove that S is normal, let x be any element of S. Then, for some $A \in \mathfrak{B}(S)$, we have

$$xS \subseteq [x]_B\,S = A[x]_B \subseteq S[x]_B \subseteq Sx\,.$$

Similarly we can prove that the converse inclusion holds. Thus we obtain that S is normal and that (13) implies (1). The rest of the proof can be easily seen. This completes the proof of the theorem.

The following corollary is immediate from Theorem 5.1.

Corollary 5.2. *Every one-sided ideal of a normal semigroup is a two-sided ideal.*

6. COMPLETELY REGULAR SEMIGROUPS

The following characterization of a completely regular semigroup is well-known.

Lemma 6.1. ([1] p. 112, [11] p. 425 $\sim$ *p.* 426 and [12] p. 105). *For a semigroup S the following conditions are equivalent*:

(1) *S is completely regular,*
(2) *$a \in a^2Sa^2$ for any $a \in S$,*
(3) *S is left and right regular.*

Following the terminology of A. H. Clifford and G. B. Preston [1] we say that a subset X of a semigroup S is *semiprime* if $a^2 \in X$, $a \in S$ implies $a \in X$. Now we give another criterion for a completely regular semigroup.

Theorem 6.2. *A semigroup S is completely regular if and only if every bi-ideal of S is semiprime.*

Proof. First we assume that S is completely regular. Let A be any bi-ideal of S. Let $a^2 \in A$ and $a \in S$. Then it follows from Lemma 6.1 that

$$a \in a^2Sa^2 \subseteq ASA \subseteq A .$$

This means that A is semiprime.

Conversely, every bi-ideal of S is semiprime. Then, since any one-sided ideal of a semigroup is a bi-ideal, every left and right ideal of S is semiprime. The it follows from Lemma 4.1 of [1] (p. 121) that S is left and right regular. Thus it follows from Lemma 6.1 that S is completely regular. This completes the proof of the theorem.

Let A be any bi-ideal of a completely regular semigroup S and a any element of A. Then by Lemma 6.1 we have

$$a \in a^2Sa^2 \subseteq A^2SA^2$$

and so we have

$$A \subseteq A^2SA^2 = A(ASA)A \subseteq A^3 \subseteq A^2 \subseteq A .$$

Thus we have

$$A^2 = A .$$

Therefore we have the following theorem:

Theorem 6.3. *Let S be a completely regular semigroup. Then $\mathfrak{B}(S)$ is idempotent.*

As is stated in Introduction, the following lemma is due to J. Luh [10].

Lemma 6.4. *For a semigroup S the following conditions are equivalent:*

(1) *S is regular,*
(2) *$\mathfrak{B}(S)$ is regular.*

Since a completely regular semigroup is regular, Theorem 6.3 and Lemma 6.4 imply the following:

Theorem 6.5. *For a completely regular semigroup S the following conditions are equivalent:*

(1) *$\mathfrak{B}(S)$ is idempotent,*
(2) *$\mathfrak{B}(S)$ is regular.*

The following theorem shows that, for a normal semigroup, the converse of Theorem 6.3 holds.

Theorem 6.6. *For a normal semigroup S the following conditions are equivalent:*

(1) *S is regular,*
(2) *S is left regular,*
(3) *S is right regular,*

(4) *S is intra-regular,*

(5) *S is completely regular,*

(6) *$a \in (aS)^n$ for every element a of S and every integer $n \geqq 2$.*

(7) *$\mathfrak{B}(S)$ is idempotent,*

(8) *$\mathfrak{B}(S)$ is completely regular,*

(9) *$\mathfrak{B}(S)$ is regular.*

Proof. Since S is normal, it can be easily seen that (1) $\sim$ (6) are equivalent. By Lemma 6.4, (1) and (9) are equivalent. By Theorem 6.3, (5) implies (7). It is clear that (7) implies (8), and (8) implies (9). This completes the proof of the theorem.

7. NORMAL REGULAR SEMIGROUPS

Theorem 7.1. *Any right (left, two-sided) ideal of a normal regular semigroup is regular.*

Proof. By Corollary 5.2, it suffices to prove that any twosided ideal A of a normal semigroup S is regular. Let a be any element of A. Then it follows from Theorem 6.6 that

$$a \in (aS)^3 \subseteq a(SASS)\, a \subseteq aAa\,.$$

This means that A is regular.

Theorem 7.2. *The center of a normal regular semigroup is regular.*

Proof. Let a be any element of the center Z of a normal regular semigroup S. Then, by Theorem 6.6, we have

$$a \in (aS)^3 \subseteq a(aS)\, a$$

and so there exists an element x in S such that

$$a = a(ax)\, a\,.$$

In order to prove that $ax \in Z$, let b be any element of S. Then we have

$$\begin{aligned} b(ax) = (ba)\,x = (ab)\,x = a(bx) = (aaxa)\,(bx) = aax(ab)\,x = \\ = aax(ba)\,x = aa(xb)\,(ax) = a(a(xb))\,(ax) = a(a(xb))\,(xa) = \\ = a((xb)\,a)\,(xa) = (a(xb))\,(axa) = ((xb)\,a)\,axa = (xb)\,(aaxa) = \\ = (xb)\,a = a(xb) = (ax)\,b\,. \end{aligned}$$

Therefore we have

$$ax \in Z$$

and so

$$a \in aZa\,.$$

This means that the center Z of S is regular. The proof is complete.

Now we give the following two lemmas obtained by S. Lajos.

Lemma 7.3. ([8] Theorem 2). *For a semigroup S the following conditions are equivalent*:

(1) *S is a semilattice of groups,*

(2) *S is regular and $eS = Se$ for all idempotent elements e of S.*

Lemma 7.4. ([5] Theorem 12). *If a semigroup S is a semilattice of groups, then every bi-ideal of S is two-sided.*

A semigroup S is called *viable* if $ab = ba$ whenever ab and ba are idempotents. A viable semigroup is studied by M. S. PUTCHA and J. WEISSGLASS [13].

Lemma 7.5. ([13] Theorem 6). *If a semigroup S is a semilattice of groups, then it is viable.*

Theorem 7.6. *For a regular semigroup S the following conditions are equivalent to each other*:

(1) *S is normal,*

(2) $eS = Se$ *for all idempotents e of S,*

(3) $[e]_B S = S[e]_B$ *for all idempotents e of S,*

(4) $[s]_B S = S[e]_L$ *for all idempotents e of S,*

(5) $[e]_B S = Se$ *for all idempotents e of S,*

(6) $[e]_R S = S[e]_B$ *for all idempotents e of S,*

(7) $[e]_R S = S[e]_L$ *for all idempotents e of S,*

(8) $[e]_R S = Se$ *for all idempotents e of S,*

(9) $eS = S[e]_B$ *for all idempotents e of S,*

(10) $eS = S[e]_L$ *for all idempotents e of S,*

(11) $[e]_B \mathfrak{B}(S) = \mathfrak{B}(S)[e]_B$ *for all idempotents e of S,*

(12) $[e]_B \mathfrak{B}(S) = \mathfrak{B}(S)[e]_L$ *for all idempotents e of S,*

(13) $[e]_B \mathfrak{B}(S) = \mathfrak{B}(S)e$ *for all idempotents e of S,*

(14) $[e]_R \mathfrak{B}(S) = \mathfrak{B}(S)[e]_B$ *for all idempotents e of S,*

(15) $[e]_R \mathfrak{B}(S) = \mathfrak{B}(S)[e]_L$ *for all idempotents e of S,*

(16) $[e]_R \mathfrak{B}(S) = \mathfrak{B}(S)e$ *for all idempotents e of S,*

(17) $e\mathfrak{B}(S) = \mathfrak{B}(S)[e]_B$ *for all idempotents e of S,*

(18) $e\mathfrak{B}(S) = \mathfrak{B}(S)[e]_L$ *for all idempotents e of S,*

(19) $e\mathfrak{B}(S) = \mathfrak{B}(S)e$ *for all idempotents e of S,*

(20) $\mathfrak{B}(S)$ *is viable.*

Proof. Since the semigroup S itself is an ideal of S, it follows from Theorem 3.4 that (1) $\sim$ (10) are equivalent. From this and Theorem 5.1, we can easily seen that (1) and (11) $\sim$ (19) are equivalent. We note that it follows from Lemma 7.3 that a semigroup S is normal and regular if and only if it is a semilattice of groups (see, D. Latorre [9]). Assume that S is normal. Then it follows from Theorem 5.1 and Lemma 6.4 that $\mathfrak{B}(S)$ is normal and regular. Thus $\mathfrak{B}(S)$ is a semilattice of groups. Therefore it follows from Lemma 7.5 that $\mathfrak{B}(S)$ is viable. Thus we obtain that (1) implies (20). Conversely we assume that $\mathfrak{B}(S)$ is viable. Since S is regular, by Lemma 6.4 $\mathfrak{B}(S)$ is regular. Let A be any bi-ideal of S. Then for some $X \in \mathfrak{B}(S)$ we have

$$A = AXA \subseteq ASA \subseteq A$$

and so we have

$$A = ASA$$

and both AS and SA are idempotents of $\mathfrak{B}(S)$. Since $\mathfrak{B}(S)$ is viable, we have

$$AS = SA\,.$$

Since A is any bi-ideal of S, it follows from Theorem 5.1 that S is normal. Therefore we obtain that (20) implies (1). This completes the proof of the theorem.

Finally we give a characterization of semigroups which are semilattices of groups.

Theorem 7.7. *The following conditions concerning a semigroup S are equivalent to each other*:

(1) *S is a semilattice of groups,*

(2) *S is regular with any one of the conditions* (1) $\sim$ (21) *of Theorem* 5.1 *or* (2) $\sim$ (20) *of Theorem* 7.6,

(3) $\mathfrak{B}(S)$ *is a semilattice of groups,*

(4) $\mathfrak{B}(S)$ *is regular and normal,*

(5) $\mathfrak{B}(S)$ *is regular and viable,*

(6) $\mathfrak{B}(S)$ *is a completely regular semigroup such that every bi-ideal of S is two-sided.*

Proof. It suffice to prove that (2) and (6) are equivalent. Assume that (6) holds. Since $\mathfrak{B}(S)$ is regular, S is regular by Lemma 6.4. In order to prove that S is normal, let A be any bi-ideal of S. Since $\mathfrak{B}(S)$ is completely regular, there exists a bi-ideal X of S such that

$$A = AXA \quad \text{and} \quad AX = XA\,.$$

Then, since every bi-ideal of S is two-sided, we have

$$AS = (AXA)\,S = (AX)\,(AS) = (XA)\,(AS) = X(AAS) \subseteq XA \subseteq SA\,.$$

In a similar way we can prove that the converse inclusion holds. Therefore we have

$$AS = SA$$

for all $A \in \mathfrak{B}(S)$. Then it follows from Theorem 5.1 that S is normal. Therefore we obtain that (6) implies (2). Conversely we assume that S is normal and regular. Then it follows from Theorem 6.6 that $\mathfrak{B}(S)$ is completely regular. Since S is a semilattice of groups, it follows from Lemma 7.4 that every bi-ideal of S is two-sided. Therefore we obtain that (2) implies (6). This completes the proof of the theorem.

The following well-known result is immediate from (1) and (2) of Theorem 7.7.

Corollary 7.8 ([12] Corollary II. 4.12). *For an idempotent semigroup S the following conditions are equivalent*:

(1) *S is normal,*

(2) *S is commutative.*

Acknowledgement. I am indebted to Mr. H. Komuro for some valuable comments.

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